Correction problems in electronic distance measurements

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ABSTRACT

In electronic distance measurements the transit time for the signal from the instrument to the far point and back again is determined. From this, the slope distance between the points is computed, taking into account the variability of the refractive index along the path (Part 1). The slope distance is then reduced to the geodesic on the reference ellipsoid (Part 2).

List of symbols

a	major axis of reference ellipsoid
b	geodesic (or normal section) on reference ellipsoid joining A_0 and B_0
c	velocity of light in a vacuum (=299792.5 kms/sec)
d_b	chord joining A and B
d_e	chord joining A_0 and B_0
$d\sigma$	projection of the infinitesimal increase ds onto the equi-index surface
e^2	eccentricity of reference ellipsoid
e'^2	second eccentricity $= e^2/(1-e^2)$
$\mathbf{e}_1$	unit normal vector to the ellipsoid at A_0
$\mathbf{e}_2$	unit normal vector to the ellipsoid at B_0
h	height of the point over the ref. ellipsoid (except in eqs. (1) and (2))
$\mathbf{i}$	unit vector along d_b
$\mathbf{j}$	unit vector along d_e
k	coefficient of refraction $= \varkappa/\varkappa_n$
n	refractive index
s	ray path curve joining A and B
t	half transit time
$\mathbf{v}_1$	unit tangent vector for s
$\mathbf{v}_2$	unit normal vector for s
$\mathbf{v}_3$	unit binormal vector for s
$\mathbf{w}_1$	unit vector on surface, along the curve
$\mathbf{w}_2$	unit vector on surface, perpendicular to the curve
$\mathbf{w}_3$	unit normal vector to the surface, downwards
A, B	end points of the ray
A_0, B_0	projections of the end points onto the reference ellipsoid (with normals to ellipsoid)

E	correction term see eq. (20a)
M	radius of curvature along meridian
N	radius of curvature perpendicular to meridian
R	radius of curvature along azimut α. $R \sim 6400$ kms
W^2	$1 - e^2 \sin^2 \phi$
α	azimuth of geodesic b
β	in part 1: angle between $\mathbf{v}_1$ and $-\mathbf{w}_3$, usually zenithal distance of ray path; in part 2: reduced latitude
$\varkappa$	curvature of ray path (space curve)
$\varkappa_n$	normal curvature of surface curve
λ	difference of longitude between A and B
$\mu + 90°$	angle between normal and chord, see Fig. 3
σ	centre angle, i.e. the third side in a spherical triangle with sides $(90 - \phi_A)$ and $(90 - \phi_B)$, with the enclosed angle λ
τ_g	geodetic torsion
ϕ	geographical latitude
Δs	correction term see eq. (10)

Symbols with index A refer to point A.
Symbols with index B refer to point B.
Symbols without index are usually the means of the above.
Symbols with prefix Δ refer to differences between the above values.
Prefix δ is used to indicate the error.

The accuracy obtainable with electro-optical and electro-magnetic distance measuring techniques depends on both internal and external factors. Instrument construction techniques, accuracy of the phase resolution, crystal stability, zero errors and, in electro-magnetic equipment, choice of carrier frequency and the associated beaming accuracy can be included under the first heading. The determination of the velocity of the carrier signal through the atmosphere is the principal external factor which influences accuracy.

Considerable advances have been made in reducing or eliminating the errors which are due to internal factors; and instrument accuracy even over relatively short distances is, at the present time, superior to the accuracy with which the refractive index can be determined.

To directly determine the refractive index along the signal path is difficult and expensive. The field applications of dispersion techniques in connection with electro-optical measurements are, as yet, not fully developed; and helicopter-borne refractometers can rarely be used in normal field work.

In this paper methods for determining refractive index are not discussed. It is assumed that representative values for the refractive index are available at both terminals see eq. (4).

Part 1 deals with the reduction of measured transit time to chord distance. The treatment here is based to a certain extent on a paper "Curvature corrections in electronic distance measurements" by M. Hotine which was read at the IAG congress on Lucerne in 1967. Part 2 deals with the reduction to the reference ellipsoid.

Part 1. Computation of the length of the ray path

For a continuous, differentiable function, the Taylor expansion to a fourth order is

$$F(B) = F(A+h) = F(A) + hF'(A) + \frac{h^2}{2}F''(A) + \frac{h^3}{3!}F'''(A) + \frac{h^4}{4!}F^{IV}(A). \qquad (1)$$

The superscripts refer to successive derivatives of F with respect to h. Successive derivations of (1) give

$$F'(B) = F'(A) + hF''(A) + \frac{h^2}{2}F'''(A) + \frac{h^3}{6}F^{IV}(A)$$

$$F''(B) = \qquad F''(A) + hF'''(A) + \frac{h^2}{2}F^{IV}(A).$$

We have also

$$\frac{h}{2}\{F'(A)+F'(B)\} = hF'(A) + \frac{h^2}{2}F''(A) + \frac{h^3}{4}F'''(A) + \frac{h^4}{12}F^{IV}(A)$$

$$\frac{h^2}{12}\{F''(A)-F''(B)\} = \qquad -\frac{h^3}{12}F'''(A) - \frac{h^4}{24}F^{IV}(A)$$

and (1) can be written as follows

$$F(B) - F(A) = \frac{h}{2}\{F'(A)+F'(B)\} + \frac{h^2}{12}\{F''(A)-F''(B)\}. \qquad (2)$$

In electronic distance measurements the value for half transit time t is given by the curve integral

$$t = \frac{1}{c}\int_A^B n\,ds$$

where c = velocity of light in a vacuum
n = refractive index

The integral is computed over the curved line s from the initial point A to the far point B. It should be noted that, generally, n is variable along the whole curve.

