On the boundary value problem of physical geodesy

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ABSTRACT

A solution of the principal boundary value problem of physical geodesy is made using spherical harmonics for the global approach and gravity reduction by an integral equation for the local studies, all in a compatible system. This procedure should be unique. In the normal application gravity is reduced to the reference sphere (ellipsoid) which coincides with mean sea level.

For a computation of the vertical deflection a final gravity reduction is made to a reference surface through the actual point. An "isometric" gravity anomaly is introduced. The isostatic, Bouguer and Rudzky anomalies are interpreted in a new way.

1. Introduction

The boundary value problem for physical geodesy was solved by Stokes (1849) for a sphere. Elementary reductions of gravity anomalies from the physical surface of the earth were made in order to give the "physical geoid" in the interior of the earth. Molodensky (1948, 1960, 1962) showed that the classical approach had no solution for a non-spherical earth and solved the problem using the *topographical* surface as the reference surface thus avoiding all reduction below this surface. This technique is elegant but very complicated.

In our new approach we obtain the global solution directly from the free air anomalies using an expansion in spherical harmonics. Gravity reduction for local studies of the vertical deflection is made with the use of an integral equation.

The basic equation for *gravity reduction* can be derived from Green's integral equation for potentials or Poisson's equation by replacing the disturbance potential with a harmonic function of the gravity anomaly.

These two equations have been known for over 100 years but were not considered for geodetic gravity reduction. Analyses by *Molodensky* (1960, 1962) and *Moritz* (1961) proved that no strict solution of "analytical continuation" was possible according to classical methods. This means that "analytical gravity reduction" to an interior surface cannot generally be made. However, it should be noted that our method of gravity reduction is not analytical. Normally, it will be too time-consuming to use a fully analytical solution of an integral equation. Instead we make use of a system of linear equations of integrals which always can be obtained in such a way that the boundary values are satisfied at all given points. ("Problem of Bjerhammar" cf. Moritz, 1964.) In our primary study an explicit "resolvent function" according to Stokes was used. The discrete approach was studied in a numerical solution of a system of linear equations with 36 unknowns (Bjerhammar, 1960*a*). Generalized formulas for the potential and the vertical deflections were then computed (Bjerhammar, 1960*c*). This "resolvent" required a previous gravity reduction which was defined in an integral equation and a method for an iterative solution was given (Bjerhammar, 1962). Also non-iterative methods were included in a rather complete analysis. Numerical computations from models in three dimensions were presented (Bjerhammar, 1963*b*). Russian scientists studied the same approach for geophysical gravity reduction (Aronov, 1963). The geodetic application was later recognized (Aronov, 1964). Förstner (1964, 1966) made the first geodetic application on a practical model (Cypres). Reit (1966*b*, 1967) showed in model tests that the method could be used also for enormous point masses between the reference surface and the physical surface. A numerical study for the West Alps was made using 1600 unknowns (Bjerhammar, 1968). Baranov (1957) studied the problem of gravity reduction without using integral equations. Cf. Gottschalk (1967).

2. Boundary value conditions

In our system we use the definitions (cf. Fig. 1)

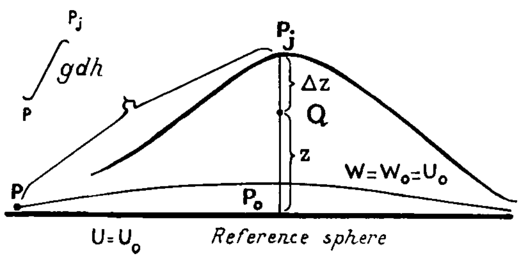


Fig. 1. Relations between potentials and heights. The potential of the point P_j in the true gravity field is identical with the potential of the reference point Q in the theoretical gravity field.

U_0 = theoretical potential of the reference sphere (ellipsoid)
W_0 = true potential at mean sea level
U_j = theoretical potential at the fixed point P_j
W_j = true potential at the fixed point P_j.

Here we have

$$T_j = W_j - U_j = \text{disturbance potential} \tag{1}$$

and

$$W_j = W_0 - \int_P^{P_j} g\,dh, \quad \text{(cf. eq. (5))} \tag{2}$$

where

$$P = \text{any point at mean sea level.}$$

Furthermore

$$U_j = U_0 - \int_P^{P_j} g\,dh - T_j \tag{3}$$

h = height in the true gravity field.

The theoretical gravity field and the true one are linked by the relation

$$W_0 = U_0. \tag{4}$$

The true potential difference between P and P_j can be evaluated in the theoretical field of the reference sphere (ellipsoid) using the equation

$$\int_P^{P_j} g\,dh = \int_{P_0}^{Q} \gamma\,dz \tag{5}$$

P_0 = point at the reference sphere (ellipsoid)

where

$$g = -\frac{\partial W}{\partial r}$$

$$\gamma = -\frac{\partial U}{\partial r}$$

z = height in the theoretical gravity field,
r = direction normal to the reference sphere (ellipsoid).

The angle between the surface normal of the reference sphere and the plumb line is considered negligible when forming the boundary value condition.

The true potential of P_j is now equal to the theoretical potential of Q. Both points are situated on the same vertical in the theoretical fields.