Now it follows that

$$ct = \int_A^B n\,ds = F(B) - F(A). \qquad (3)$$

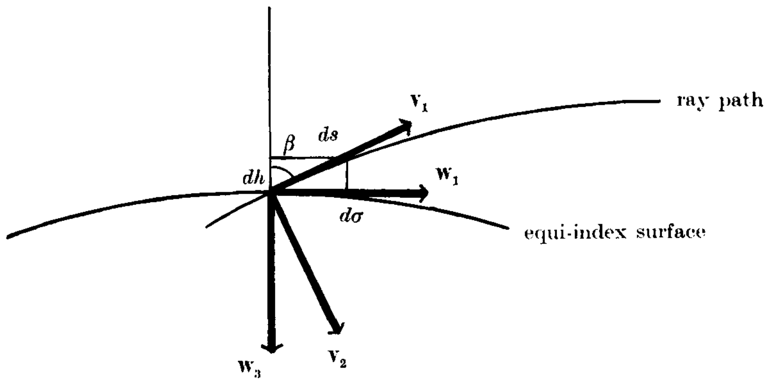

Fig. 1

For n the expansion

$$n = n_0 + s\left(\frac{dn}{ds}\right)_0 + \frac{s^2}{2}\left(\frac{d^2n}{ds^2}\right)_0 + \frac{s^3}{6}\left(\frac{d^3n}{ds^3}\right)_0 \tag{4}$$

is assumed to be valid between A and B: the values with index 0 are taken for an arbitrary point in the internal A and B. Eqs. (2), (3), (4) then give

$$ct = \frac{s}{2}(n_A + n_B) + \frac{s^2}{12}\left\{\left(\frac{dn}{ds}\right)_A - \left(\frac{dn}{ds}\right)_B\right\} \tag{5}$$

To express the derivatives dn/ds it is necessary to introduce Frenet's formulae.

For a space curve, the unit vectors are: $\mathbf{v}_1$ (tangent), $\mathbf{v}_2$ (principal-normal) and $\mathbf{v}_3$ (binormal).

Points with equal n build an equi-index surface. The normal to this surface has a unit vector $\mathbf{w}_3$. The plane of the paper in Fig. 1 contains vectors $\mathbf{v}_1$ and $\mathbf{w}_3$. The intersection of this plane with equi-index surface is a surface curve with unit vector $\mathbf{w}_1$, along its tangent. $\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3$ form the moving trihedron for a space curve, $\mathbf{w}_1\mathbf{w}_2\mathbf{w}_3$ for a surface curve; definitions according to [2] and [3].

According to the laws of refraction, $\mathbf{v}_2$ is also in the plane of the paper. The angle between $\mathbf{w}_3$ and $\mathbf{v}_1$ is $(180° - \beta)$. An infinitesimal increase ds along the ray gives rise to increases $d\sigma$ along $\mathbf{w}_1$ and $-dh$ along $\mathbf{w}_3$. Using these definitions the following equation is obtained:

$$\frac{dn}{ds} = \operatorname{grad} n \cdot \mathbf{v}_1 = |\operatorname{grad} n|\, \mathbf{w}_3 \cdot \mathbf{v}_1 = |\operatorname{grad} n| \cos(180 - \beta) = -|\operatorname{grad} n| \cos\beta.$$

Eq. (5) can be written

$$ct = \frac{s}{2}(n_A + n_B) - \frac{s^2}{12}(|\operatorname{grad} n_A| \cos\beta_A - |\operatorname{grad} n_B| \cos\beta_B). \tag{6}$$

As this formula is not suitable for practical purposes, a number of assumptions must be made.

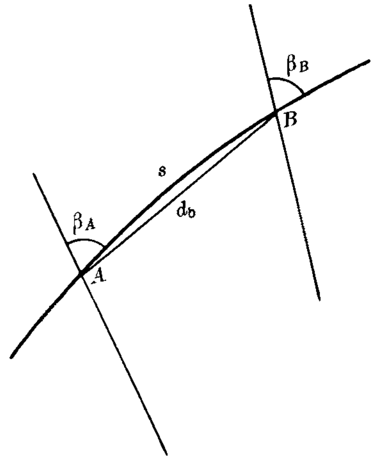

Fig. 2

In points A and B a horizontal stratification of the atmosphere is assumed, thus the direction of $\mathbf{w}_3$ coincides with the nadir direction. The angle β becomes the zenithal distance, measured along the direction of increase in s (see Fig. 2).
Also

$$|\operatorname{grad} n| = -\frac{dn}{dh}$$

in which h is the height of the point.

Eq. (6) then becomes

$$ct = \frac{s}{2}(n_A + n_B) - \frac{s^2}{12}\left\{\left(\frac{dn}{dh}\cos\beta\right)_B - \left(\frac{dn}{dh}\cos\beta\right)_A\right\}. \tag{7}$$

Using electro-optical measuring systems β can be determined with a theodolite. The determination of dn/dh using probes is a time-consuming and difficult procedure, and, therefore, eq. (7) can be simplified.

With reference to Fig. 1

$$-\cos\beta = \mathbf{w}_3 \cdot \mathbf{v}_1.$$

Differentiating gives

$$\sin\beta\,\frac{d\beta}{ds} = \frac{d\mathbf{w}_3}{d\sigma}\,\frac{d\sigma}{ds}\cdot\mathbf{v}_1 + \mathbf{w}_3\cdot\frac{d\mathbf{v}_1}{ds}. \tag{8}$$

According to Frenet's formulae for space curves

$$\frac{d\mathbf{v}_1}{ds} = \varkappa\cdot\mathbf{v}_2$$

where $\varkappa$ is the curvature of the space curve.

Frenet's formulae for a surface curve, i.e. intersection line of the equi-index surface with the plane give

$$\frac{d\mathbf{w}_3}{d\sigma} = -\varkappa_n \mathbf{w}_1 - \tau_g \mathbf{w}_2$$

in which $\varkappa_n$ = normal curvature

τ_g = geodetic torsion

$\mathbf{w}_2 = \mathbf{w}_3 \times \mathbf{w}_1$ and is consequently vertical to $\mathbf{v}_1$.

According to Fig. 1 $d\sigma/ds = \sin\beta$.