From equations (1–3) we obtain after series expansion

$$W_j = W_0 - \int_P^{P_j} g\,dh = U_0 + \frac{\partial U_0}{\partial r} z + \frac{\partial^2 U_0}{2 \cdot \partial r^2} z^2 + \dots \tag{6}$$

Thus we have the planar solution

$$z = \frac{\int g\,dh}{\gamma_0}. \tag{7}$$

From equation (1) we obtain

$$\frac{\partial T_j}{\partial r} = \frac{\partial W_j}{\partial r} - \frac{\partial U_j}{\partial r} = -(g_j - \gamma_j) \text{ (gravity disturbance).} \tag{8}$$

A series expansion of γ_j gives

$$\gamma_j = \gamma_0 + \frac{\partial \gamma_0}{\partial r}(z + \Delta z) + \frac{\partial^2 \gamma_0}{2\partial r^2}(z + \Delta z)^2 + \dots \tag{9}$$

or

$$\gamma_j = \gamma_z + \frac{\partial \gamma_z}{\partial r}\Delta z + \dots \tag{9a}$$

Here we use the relation

$$T_j = \int_Q^{P_j} \gamma\,dz \simeq \gamma_z \Delta z \tag{10}$$

or

$$\Delta z = T_j/\gamma_z. \tag{11}$$

Then we have

$$\frac{\partial T_j}{\partial r} = -\left(g_j - \gamma_z - \frac{T_j \partial \gamma}{\gamma_z \partial r}\right) \tag{12}$$

or

$$\frac{\partial T_j}{\partial r} - \frac{T_j}{\partial \gamma_z} \frac{\partial \gamma}{\partial r} = -(g_j - \gamma_z) = -\Delta g_j. \tag{13}$$

For a sphere we have the approximation

$$\boxed{\frac{\partial T_j}{\partial r} + \frac{2T_j}{r_j} = -\Delta g_j} \tag{14}$$

where

r_j = geocentric distance of the fixed point.

Δg_j = free air anomaly.

Equation (14) gives the boundary value condition of modern physical geodesy. Bruns and Molodensky have given the final definition.

There is no unique solution of the disturbance potential from the boundary value alone. However, from the potential theory we know that the potential has to fulfil Laplace condition all over in space (harmonic function).

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0} \tag{15}$$

A strict solution has furthermore to satisfy the relation

$$\boxed{\begin{array}{l} \text{limes } T_j\, r_j^2 = 0 \\ \quad \text{for } r_j \to \infty \end{array}} \tag{16}$$

For our further analysis we derive from Poisson's integral for the harmonic parameter $r\Delta g$

$$r_j \Delta g_j = \frac{r_j^2 - r_0^2}{4\pi r_0} \iint \frac{r_0 \Delta g^*}{r_{ij}^3} dS \tag{17}$$

where

r_0 = radius of the earth
S = surface of the reference sphere
r_{ij} = distance between the fixed point P_j and the moving point at the reference sphere
Δg^* = gravity at the reference sphere.

This relation is transcribed using Legendre polynomials

$$\Delta g_j = \frac{1}{4\pi r_j^2} \iint \Delta g^* \sum_{n=0}^{\infty} (2n+1) \left(\frac{r_0}{r_j}\right)^n P_n(\cos\omega)\, dS \tag{18}$$

where

$$P_n(\cos\omega) = \frac{1}{2^n n!} \frac{d^n(\cos^2\omega - 1)^n}{d(\cos\omega)^n} \quad \text{(Legendre polynomial)}$$

ω = angular geocentric distance between the fixed and the moving point.

The disturbance potential T_j is now written

$$T_j = \frac{1}{r_j} \iint \Delta g^* \sum \Omega \left(\frac{r_0}{r_j}\right)^n P_n(\cos\omega)\, dS \tag{19}$$

where

Ω = unknown quantity to be determined.

From equations (14) and (19) we obtain

$$-\frac{1}{r_j^2} \iint \Delta g^* \sum (n-1)\Omega \left(\frac{r_0}{r_j}\right)^n P_n(\cos\omega)\, dS$$

$$= -\frac{1}{4\pi r_j^2} \iint \Delta g^* \sum (2n+1) \left(\frac{r_0}{r_j}\right)^n P_n(\cos\omega)\, dS. \tag{20}$$

Thus we have

$$\Omega = \frac{2n+1}{4\pi(n-1)} \quad \text{for} \quad n \geqslant 2.$$

Now we have the final solution

$$\boxed{T_j = \frac{1}{4\pi r_j} \iint \Delta g^* \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \left(\frac{r_0}{r_j}\right)^n P_n(\cos\omega)\, dS} \tag{21}$$

(Cf. Stokes, 1849)

where

P_n = Legendre polynomials of order n
T_j = disturbance potential of the fixed point P_j.

It is easily verified that the conditions (14), (15) and (16) are fulfilled for this solution.

Also for a non-spherical surface a direct solution of the disturbance potential for the unknown physical surface can be considered. This problem has been comtemplated by Molodensky (1948, 1960). The convergence of the resulting iterative solution has been questioned for steep models and there is still no proof for the regularity according to equation (16) above. Anyway the merits of the solution are well known.

In our approach we consider that gravity data are only known at discrete points. Therefore we are forced to include some additional conditions in order to obtain a unique solution. This is done by extending the Laplace condition to be valid also between the reference sphere (ellipsoid) and the physical surface. Then we make use of an expansion in spherical harmonics

$$T_j = \frac{GM}{r_j} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{r_0}{r_j}\right)^n P_{nm}(\sin\phi)\,(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda) \tag{22}$$

where

$$P_{nm}(\sin\phi) = (1-\sin^2\phi)^{m/2}\frac{d^m P_n(\sin\phi)}{d(\sin\phi)^m},$$

ϕ = latitude,

λ = longitude,

M = mass of the earth,

G = Newton's gravitational constant.