Eq. (8) can thus be written

$$\sin\beta\,(d\beta/ds) = \sin\beta(-\varkappa_n \mathbf{w}_1 - \tau_g \mathbf{w}_2)\cdot\mathbf{v}_1 + \varkappa\mathbf{w}_3\cdot\mathbf{v}_2.$$

From Fig. 1 we obtain

$$\mathbf{w}_1\cdot\mathbf{v}_1 = \cos(90^\circ - \beta) = \sin\beta$$

$$\mathbf{w}_3\cdot\mathbf{v}_2 = \cos(90^\circ - \beta) = \sin\beta.$$

Since $\mathbf{w}_2$ is vertical to $\mathbf{v}_1$, $\mathbf{w}_2\cdot\mathbf{v}_1 = 0$ and the equation above becomes

$$\sin\beta(d\beta/ds) = \sin\beta\,(-\varkappa_n \sin\beta + \varkappa)$$

$$d\beta/ds = -\varkappa_n \sin\beta + \varkappa. \tag{9}$$

It is further assumed that the curvatures $\varkappa$ and $\varkappa_n$ are constant between A and B. This implies:

(1) the ray is a circular arc

(2) equi-index surfaces are concentric spheres with a curvature

$$\varkappa_n = \frac{1}{R+h} \sim \frac{1}{R}$$

since $h \ll R$ and where R is the radius of the earth. The ratio $\varkappa/\varkappa_n = k$ is the coefficient of refraction.

Eq. (9) can thus be written

$$\frac{d\beta}{ds} = -\varkappa_n\left(\sin\beta - \frac{\varkappa}{\varkappa_n}\right) = -\frac{1}{R}(\sin\beta - k)$$

and by introducing

$$\beta_B + \beta_A = 2\beta \qquad \beta_B - \beta_A = \Delta\beta$$

$$\left(\frac{dn}{dh}\right)_B + \left(\frac{dn}{dh}\right)_A = 2\frac{dn}{dh} \qquad \left(\frac{dn}{dh}\right)_B - \left(\frac{dn}{dh}\right)_A = \Delta\, dn$$

the second bracket in eq. (7) can be written

$$\left(\frac{dn}{dh}\cos\beta\right)_B - \left(\frac{dn}{dh}\cos\beta\right)_A$$

$$= \left(\frac{dn}{dh}+\frac{\Delta\, dn}{2}\right)\cos\left(\beta+\frac{\Delta\beta}{2}\right) - \left(\frac{dn}{dh}-\frac{\Delta\, dn}{2}\right)\cos\left(\beta-\frac{\Delta\beta}{2}\right)$$

$$= \frac{dn}{dh}\left\{\cos\left(\beta+\frac{\Delta\beta}{2}\right) - \cos\left(\beta-\frac{\Delta\beta}{2}\right)\right\} + \frac{\Delta\, dn}{2}\left\{\cos\left(\beta+\frac{\Delta\beta}{2}\right) + \cos\left(\beta-\frac{\Delta\beta}{2}\right)\right\}$$

$$= -\frac{dn}{dh}\sin\beta\,\Delta\beta + \Delta\, dn\cos\beta + \text{terms of higher order.}$$

Putting
$$\Delta\beta = s\frac{d\beta}{ds} = -\frac{s}{R}(\sin\beta - k)$$

eq. (7) becomes

$$ct = \frac{s}{2}(n_A+n_B) - \frac{1}{12}s^2\left\{\frac{dn}{dh}\sin\beta\,\frac{s}{R}(\sin\beta-k) + \Delta\, dn\cos\beta\right\}. \tag{10}$$

In this equation dn/dh, β, k and Δdn are not yet determined. β is the mean of zenithal distances in A and B.

Thus

$$\cos\beta = \frac{h_B - h_A}{s} \qquad \sin\beta = \sqrt{1-\cos^2\beta}\,.$$

In the final term Δdn is a small quantity, and it is multiplied by $\cos\beta$ which is also small. With the formula (24) in [1] p. 225

$$(n-1)\cdot 10^6 = 318.5 - 41.5\,h + 2.45\,h^2$$

in which h is in kilometres, the dimension of the term $(s^2/12)\Delta dn\cos\beta = \Delta s$ can be estimated:

$$10^6\left(\frac{dn}{dh}\right)_B = -41.5 + 4.9\,h_B \qquad 10^6\left(\frac{dn}{dh}\right)_A = -41.5 + 4.9\,h_A$$

$$\Delta\, dn = 4.9\,(h_B - h_A)\,10^{-6}$$

$$\Delta s = \frac{s^2}{12}\cdot 4.9\,(h_B-h_A)\,10^{-6}\cdot\frac{h_B-h_A}{s}$$

or, with s and h in kilometres, Δs in millimetres

$$\Delta s = 0.4\,s\,(h_B - h_A)^2.$$

In areas with large altitude differences this term should not be omitted. For instance, if $s = 10$ km, $h_B - h_A = 2$ km, Δs will be 16 mm.

In Sweden it is not necessary to take it into account: in the present primary network the maximum value would be only 6 mm (the side Ripainen–Kistefjell: $s=30$ km, $h_B-h_A=0.7$ km).

The relationship between dn/dh and k is given in the spherical refraction law

$$(R+h)n\sin\beta = \text{const}$$

Derivation with respect to h gives

$$n\sin\beta+(R+h)\frac{dn}{dh}\sin\beta+(R+h)n\cos\beta\frac{d\beta}{ds}\frac{ds}{dh}=0 \tag{12}$$

with
$$\frac{d\beta}{ds}=-\frac{1}{R}(\sin\beta-k)$$

$$\frac{ds}{dh}=\frac{1}{\cos\beta}$$

and
$$R+h\sim R.$$

Eq. (12) can be written

$$n\sin\beta+R\frac{dn}{dh}\sin\beta=Rn\cos\beta\,\frac{1}{R}(\sin\beta-k)\frac{1}{\cos\beta}$$

$$\frac{dn}{dh}=-\frac{k}{R}\frac{n}{\sin\beta}. \tag{13}$$

Substituting eq. (13) into eq. (10)

$$ct=\frac{s}{2}(n_A+n_B)+\frac{s^3}{12R^2}nk(\sin\beta-k)-\Delta s. \tag{14}$$

Usually in the second term of eq. (14) it can be assumed that

$$n\sim 1$$
$$\sin\beta\sim 1$$
$$s\sim ct.$$

Omitting Δs the formula given by Höpcke [5] is obtained

$$s=\frac{2ct}{n_A+n_B}-\frac{(ct)^3}{12R^2}(k-k^2). \tag{15}$$

To obtain the length d_b of the straight line joining the end points, the correction for path curvature $-k^2(s^3/24R^2)$ is added which gives

$$d_b=\frac{2ct}{n_A+n_B}-\frac{(ct)^3}{24R^2}(2k-k^2)+\Delta s \tag{16}$$

The value of k remains to be determined.