Thus we have from (14) and (22)

$$\boxed{\Delta g_j = \frac{GM}{r_j^2} \sum_{n=2}^{\infty} \sum_{m=0}^{n} (n-1)\left(\frac{r_0}{r_j}\right)^n P_{nm}(\sin\phi)\,(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)} \tag{23}$$

This solution gives no difficulties in a global terrestrial application for harmonics as high as 1000. The solution is unique when the number of gravity values is identical with the number of unknown coefficients. In a practical approach it seems convenient to use gravity data representing "squares" or rather "blocks" of a uniform grid.

In a number of applications it will be more natural with a *local* solution. This is the case when the vertical deflection is needed. Then we make a preceeding "harmonic" *gravity reduction* down to the reference sphere (ellipsoid) according to eq. (18).

Our gravity reduction is defined in such a way that all mass outside the reference sphere is "translocated" inside the reference sphere without changing the boundary values at the physical surface.

In our solution we obtain an equipotential surface coinciding with mean sea level. This equipotential surface is a "geoid" in the external gravity field of the earth.

The internal (physical) geoid cannot be computed from gravity data at the physical surface.

For very steep models our Δg^*-solution is sometimes somewhat "unstable" when using surface elements smaller than $5' \times 5'$. In such cases we can replace the free air anomaly by the "isometric anomaly" for a refined study of the details of the topography. Cf. Chapter 16.

In our study we finally make a "local solution" *without* any gravity reduction. Instead we introduce a direct determination of the disturbance potential at the reference sphere (T^*). Cf. Chapter 17.

3. Discrete procedure

There is no general *analytical* solution of our integral equation for gravity reduction (17). However, gravity is only known in discrete points and we can always find a solution that satisfies the boundary values at all given discrete points.

Determination of the discrete boundary values

In our approach we have to select a finite number of discrete boundary points to represent the total gravity field. Different methods of sampling can be used for this procedure.

"Random" discrete boundary values

All gravity measurements are normally related to discrete points at the surface of the earth with a semi-random distribution. Thus we find that only the continents are well covered with measurements. All these primary discrete measurements can be considered for a simultaneous solution of our basic integral equation. However, the computational problems cannot be taken care of straight forward because no computer can handle systems with several hundred thousand unknowns. Furthermore, there is no need to form individual observation equations for every measurement.

The kernel of the integral equation decreases very rapidly and therefore it will be convenient to use a sampling technique where the gravity over a limited area is represented by just one value.

"Non-random" discrete boundary values

In most applications it is an advantage to replace the primary random discrete boundary values by new discrete values which represent surface elements of a more or less uniform size. For this purpose a suitable grid has to be selected. A "rectangular grid" will be useful for a number of studies (for example $5' \times 5'$). There will normally be no gravity value available for the center of this surface element but such a value has to be obtained by statistical methods.

A suitable technique is obtained in the following way. Inside the surface element we make a regression analysis between gravity and height.

Thus we obtain

$$\Delta g = c + hc_1 + h^2 c_2 + \dots \tag{1}$$

Furthermore we compute the discrete "grid" height ($\bar{h}$) of each individual surface element (ΔS) of the grid

$$\bar{h} = \frac{\iint_{\Delta S} h \, dS}{\iint_{\Delta S} dS}. \tag{2}$$

Then we have the discrete "grid" gravity value ($\Delta \bar{g}$)

$$\Delta \bar{g} = c + (\bar{h}) c_1 + (\bar{h})^2 c_2 + \dots \tag{3}$$

In our study we now represent the actual surface element by the discrete "grid" values.

Also covariance analysis can be used for this type of study (Moritz).

3.1. *Discrete solution of the gravity reduction*

In accordance with equation (2:17) our gravity reduction under consideration is defined by the integral equation (cf. Fig. 2).

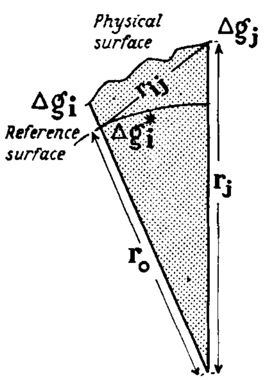


Fig. 2. Reduced and unreduced gravity anomalies.

$$\Delta g_j = \frac{r_j^2 - r_0^2}{4\pi r_j} \iint_S \frac{\Delta g_i^*}{r_{ij}^3} dS \tag{1}$$

where

Δg_j = gravity anomaly at the surface of the earth
Δg_i^* = reduced gravity anomaly at the reference surface
r_j = geocentric distance to the fixed point
r_0 = geocentric distance to the reference sphere
r_{ij} = distance between the fixed point and the moving point at the reference sphere
S = reference surface (spherical)

Our solution is now defined in such a way that we consider Δg^* *constant* over uniquely defined surface elements. For most applications a rectangular grid will be useful.

Thus we obtain the "discrete integral equation"

$$\Delta g_j = \frac{r_j^2 - r_0^2}{4\pi r_j} \iint_{\Delta S_1} \frac{\Delta g_1^*}{r_{ij}^3} dS_1 + \frac{r_j^2 - r_0^2}{4\pi r_j} \iint_{\Delta S_2} \frac{\Delta g_2^*}{r_{ij}^3} dS_2 + \dots \tag{2}$$

where

ΔS_1, ΔS_2 ... = surface elements
Δg_1^*, Δg_2^* ... = reduced gravity anomaly of the surface element.