According to [5], the refraction coefficient k is for heights of 40 to 100 m above ground

$k_M = 0.25$ daytime throughout the year and on cloudy winter nights;
0.5 nights June to October and on clear winter nights.

$k_L = 0.2$ daytime and cloudy nights throughout the year;
0.3 clear nights;
0.13 daytime with a cloudless sky.

in which the index M refers to microwave and index L to visible light.

What is the required accuracy for k? Derivation of eq. (16) with respect to k gives

$$\frac{d}{dk}(d_b) = -\frac{d_b^3}{12R^2}(1-k).$$

An alteration of k with $\delta k = -0.01$ would thus increase the length of d_b as shown in the table below (all values in mm).

Table 1

d_b	10 km	20 km	30 km	40 km	50 km	60 km
k						
0.1	0.02	0.15	0.50	1.18	2.31	3.99
0.2	0.02	0.13	0.44	1.05	2.05	3.55
0.3	0.01	0.12	0.39	0.92	1.80	3.11

The required accuracy for k may also be expressed as follows:

In order that the relative error should not exceed 10^{-7}, $|\delta k|$ should not exceed

0.50 for $d_b = 10$ km
0.15 for $d_b = 20$ km
0.07 for $d_b = 30$ km
0.04 for $d_b = 40$ km
0.02 for $d_b = 50$ km
0.016 for $d_b = 60$ km

In the above, the following assumptions have been made:

(1) horizontal stratification
(2) spherical equi-index surfaces
(3) a variation of the refractive index along the ray path according to formula (4).

The degree of validity of the assumptions is, naturally, open for discussion. This is particularly the case with formula (4). If, however, they are valid the choice of k value is by no means critical for distances less than 30–40 km.

Part 2. Reduction from chord to geodesic

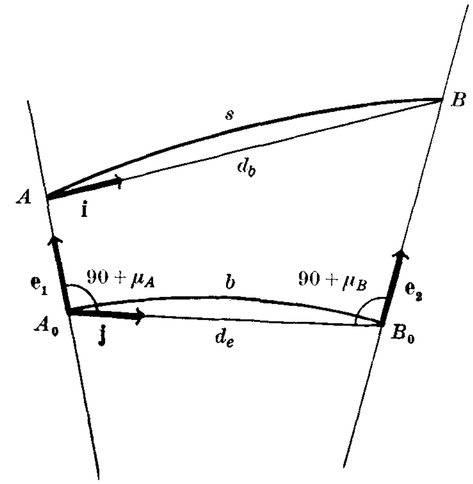

Fig. 3

A, B	end points of the line
A_0, B_0	projections of the end points onto the reference ellipsoid (with normals to ellipsoid)
s	ray path joining A and B
d_b	chord joining A and B
d_e	chord joining A_0 and B_0
b	geodesic joining A_0 and B_0
$\mathbf{e}_1$, $\mathbf{e}_2$	unit normal vectors to the ellipsoid
$\mathbf{i}$	unit vector along d_b
$\mathbf{j}$	unit vector along d_e
h_A, h_B	heights of end points above the ellipsoid

It should be noted that, generally, the four vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{e}_1$ and $\mathbf{e}_2$ are not coplanar.

A. *Reduction from chord d_b to chord d_e*

1. *Introduction*

The relation between d_b and d_e is given by vector addition

$$d_b \mathbf{i} = d_e \mathbf{j} - h_A \mathbf{e}_1 + h_B \mathbf{e}_2$$

and after squaring

$$d_b^2 = d_e^2 + h_A^2 + h_B^2 + 2(-d_e h_A \mathbf{j}\cdot\mathbf{e}_1 + d_e h_B \mathbf{j}\cdot\mathbf{e}_2 - h_A h_B \mathbf{e}_1\cdot\mathbf{e}_2).$$

Introducing the angles μ_A and μ_B, ref. Fig. 3

$$d_b^2 = d_e^2 + h_A^2 + h_B^2 + 2(d_e h_A \sin\mu_A + d_e h_B \sin\mu_B - h_A h_B \mathbf{e}_1\cdot\mathbf{e}_2) \tag{17}$$

On the sphere the angle between $\mathfrak{e}_1$ and $\mathfrak{e}_2$ is the centre angle σ and

$$\mu_A = \mu_B = \frac{\sigma}{2} = \mu.$$

If the radius of the sphere is R, then

$$\sin \mu = \frac{d_e}{2R}$$

$$\mathfrak{e}_1 \cdot \mathfrak{e}_2 = \cos \sigma = 1 - 2 \sin^2 \mu = 1 - \frac{d_e^2}{2R^2}$$

and eq. (17) becomes

$$d_b^2 = d_e^2 + h_A^2 + h_B^2 + \frac{h_A}{R} d_e^2 + \frac{h_B}{R} d_e^2 - 2h_A h_B + \frac{h_A h_B}{R^2} d_e^2$$

$$= (h_A - h_B)^2 + d_e^2 \left(1 + \frac{h_A}{R} + \frac{h_B}{R} + \frac{h_A h_B}{R^2}\right)$$

giving the well known formula

$$d_e^2 = \frac{d_b^2 - (h_A - h_B)^2}{\left(1 + \frac{h_A}{R}\right)\left(1 + \frac{h_B}{R}\right)}. \tag{18}$$

The validity and accuracy of formula (18) for computation on the ellipsoid is discussed below. It is assumed that d_e is less than 100 km and that h is less than 2 km. It is seldom that these values are exceeded in practice.