With Δg^* *constant* over each individual surface element we obtain a system of linear equations which can be transcribed using matrix notations.

$$\underset{n1}{\Delta \mathbf{g}} = \underset{nm}{\mathbf{K}}\ \underset{m1}{\Delta \mathbf{g}^*}. \tag{3}$$

First, we assume that there is a solution satisfying this matrix equation.
Then we have

$$\underset{m1}{\Delta \mathbf{g}^*} = \underset{mn}{\mathbf{K}^{-1}} \underset{n1}{\Delta \mathbf{g}} + \underset{mm}{(\mathbf{I}} - \underset{mn}{\mathbf{K}^{-1}} \underset{nm}{\mathbf{K})} \underset{m1}{\mathbf{M}} \tag{4}$$

where

$$\mathbf{K}\mathbf{K}^{-1}\mathbf{K} = \mathbf{K} \tag{5}$$

$\mathbf{K}^{-1}$ = generalized inverse
$\mathbf{M}$ = arbitrary matrix of adequate order.

Equations (3) and (4) give

$$\Delta \mathbf{g} = \mathbf{K}\mathbf{K}^{-1}\Delta \mathbf{g} + (\mathbf{K} - \mathbf{K}\mathbf{K}^{-1}\mathbf{K})\mathbf{M}. \tag{6}$$

Thus we obtain the *necessary* condition for a solution of equation (3) (necessary and sufficient)

$$\Delta \mathbf{g} = \mathbf{K}\mathbf{K}^{-1}\Delta \mathbf{g}. \tag{7}$$

We have now to prove that all solutions are included. For this proof we are making use of the following sets:

(1:o) The set of all solutions satisfying equations (3) is denoted: $\Delta \tilde{\mathbf{g}}^*$.
(2:o) The set of solutions defined by equation (4) is denoted: $\mathbf{K}^{-1}\Delta \mathbf{g} + (\mathbf{I} - \mathbf{K}^{-1}\mathbf{K})\mathbf{M}$.

Our solution must be a *subset* of the set of all solutions and we have

$$\mathbf{K}^{-1}\Delta \mathbf{g} + (\mathbf{I} - \mathbf{K}^{-1}\mathbf{K})\mathbf{M} \subset \Delta \tilde{\mathbf{g}}^*. \tag{8}$$

From our subset (2:o) we can separate a new *subset* by replacing $\mathbf{M}$ with $\Delta \tilde{\mathbf{g}}^*$

(3:o) $$\mathbf{K}^{-1}\Delta \mathbf{g} + (\mathbf{I} - \mathbf{K}^{-1}\mathbf{K})\Delta \tilde{\mathbf{g}}^*. \tag{9}$$

Obviously we have

$$\mathbf{K}^{-1}\Delta \mathbf{g} + (\mathbf{I} - \mathbf{K}^{-1}\mathbf{K})\Delta \tilde{\mathbf{g}}^* \subset \mathbf{K}^{-1}\Delta \mathbf{g} + (\mathbf{I} - \mathbf{K}^{-1}\mathbf{K})\mathbf{M} \tag{10}$$

From our subset (3:o) we can separate the *subset*

(4:o) $$\Delta \tilde{\mathbf{g}}^* \subset \mathbf{K}^{-1}\Delta \mathbf{g} + (\mathbf{I} - \mathbf{K}^{-1}\mathbf{K})\Delta \tilde{\mathbf{g}}^* \tag{11}$$

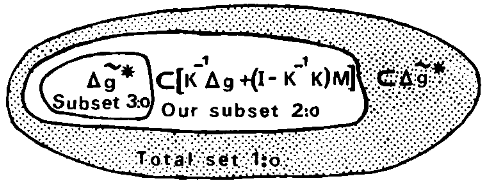


Fig. 3. Sets and subsets of the general solution for gravity reduction.

because

$$\mathbf{K}\Delta\tilde{\mathbf{g}}^* = \Delta\mathbf{g}. \tag{12}$$

Then we have

$$\Delta\tilde{\mathbf{g}}^* \subset \mathbf{K}^{-1}\Delta\mathbf{g} + (\mathbf{I} - \mathbf{K}^{-1}\mathbf{K})\mathbf{M} \subset \Delta\tilde{\mathbf{g}}^*. \tag{13}$$

Evidently we have proved that our set of solutions is identical with the total set of solutions (cf. Fig. 3).

Thus the "discrete problem" has always a solution, because it has been proved that any matrix has a generalized inverse. The necessary and sufficient condition (7) can always be fulfilled since any *linear* dependence is avoided with a proper subdivision. (Unlinear kernel of third power.)

3.2. *Degenerative solutions*

The total set of solutions for fully discrete gravity reduction is included in the expression (cf. Fig. 4).

$$\underset{m1}{\Delta\mathbf{g}^*} = \underset{mn}{\mathbf{K}^{-1}}\underset{n1}{\Delta\mathbf{g}} + \underset{mm}{(\mathbf{I}} - \underset{mn}{\mathbf{K}^{-1}}\underset{nm}{\mathbf{K})}\underset{m1}{\mathbf{M}} \quad \text{(Bjerhammar, 1951)} \tag{1}$$

where

n = number of observations
m = number of unknown reduced gravity values.

This set of solutions is infinite for

$$m > n$$

and

$$\text{rank}\ (\mathbf{K}) = n.$$

The number of degenerations is given by

$$m - n.$$

Furthermore, we note that this infinite set of solutions can be partitioned in a number of *subsets* which are all *infinite* just by varying n.

In a number of applications it will be justified to operate with degenerative conditions.

If all degenerations are excluded we have in an artificial way limited our possible solutions. This is the case if we select a system with the same number of observa-

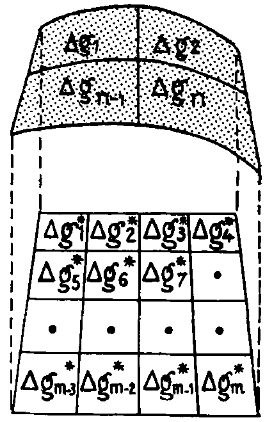


Fig. 4. Degenerative approach.

tions and unknowns (non-singular). The justification for this type of solution is often weak and we can hardly accept that the number of measurements at the physical surface has to be identical with the number of unknowns at the reference surface. This procedure may force the solution in such a way that extremely unrealistic solutions can be expected. Solutions of this type have sometimes given reduced gravity values of 10^6 milligals and more for a random distribution of the gravity points. When using gravity distributed in a grid system excellent convergence has been found. (Cf. Appendix.)

Sometimes the solutions can be improved using more refined methods which will be discussed here.

The first degenerative solution of interest is obtained straight forward according to methods of least squares

$$\mathbf{K}\Delta\mathbf{g}^* = \Delta\mathbf{g} \tag{2}$$

$$\Delta\mathbf{g}^* = \mathbf{K}^*(\mathbf{K}\mathbf{K}^*)^{-1}\Delta\mathbf{g} \tag{3}$$

or with weights P

$$\Delta\mathbf{g}^* = \mathbf{P}^{-1}\mathbf{K}^*(\mathbf{K}\mathbf{P}^{-1}\mathbf{K}^*)^{-1}\Delta\mathbf{g}. \tag{4}$$

This type of solution will minimize the sum of squares

$$\underset{1m}{(\Delta\mathbf{g}^*)^*}\,\underset{mm}{\mathbf{P}}\,\underset{m1}{(\Delta\mathbf{g}^*)} = \sum_{i=1}^{m} p_i(\Delta g_i^*)^2. \qquad (5)^1$$

Thus we can expect a considerable smoothing of the $\Delta\mathbf{g}^*$-values compared with the non-degenerative solution. However this type of solution seems rather unjustified from a physical point of view. More natural is the following approach

$$\mathbf{K}(\Delta\mathbf{g} + \delta(\Delta\mathbf{g})) = \Delta\mathbf{g}, \tag{6}$$

$$\delta(\Delta\mathbf{g}) = \mathbf{P}^{-1}\mathbf{K}^*(\mathbf{K}\mathbf{P}^{-1}\mathbf{K}^*)^{-1}(\Delta\mathbf{g} - \mathbf{K}\Delta\mathbf{g}) \tag{7}$$

[1] Notations: $(\Delta\mathbf{g})^*$ = transposed $\Delta\mathbf{g}$-matrix (unreduced gravity); $(\Delta\mathbf{g}^*)^*$ = transposed $\Delta\mathbf{g}^*$-matrix (reduced gravity).

minimizing the variance (cf. Fig. 5) and

$$\underset{1m}{[\delta(\Delta g)]^*}\ \underset{mm}{\mathbf{P}}\ \underset{m1}{[\delta(\Delta g)]}. \tag{8}$$

This solution will minimize the sum of the squares of the corrections in relation to the measured gravity values. With weights inversly proportional to the heights we have taken care of the available physical information in a more natural way.

The practical consequences of the number of degenerations have to be discussed. We easily find that all solutions discussed here have zero "unreduced" residual variances in measured points. Furthermore we find that the "reduced" gravity variance according to (5) and (8) decreases for increasing m

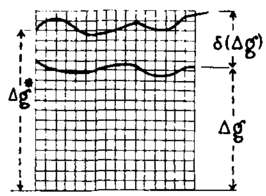


Fig. 5. Free air anomaly $= \Delta g$; reduced gravity $= \Delta g^*$; $\Delta g + \delta(\Delta g) = \Delta g.^*$

$$\underset{1m}{(\Delta g^*)^*}\ \underset{mm}{\mathbf{P}}\ \underset{m1}{(\Delta g^*)} \to \min \tag{9}$$

$$m \to \infty$$

$$\underset{1m}{[\delta(\Delta g)]^*}\ \underset{mm}{\mathbf{P}}\ \underset{m1}{[\delta(\Delta g)]} \to \min \tag{10}$$

$$m \to \infty$$

because all smaller surface elements inside a larger surface can as a special case accept the value of the larger surface.

Thus we find that a large number of degenerations gives a smooth performance of Δg^*. Furthermore any computed Δg-values for points between the measured values at the physical surface are likely to be close to the measured values.

Our study has indicated that the degenerative solutions are ideal from several points of view. However there is a *practical limitation*. In most cases it will be impossible to fulfil a straight forward solution with degenerative methods for a larger area. Only iterative methods can handle systems with many thousand unknowns in a practical way. The final procedure can then be completed in the following manner:

(1:o) Regular solution with *iterative* methods for surface elements of a given minimum size. (Adequate down to at least $5' \times 5'$. Cf. below. The iterative surface element = "block".)

(2:o) Additional corrections with *degenerative* methods inside the individual surface element after subdivision into t compartments. (The residuals are eliminated for all discrete points inside the block.)

Here we minimize the variance

$$[\delta(\underset{1t}{\Delta \mathbf{g}^*})]^* [\delta(\underset{t1}{\Delta \mathbf{g}^*})]$$

t = number of gravity values inside the block
$\delta(\Delta \mathbf{g}^*)$ = additional correction to the iterative solution.