2. *Computation of* $\sin \mu_A$ *and* $\sin \mu_B$

From [3] § 15

$$d_e \sin \mu_A = a W_A (1 - \sin \beta_A \sin \beta_B - \cos \beta_A \cos \beta_B \cos \lambda) \tag{19a}$$

$$d_e \sin \mu_B = a W_B (1 - \sin \beta_A \sin \beta_B - \cos \beta_A \cos \beta_B \cos \lambda) \tag{19b}$$

is obtained in which

a	major axis of the ellipsoid
W_i	$\sqrt{1 - e^2 \sin^2 \phi_i}$
β	reduced latitude
λ	difference in longitude between A and B
ϕ	geographical latitude
e^2	square of first eccentricity
e'^2	square of second eccentricity

When computing the errors the value 2/300 may be used for e^2 and e'^2.

Since

$$W \sin\beta = \sin\phi\sqrt{1-e^2} \qquad W\cos\beta = \cos\phi$$

eq. (19a) becomes

$$d_e \sin\mu_A = \frac{aW_A}{W_A W_B}(W_A W_B + e^2 \sin\phi_A \sin\phi_B - \cos\sigma) \tag{20}$$

where σ is the third side in a spherical triangle with sides $(90-\phi_A)$ and $(90-\phi_B)$, with the enclosed angle λ.

What is the effect on d_e of an error in $\sin\mu_A$? Derivation of eq. (17) with respect to $\sin\mu_A$ gives

$$0 = 2(d_e + h_A \sin\mu_A + h_B \sin\mu_B)\,\delta(d_e) + 2d_e h_A\,\delta(\sin\mu_A)$$

Since $h_A \sin\mu_A$ and $h_B \sin\mu_B$ are small in relation to d_e, they may be omitted, hence

$$-\delta(d_e) = h_A\,\delta(\sin\mu_A).$$

An error 10^{-7} in $\sin\mu_A$ thus gives rise to an error of less than 0.2 mm in d_e; and the same for $\delta(\sin\mu_B)$.

Series expansion of the two first terms in the bracket in eq. (20) gives

$$\begin{aligned} W_A W_B - e^2 \sin\phi_A \sin\phi_B &= \sqrt{1-e^2\sin^2\phi_A}\cdot\sqrt{1-e^2\sin^2\phi_B} + e^2\sin\phi_A\sin\phi_B \\ &= 1 - \frac{e^2}{2}(\sin^2\phi_A + \sin^2\phi_B) + e^2 \sin\phi_A \sin\phi_B + E \end{aligned}$$

where

$$E = -\frac{e^4}{8}(\sin\phi_A - \sin\phi_B)^2(\sin\phi_A + \sin\phi_B)^2. \tag{20a}$$

The effect of the terms with factor e^6 is

$$\frac{e^6}{16}(\sin^2\phi_A - \sin^2\phi_B)^2(\sin^2\phi_A + \sin^2\phi_B) \tag{20b}$$

and may thus be neglected (see end of this subsection). Putting $a/W = N$ (perpendicular curvature) eq. (19) becomes

$$d_e \sin\mu_A = N_B\left\{2\sin^2\frac{\sigma}{2} - \frac{e^2}{2}(\sin\phi_A - \sin\phi_B)^2 + E\right\} \tag{21a}$$

$$d_e \sin\mu_B = N_A\left\{2\sin^2\frac{\sigma}{2} - \frac{e^2}{2}(\sin\phi_A - \sin\phi_B)^2 + E\right\}. \tag{21b}$$

Introducing R = radius of curvature for latitude $\phi = (\phi_A + \phi_B)/2$ and azimuth for $\alpha = (\alpha_{\bar{A}B} + \alpha_{\bar{B}A} + 180°)/2$ eq. (21a) can be written

$$d_e \sin\mu_A = \frac{N_B}{2R^2}\left\{\left(2R\sin\frac{\sigma}{2}\right)^2 - e^2R^2(\sin\phi_A - \sin\phi_B)^2 + 2R^2E\right\} \tag{22}$$

$2R \sin(\sigma/2)$ is the length of a chord between two points with coordinates $(\phi_1, 0)$ and (ϕ_2, λ) on a sphere with radius R. d_e is the length between two corresponding points on the ellipsoid. Putting

$$2R \sin\frac{\sigma}{2} = d_e + \varepsilon$$

ε will be a few cms and gives rise to an error in $\sin \mu_A$

$$\varepsilon \cdot \frac{N_b}{R^2} \sim \frac{\varepsilon}{R} < \frac{0.05}{6000000} < 10^{-8}$$

and may thus be neglected. Eq. (22) now becomes

$$\sin \mu_A = \frac{d_e}{2R}\left\{1 - e^2\left(\frac{R}{d_e}\right)^2 (\sin\phi_A - \sin\phi_B)^2 + 2\left(\frac{R}{d_e}\right)^2 E\right\}\frac{N_B}{N}\cdot\frac{N}{R} \qquad (23)$$

where N is the perpendicular curvature for latitude ϕ.

Omitting higher terms the bracket in eq. (23) can be simplified as follows

$$\frac{d_e}{R} = 2\sin\frac{\sigma}{2} = \sigma - \frac{\sigma^3}{24}$$

$$\left(\frac{R}{d_e}\right)^2 = \frac{1}{\sigma^2}\left(1 + \frac{\sigma^2}{12}\right)$$

$$\sin\phi_A - \sin\phi_B = 2\sin\frac{\Delta\phi}{2}\cos\phi = \Delta\phi\left(1 - \frac{\Delta\phi^2}{24}\right)\cos\phi$$

$$\left(\frac{R}{d_e}\right)^2 (\sin\phi_A - \sin\phi_B)^2 = \left(\frac{\Delta\phi}{\sigma}\right)^2\left(1 + \frac{\sigma^2}{12}\right)\left(1 - \frac{\Delta\phi^2}{12}\right)\cos^2\phi.$$

With reference to [3] formula V. 37

$$\Delta\phi = \sigma\cos\alpha\left(1 + \frac{\sigma^2}{12}\sin^2\alpha\right)$$

the second term in the bracket in eq. (23) becomes

$$e^2\left(\frac{R}{d_e}\right)^2(\sin\phi_A - \sin\phi_B)^2 = e^2\cos^2\alpha\left(1 + \frac{\sigma^2}{6}\sin^2\alpha\right)\left(1 + \frac{\sigma^2}{12}\right)\left(1 - \frac{\sigma^2}{12}\cos^2\alpha\right)\cdot\cos^2\phi$$

$$= e^2\cos^2\alpha\cos^2\phi\left(1 + \frac{\sigma^2}{4}\sin^2\alpha\right) = e^2\cos^2\alpha\cos^2\phi. \qquad (24)$$

By assuming that $d_e < 100$ km, both σ and ϕ will be less than $1°$, and thus the effect of the omitted term on $\sin\mu_A$ becomes

$$\frac{d_e}{2R}\cdot e^2\cdot\cos^2\alpha\cdot\cos^2\phi\cdot\frac{\sigma^2}{4}\sin^2\alpha < \frac{100}{2\cdot 6400}\cdot\frac{2}{300}\cdot\frac{1}{4}\left(\frac{\pi}{180}\right)^2 < 5\cdot 10^{-10}$$

and may be neglected.

The final term in bracket in eq. (23) is

$$2\left(\frac{R}{d_e}\right)^2 \cdot E = -\frac{2}{\sigma^2}\frac{e^4}{8}(\sin\phi_A - \sin\phi_B)^2(\sin\phi_A + \sin\phi_B)^2$$

$$= -\frac{2}{\sigma^2}\cdot\frac{e^4}{8}(\Delta\phi\cos\phi)^2(2\sin\phi)^2 = -e^4\cos^2\alpha\cos^2\phi\sin^2\phi. \quad (25)$$

Further

$$\frac{N}{R} = 1 + e'^2\cos^2\alpha\cos^2\phi = 1 + e^2\cos^2\alpha\cos^2\phi + e^4\cos^2\alpha\cos^2\phi \quad (26)$$

$$\frac{N_B}{N} = \sqrt{1-e^2\sin^2\phi_B} : \sqrt{1-e^2\sin^2\phi}$$

$$= \left(1 - \frac{e^2}{2}\sin^2\phi_B - \frac{e^4}{8}\sin^4\phi_B\right)\left(1 + \frac{e^2}{2}\sin^2\phi + \frac{3e^4}{8}\sin^4\phi\right)$$

$$= 1 + \frac{e^2}{2}(\sin^2\phi - \sin^2\phi_B) + \frac{e^4}{8}(\sin^2\phi - \sin^2\phi_B)(3\sin^2\phi + \sin^2\phi_B). \quad (27)$$

Since

$$(\sin^2\phi - \sin^2\phi_B) = (\sin\phi - \sin\phi_B)(\sin\phi + \sin\phi_B)$$

$$\sim (\phi - \phi_B)\cos\phi\cdot 2\sin\phi = (\phi - \phi_B)\sin 2\phi \quad (28)$$

$$|\phi - \phi_B| < 0.5^\circ = \frac{\pi}{360}$$

the final term in eq. (27) can be written

$$\frac{e^4}{8}|\phi - \phi_B|\sin 2\phi\cdot 4\sin^2\phi < \frac{1}{8}\left(\frac{2}{300}\right)^2\cdot\frac{\pi}{360}\cdot 1\cdot 4 < 2\cdot 10^{-7}.$$

Its effect in $\sin\mu_A$ is less than $2\cdot 10^{-9}$ and may be omitted. Substituting eqs. (24)–(27) into eq. (23)

$$\sin\mu_A = \frac{d_e}{2R}(1 - e^2\cos^2\alpha\cos^2\phi - e^4\cos^2\alpha\cos^2\phi\sin^2\phi)$$

$$\times\left(1 + e^2\frac{\sin^2\phi - \sin^2\phi_B}{2}\right)(1 + e^2\cos^2\alpha\cos^2\phi + e^4\cos^2\alpha\cos^2\phi).$$

Carrying out multiplication and omitting terms with e^6 and higher

$$\sin\mu_A = \frac{d_e}{2R}\left(1 + e^2\frac{\sin^2\phi - \sin^2\phi_B}{2} + \frac{e^4}{4}\sin^2 2\alpha\cos^4\phi\right) \quad (29)$$

is obtained.

The final term reaches a maximum value for $\alpha = 45^\circ$, $\phi = 0^\circ$ and is then

$$\frac{100}{2\cdot 6400}\cdot\frac{1}{4}\left(\frac{2}{300}\right)^2 < 10^{-7}$$

which implies an error of 0.2 mm in d_e at the Equator. In Sweden, $\phi > 55^\circ$, $\cos^4\phi < 0.1$ and the above term may therefore be omitted.

The second term in eq. (29) becomes, with reference to eq. (28),

$$\frac{d_e}{2R}\cdot\frac{e^2}{2}(\phi-\phi_B)\sin 2\phi < \frac{100}{2\cdot 6400}\cdot\frac{2}{2\cdot 300}\cdot\frac{\pi}{360} < 2.5\cdot 10^{-7}$$

and the error in d_e will not exceed 0.5 mm.

The above reasoning can also be applied to $\sin\mu_B$. Thus, for $d_e < 100$ km, $h < 2$ km, the substitution

$$\sin\mu_A = \sin\mu_B = \frac{d_e}{2R} \tag{30}$$

will give rise to an error in d_e not exceeding 1 mm.

Note: the effect of the remainder term (20b) on $\sin\mu_A$ will be, see eq. (28),

$$\frac{a}{d_e W_B}\cdot\frac{e^6}{16}(\phi_A-\phi_B)^2\sin^2 2\phi\,(2\sin^2\phi) < \frac{R}{d_e}\cdot\frac{e^6}{16}\Delta\phi^2\cdot 2$$

$$= \frac{\Delta\phi}{\sigma}\cdot\frac{e^6}{16}\Delta\phi\cdot 2 < \frac{1}{8}\left(\frac{2}{300}\right)^3\cdot\frac{\pi}{180} < 10^{-9}$$

which indicates that it was permissible to omit it.