An exact solution requires a simultaneous solution for all blocks and therefore only approximate solutions are obtained when the individual block is adjusted in a separate operation. Then interaction with adjacent blocks cannot be avoided. Iterative methods are studied in chapter 5.

3.3. *Inconsistent solutions*

Sometimes it is of interest to consider inconsistent equations for the gravity reduction

$$\underset{nm}{\mathbf{K}} \; \underset{m1}{\Delta \mathbf{g}^*} \neq \underset{n1}{\Delta \mathbf{g}} \tag{1}$$

where no exact solutions of the equation generally exist. This type of problem can be created in the following way:

For $n = m$

and
$$\text{rank } (\mathbf{K}) < m \tag{2}$$

a solution only exists for

$$\mathbf{K}\mathbf{K}^{-1}\Delta \mathbf{g} = \Delta \mathbf{g} \tag{3}$$

For an *arbitrary* Δg there is no solution in a classical sense!

In a more sophisticated way we have for *any* matrix $\mathbf{K}$

$$\underset{m1}{\Delta \mathbf{g}^*} = \underset{mn}{\mathbf{K}^*}(\underset{nm}{\mathbf{K}}\,\underset{mn}{\mathbf{K}^*})^{-1}\,\underset{nm}{\mathbf{K}}\,(\underset{mn}{\mathbf{K}^*}\,\underset{nm}{\mathbf{K}})^{-1}\,\underset{mn}{\mathbf{K}^*}\,\underset{n1}{\Delta \mathbf{g}} \tag{4}$$

where
$$(\Delta \mathbf{g}^*)^* \Delta \mathbf{g}^* = \min \tag{5}$$

and
$$(\Delta \mathbf{g} - \mathbf{K}^\circ \Delta \mathbf{g})^* (\Delta \mathbf{g} - \mathbf{K}^\circ \Delta \mathbf{g}) = \min \tag{6}$$

with
$$\mathbf{K}^\circ = \mathbf{K}\mathbf{K}^*(\mathbf{K}\mathbf{K}^*)^{-1}\mathbf{K}(\mathbf{K}^*\mathbf{K})^{-1}\mathbf{K}^*. \tag{7}$$

For rank $(\mathbf{K}) = m$ we have

$$\Delta \mathbf{g}^* = (\mathbf{K}^*\mathbf{K})^{-1}\mathbf{K}^*\Delta \mathbf{g} \tag{8}$$

minimizing
$$(\Delta \mathbf{g} - \mathbf{K}^\circ \Delta \mathbf{g})^* (\Delta \mathbf{g} - \mathbf{K}^\circ \Delta \mathbf{g}). \tag{9}$$

For $n > m$ we only have general solutions according to the method of least squares.

4. Semi analytical procedures

4.1. *Power series solutions*

When using power series for a solution of the gravimetric boundary value problem we obtain a semi-analytical representation of the reduced gravity anomaly. With only heights as parameters this semi-analytical relation is only formal, because no analytical height information will be available in practical applications. The boundary values of gravity will only be known at discrete points and therefore we just obtain a semi-analytical relation.

In a number of applications power series can be used for a representation of the Δg^*. The reduced gravity anomaly has been expanded in the following way (cf. Fig. 6)

$$\Delta g^* = \Delta g + c_0 h^0 + c_1 h^1 + c_2 h^2 + c_3 h^3 + \ldots \qquad (1)^1$$

where

h = height above the reference surface.

Or

$$\Delta g^* = \Delta g + \sum_{\nu=0}^{n} c_\nu h^\nu. \qquad (2)$$

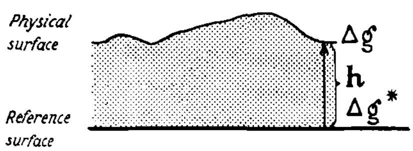


Fig. 6. Parameters of the power series solution.

This type of series expansion has been rather useful for theoretical models with a simple topography. Practical models with a complicated topography have a much more detailed structure and will not always permit an application of this simple technique. It is rather well known that power series of higher order than 20 start to degenerate and no further increase in accuracy is obtained.

Our power series (1) requires a regression analysis for a practical solution and we will normally have many degrees of freedom for the numerical estimation of the unknown parameters. With several thousands of observations we can only expect a limited improvement using a power series of this type.

In order to obtain greater flexibility without using higher powers we can also include fractional powers.

For example

$$\Delta g^* = \Delta g + c_0 h^{0.1} + c_1 h^{0.2} + c_2 h^{0.3} + c_3 h^{0.4} + \ldots \qquad (3)$$

Or in a more general way

$$\Delta g^* = \Delta g + c_0 h^0 + c_1 h^{1\alpha} + c_2 h^{2\alpha} + c_3 h^{3\alpha} + c_4 h^{4\alpha} + \ldots \qquad (4)$$

where

α = fractional power.

[1] Indices of Δg and Δg^* are omitted in the following pages when not needed for a proper understanding. Detailed studies with power series have been made by Förstner (1966) and Reit (1966 and 1967). Cf. References.

Simultaneous use of several different power series gives the following relation

$$\begin{aligned}\Delta g^* = \Delta g + c_0 h^0 + c_1 h^{1\alpha} + c_2 h^{2\alpha} + c_3 h^{3\alpha} + c_4 h^{4\alpha} + \dots \\ + d_1 h^{1\beta} + d_2 h^{2\beta} + d_3 h^{3\beta} + d_4 h^{4\beta} + \dots \\ + e_1 h^{1\gamma} + e_2 h^{2\gamma} + e_3 h^{3\gamma} + e_4 h^{4\gamma} + \dots\end{aligned} \tag{5}$$

where

α, β, γ = fractional powers (identities of the powers have to be avoided in the series).