3. *Computation of* $\mathbf{e}_1\cdot\mathbf{e}_2$

Referring to the symbols in [2] and [3]

$$\mathbf{e}_1 = \mathbf{w}_3^{(A)} \quad\text{and}\quad \mathbf{e}_2 = \mathbf{w}_3^{(B)}.$$

Series development with the mid-point of geodesic as origin

$$\mathbf{w}_3^{(A)} = \mathbf{w}_3 - \mathbf{w}_3'\frac{b}{2} + \mathbf{w}_3''\frac{b^2}{8}$$

$$\mathbf{w}_3^{(B)} = \mathbf{w}_3 + \mathbf{w}_3'\frac{b}{2} + \mathbf{w}_3''\frac{b^2}{8}$$

$$\mathbf{w}_3' = \frac{d\mathbf{w}_3}{db} = -\varkappa_n\mathbf{w}_1 - \tau_g\mathbf{w}_2$$

$$\mathbf{w}_3'' = \frac{d^2\mathbf{w}_3}{db^2} = -\varkappa_n'\mathbf{w}_1 - \tau_g'\mathbf{w}_2 - \varkappa_n\mathbf{w}_1' - \tau_g\mathbf{w}_2'$$

$$= -\varkappa_n'\mathbf{w}_1 - \tau_g'\mathbf{w}_2 - \varkappa_n(\varkappa_g\mathbf{w}_2 + \varkappa_n\mathbf{w}_3) - \tau_g(-\varkappa_g\mathbf{w}_1 + \tau_g\mathbf{w}_3).$$

The above gives

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{w}_3 \cdot \mathbf{w}_3 - \mathbf{w}_3' \cdot \mathbf{w}_3' \frac{b^2}{4} + \mathbf{w}_3 \cdot \mathbf{w}_3'' \frac{b^2}{4} = 1 - (\varkappa_n^2 + \tau_g^2)\frac{b^2}{4} + (-\varkappa_n^2 - \tau_g^2)\frac{b^2}{4}$$

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = 1 - \varkappa_n^2 \frac{b^2}{2} - \tau_g^2 \frac{b^2}{2} \tag{31}$$

where according to [3] eq. V. 48

$$\tau_g = \left(\frac{1}{N} - \frac{1}{M}\right) \sin\alpha \cos\alpha.$$

In subsection 2, R was defined as the radius of curvature at point

$$\phi = \frac{\phi_A + \phi_B}{2}.$$

$\varkappa_n$ is the curvature at the mid-point of the geodesic. Generally these two points do not coincide, but the difference is small and (see section C. Required Accuracy...) thus the following substitution can be made

$$\varkappa_n = \frac{1}{R}.$$

What error in d_e does an error in $\mathbf{e}_1 \cdot \mathbf{e}_2$ give rise to? Derivation of eq. (17) gives

$$0 = 2d_e \delta(d_e) + 2(h_A \sin\mu_A + h_B \sin\mu_B)\,\delta(d_e) - 2h_A h_B \,\delta(\mathbf{e}_1 \cdot \mathbf{e}_2)$$

or, omitting the second term, which is considerably smaller than the first and third terms:

$$\delta(d_e) = \frac{h_A h_B}{d_e} \delta(\mathbf{e}_1 \cdot \mathbf{e}_2). \tag{32}$$

Since $|\tau_g| < (e^2/N)$ and $b \sim d_e$, the omission of the final term in eq. (31) gives rise to an error

$$\delta(d_e) < \frac{h_A h_B}{d_e} \cdot \frac{e^4}{N^2} \cdot \frac{b^2}{2} < 2 \cdot 2 \left(\frac{2}{300}\right)^2 \left(\frac{1}{6000}\right)^2 \cdot \frac{1}{2} \cdot 10^8 < 0.01 \text{ mm}$$

which may be neglected.

The difference between b and d_e does not exceed 1 m for $d_e < 100$ km. Substituting b with d_e in the second term of eq. (31) gives rise to an error

$$\delta(d_e) = \frac{h_A h_B}{d_e} \varkappa_n^2 \frac{b^2 - d_e^2}{2} \sim h_A h_B \varkappa_n^2 (b - d_e) \sim \left(\frac{2}{6400}\right)^2 \cdot 10^3 \sim 10^{-4} \text{ mm}$$

which is omissible, and the relation

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = 1 - \frac{1}{2}\left(\frac{d_e}{R}\right)^2 \tag{33}$$

is obtained.

4. *Conclusion*

Substituting eqs. (30) and (33) into (17) results in eq. (18), which implies, that for $h < 2$ km and $d_e < 100$ km, the spherical reduction formula (18) is fully applicable also for computing on the ellipsoid. Errors due to this approximation do not exceed 1 mm.

B. *Reduction from chord d_e to geodesic b*

Referring to [3] § 15:

(1) the difference in length between both the normal sections, from A to B and from B to A, is inappreciable

(2) the relation between b and d_e is

$$b = \frac{b_{\bar{A}B} + b_{\bar{B}A}}{2} = d_e + \frac{d_e^3}{48}\left(\frac{1}{R_A^2} + \frac{1}{R_B^2}\right)$$

$$- \frac{e^2 d_e^4}{32}\left(\frac{\cos\alpha_{\bar{A}B}\sin 2\phi_A}{R_A^3} + \frac{\cos\alpha_{\bar{B}A}\sin 2\phi_B}{R_B^3}\right) + \frac{3d_e^5}{1280}\left(\frac{1}{R_A^4} + \frac{1}{R_B^4}\right). \tag{34}$$

The fourth term in eq. (34) is of dimension

$$\frac{3}{640}\left(\frac{d_e}{R}\right)^4 d_e \sim \frac{3}{640}\left(\frac{1}{64}\right)^4 \cdot 10^8 \sim 3 \cdot 10^{-2} \text{ mm}$$

and may thus be neglected.

When estimating the dimension of the third term in eq. (34), the bracket may be calculated on the sphere, i.e. $R_A = R_B$ and $\sin\alpha\cos\phi = \text{const.}$ It should be noted, that

$$-\cos\alpha_{\bar{B}A} \sim \cos\alpha_{\bar{A}B}$$

and putting

$$\alpha_{\bar{A}B} = \alpha$$

$$\alpha_{\bar{B}A} = \alpha + 180^\circ + \Delta\alpha$$

$$\phi_A = \phi$$

$$\phi_B = \phi + \Delta\phi$$

it may be written, omitting the higher terms

$$\cos \alpha_{\bar{A}B} \sin 2\phi_A + \cos \alpha_{\bar{B}A} \sin 2\phi_B =$$

$$= \cos \alpha \sin 2\phi - \cos (\alpha + \Delta\alpha) \sin 2(\phi + \Delta\phi) =$$

$$= \cos \alpha \sin 2\phi - (\cos \alpha - \Delta\alpha \sin \alpha) (\sin 2\phi + 2\Delta\phi \cos 2\phi) =$$

$$= \Delta\alpha \sin \alpha \sin 2\phi - 2\Delta\phi \cos \alpha \cos 2\phi.$$

According to [3] formula 20.5

$$\Delta\alpha = \sigma \cdot \sin \alpha \operatorname{tg} \phi$$

and assuming both σ and $\Delta\phi$ are less than 1°, the third term in eq. (34) becomes less than

$$\frac{e^2}{32}\left(\frac{d_e}{R}\right)^3 d_e \frac{2\pi}{180} (\sin^2 \alpha \sin^2 \phi - \cos \alpha \cos 2\phi)$$

$$< \frac{1}{32} \cdot \frac{2}{300}\left(\frac{1}{63}\right)^3 \cdot 10^8 \cdot \frac{2}{57} \cdot 2 < 0.01 \text{ mm}$$

and can thus be neglected.

The magnitude of the second term is

$$\frac{2}{48}\left(\frac{d_e}{R}\right)^2 d_e = \frac{1}{24} \cdot \frac{1}{4096} \cdot 10^8 \sim 10^3 \text{ mm}$$

and it is therefore permissible to put

$$\frac{1}{R_A^2} + \frac{1}{R_B^2} = \frac{2}{R^2}.$$

Thus eq. (34) can be simplified to

$$b = d_e + \frac{d_e^3}{24R^2}. \tag{35}$$

Eq. (35) is valid for the length of the normal section between A and B. However, the difference in length between a normal section and a geodesic does not exceed $2b \cdot 10^{-10}$ (see [3] § 24), i.e. less than 0.02 mm for a side of 100 km. Thus the reduction (35) is valid also for the geodesic.

C. *Required accuracy in approximate coordinates*

Putting

$$h_A - h_B = \Delta h$$

$$h_A + h_B = 2h$$

eq. (18) can be expanded to

$$d_e^2 = d_b^2 - d_b^2 \cdot \frac{2h}{R} - (\Delta h)^2 + \dots$$

Differentation gives (with d_b as an error free quantity)

$$2d_e\,\delta d_e = -2d_b^2\cdot\frac{1}{R}\cdot\delta h + 2d_b^2\frac{h}{R^2}\delta R - 2\Delta h\,\delta(\Delta h).$$

Putting
$$d_b \sim d_e \sim b$$

$$\delta b = -\frac{b}{R}\delta h + \frac{hb}{R^2}\delta R - \frac{\Delta h}{b}\delta(\Delta h)$$

δR depends on errors $\delta\phi$ and $\delta\alpha$

$$R = N : (1 + e'^2\cos^2\phi\cos^2\alpha) \sim a\left(1 + \frac{e^2}{2}\sin^2\phi - e^2\cos^2\phi\cos^2\alpha\right)$$

$$\delta R = a\left\{\frac{e^2}{2}\sin 2\phi(1 + 2\cos^2\alpha)\,\delta\phi + e^2\cos^2\phi\sin 2\alpha\,\delta\alpha\right\}.$$

In Sweden $55°20' < \phi < 69°04'$, i.e. $\sin 2\phi < 0.94$ and $\cos^2\phi < 0.33$. Thus, for $b = 10$ km, $h = 1$ km, the error in the geodesic becomes

1.6 mm for $\delta h = 1$ m
max 0.23 mm for $\delta\phi = 1°$
max 0.06 mm for $\delta\alpha = 1°$

Errors increase proportionally to b and h.

As the reduction is done to the ellipsoid and not to the geoid, the height of the geoid above the ellipsoid must have been added to the heights. An error of 1 meter in the height difference gives rise to an error of 100 mm in b, if $b = 10$ km and $\Delta h =$ 1 km. The error will proportionally increase with Δh and decrease with b.

References

[1] Jordan, Eggert & Kneissl. 1966. *Handbuch der Vermessungskunde*, Band VI, Stuttgart.
[2] Baeschlin, C. F. 1947. *Einführung in die Kurven- und Flächentheorie*, Zürich.
[3] Baeschlin, C. F. 1948. *Lehrbuch der Geodäsie*. Zürich.
[4] Hotine, M. 1967. *Curvature corrections in electronic distance measurements*. IAG, Luzern.
[5] Höpcke, W. 1964. Über die Bahnkrümmung elektromagnetischer Wellen. *Zeitschrift für Vermessungswesen*, 183–200.
[6] Rinner, K. 1956. Über die Reduktion grosser elektronisch gemessener Entfernungen. *Zeitschrift für Vermessungswesen*, 47–55.

ПРОБЛЕМЫ КОРРЕКЦИИ В ДИСТАНЦИОННЫХ ЭЛЕКТРОННЫХ ИЗМЕРЕНИЯХ

Определяется время распространения сигнала от датчика до дальней точки и обратно при электронных дистанционных измерениях. Отсюда вычисляется наклонная дальность с учётом изменений показателя преломления вдоль пути. Наклонная дальность затем сводится к геодезической на координатном эллипсоиде.