With $c_0 = \text{const}$ and $c_\nu = 0$ for $\nu \neq 0$ we obtain $(r_0/r_j = t)$

$$\Delta g^* = \Delta g_j + \frac{r_j^2 - r_0^2}{4\pi r_j t^2} \iint \frac{t^{-2}\Delta g_j - \Delta g_i}{r_{ij}^3} dS \doteq \Delta g_j + \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta g_j - \Delta g_i}{r_{ij}^3} dS. \tag{6}$$

With $c_1 = \text{const}$ and $c_\nu = 0$ for $\nu \neq 1$ we obtain

$$\Delta g^* = \Delta g_j + \frac{\dfrac{r_j^2 - r_0^2}{4\pi r_j} \displaystyle\iint \frac{t^{-2}\Delta g_j - \Delta g_i}{r_{ij}^3} dS}{\dfrac{r_j^2 - r_0^2}{4\pi r_j} \displaystyle\iint \frac{h_i}{r_{ij}^3} dS} \cdot h \tag{6a}$$

(Cf. for example: Bjerhammar, 1963; Moritz, 1966.)

In all power series we can also include fractional powers by adding the elements of equation (5). Also fractional powers of (Δg) can be included.

The power series of (5) fulfils the necessary condition

$$\Delta g^* = \Delta g$$

for

$$h = 0 \quad \text{and} \quad c_0 = 0.$$

The power series needs variations of two different types: (1) Variation with height (h); (2) variation with position (Δg).

Variation as a function of the horizontal coordinates is not directly included here. However, variation with Δg accepts indirectly this information for powers of Δg.

Power series according to (5) have the advantage that they accept a considerable number of unknowns without making use of very high powers. With maximum power equal to 10 we can include hundreds of unknowns in our study. Making use of fractional powers gives an opportunity to use a very large number of unknowns. The maximum number of unknowns that can be included is here defined in the following way.

First we introduce the parameter

$$\Delta G = \Delta g - \frac{(r_j - r_0^2)}{4\pi r_j} \iint \frac{\Delta g^*}{r_{ij}^3} dS. \tag{7}$$

As long as $\Sigma(\Delta G)^2$ is decreasing we can use additional unknowns.

The geophysical meaning of equation (5) can be questioned. If we make use of expansion with the aid of derivatives we obtain

$$\Delta g^* = \Delta g - \frac{\partial(\Delta g^*)}{\partial h} h - \frac{\partial^2(\Delta g^*)}{2\,\partial h^2} h^2 - \frac{\partial^3(\Delta g^*)}{6\,\partial h^3} h^3 - \dots \tag{8}$$

This equation will be an adequate description of our model as long as all elements are computed individually for each point.

As an alternative geophysical description we can make use of the relation

$$\Delta g^* = \Delta g + \frac{\partial(\Delta g)}{\partial h}(-h) + \tfrac{1}{2}\frac{\partial^2(\Delta g)}{\partial h^2}(-h)^2 + \tfrac{1}{6}\frac{\partial^3(\Delta g)}{\partial h^3}(-h)^3 + \dots \tag{9}$$

If we exclude all powers of h higher than one then we obtain a solution which includes merely a simplified presentation for the whole actual area.

After differentiation we obtain the "gradient"

$$\frac{\partial(\Delta g)}{\partial h} = \frac{\partial(\Delta g)}{\partial r_j} = \frac{r_j^2 + r_0^2}{4\pi r_j^2}\iint \frac{\Delta g^*}{r_{ij}^3}\,dS - \frac{3(r_j^2 - r_0^2)}{8\pi r_j^2}\iint \frac{(r_j^2 - r_0^2 + r_{ij}^2)\Delta g^*}{r_{ij}^5}\,dS. \tag{10}$$

For $r_j = r_0$ we have

$$\frac{\partial(\Delta g^*)}{\partial r_j} = \frac{\partial(\Delta g)}{\partial r_j} = \frac{1}{2\pi}\iint \frac{\Delta g^* - \Delta g_j^*}{r_{ij}^3}\,dS - \frac{2\Delta g_j^*}{r_j}. \tag{11}$$

For $r_0 = \infty$ we have (planar approach)

$$\frac{\partial(\Delta g^*)}{\partial r_j} = \frac{1}{2\pi}\iint \frac{\Delta g - \Delta g_j}{r_{ij}^3}\,dS. \tag{12}$$

If derivatives of higher order are omitted we can compute the "explicit derivative" from equation (5) for an arbitrary surface with $c_0 = 0$.

$$\frac{\partial(\Delta g)}{\partial r_j} = -\frac{\Delta g_j - \dfrac{r_j^2 - r_0^2}{4\pi r_j}\displaystyle\iint \frac{\Delta g}{r_{ij}^3}\,dS}{\dfrac{r_j^2 - r_0^2}{4\pi r_j}\displaystyle\iint \frac{h}{r_{ij}^3}\,dS} = -c, \quad \text{(Bjerhammar, 1963}\,c\text{)} \tag{13}$$

Equation (12) is obtained from (13) for $h = \text{const} \approx 0$.

5. Iterative solutions of the integral equation for gravity reduction

Smaller theoretical models can be evaluated with closed formulas under favourable conditions. When it comes to large applied models it is often impossible to fulfil the solution without using iterative methods.

Some iterative methods will be described below. The eight first methods (11–14 and 21–24) are quite comparable for short reduction distances. The following methods are more complicated but give faster convergence for a non-spherical external surface. This is of practical importance because with faster convergence we can allow smaller surface elements.

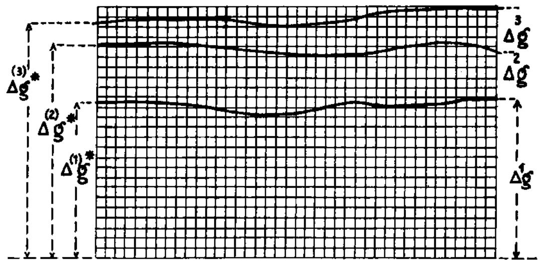


Fig. 7. Iterative notations.

These iterative solutions represent an analytical approach. A conversion to the discrete approach is made in chapter 6. The iterative notations are explained in Fig. 7.

Basic formula for the analytical solution:

$$\Delta g_j = \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta g_i^*}{r_{ij}^3}\, dS$$

where

Δg_j = free air gravity anomaly at the fixed point
Δg_i^* = reduced gravity anomaly at the moving point on the reference sphere
r_0 = radius of the reference sphere
r_j = geocentric distance to the fixed point
r_{ij} = distance between the fixed and the moving point
S = reference sphere.

Notations for the iterative solutions

$\Delta g^* = \Delta^1 g + \Delta^2 g + \Delta^3 g + \ldots$

where

$\Delta^1 g$ = initial value
$\Delta^2 g$ = first iterative correction
$\Delta^3 g$ = second iterative correction
— — — — — — — — —
$\Delta^k g$ = $(k-1)$th iterative correction
$\Delta^{(1)} g^* = \Delta^1 g$
$\Delta^{(2)} g^* = \Delta^1 g + \Delta^2 g$
$\Delta^{(3)} g^* = \Delta^1 g + \Delta^2 g + \Delta^3 g$
— — — — — —
$\Delta^{(k)} g^* = \Delta^1 g + \Delta^2 g + \ldots + \Delta^k g$

5.1. *Integration of reduced gravity*

Method: 11

$$\Delta g^* = \sum_{k=1}^{\infty} \Delta^k g = \Delta^1 g + \Delta^2 g + \Delta^3 g + \ldots$$

where
$$\Delta^1 g_j = \Delta g_j$$
$$\Delta^2 g_j = \Delta g_j - \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta^1 g_i}{r_{ij}^3} dS$$
$$\Delta^3 g_j = \Delta g_j - \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta^1 g_i + \Delta^2 g_i}{r_{ij}^3} dS$$

— — — — —

$$\Delta^k g_j = \Delta g_j - \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta^1 g_i + \Delta^2 g_i + \ldots + \Delta^{k-1} g_i}{r_{ij}^3} dS.$$

For $(k > 1)$ (cf. Bjerhammar, 1962*a*)

$$\boxed{\Delta^k g_j = \Delta g_j - \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta^{(k-1)} g_i^*}{r_{ij}^3} dS}$$

General procedure. The *residuals* at the physical surface are used as corrections to the previous estimate of the reduced gravity anomaly.

$$\Delta^{(1)} g^* = \Delta g$$
$$\Delta^{(2)} g^* = \Delta^{(1)} g^* + \Delta^2 g$$

— — — — —

$$\Delta^{(k)} g^* = \Delta^{(k-1)} g^* + \Delta^k g$$

5.2. *Integration of residuals*

Method: 12

$$\Delta g^* = \sum_{k=1}^{\infty} \Delta^k g = \Delta^1 g + \Delta^2 g + \Delta^3 g + \ldots$$

where
$$\Delta^1 g_j = \Delta g_j$$
$$\Delta^2 g_j = \Delta g_j - \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta^1 g_i}{r_{ij}^3} dS$$
$$\Delta^3 g_j = \Delta^2 g_j - \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta^2 g_i}{r_{ij}^3} dS$$

— — — — —

For $(k > 1)$ (cf. Bjerhammar, 1965*a*)

$$\boxed{\Delta^k g_j = \Delta^{k-1} g_j - \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta^{k-1} g_i}{r_{ij}^3} dS}$$

General procedure. The residuals at the physical surface are used as corrections to the previous estimate of the reduced gravity anomaly.

$$\Delta^{(1)} g^* = \Delta g$$
$$\Delta^{(2)} g^* = \Delta^{(1)} g^* + \Delta^2 g$$
$$— — — — —$$
$$\Delta^{(k)} g^* = \Delta^{(k-1)} g^* + \Delta^k g.$$

Note: In most applications buffer zones have to be used. Here $\Delta g = \Delta g^*$ and integration over the buffer zone is only needed once when using this method.

5.3. *Integration of gravity differences*

Method: 13

$$\Delta g^* = \sum_{k=1}^{\infty} \Delta^k g = \Delta^1 g + \Delta^2 g + \Delta^3 g + \dots$$

where

$$\Delta^1 g_j = \Delta g_j$$

$$\Delta^2 g_j = \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{t^{-2}\Delta g_j - \Delta^1 g_i^*}{r_{ij}^3} dS$$

$$\Delta^3 g_j = \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{t^{-2}\Delta g_j - \Delta^{(2)} g_i^*}{r_{ij}^3} dS$$

$$— — — — —$$

For (k > 1) (cf. Bjerhammar, 1963c, 1964 and Moritz, 1964)

$$\boxed{\Delta^k g_j = \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{t^{-2}\Delta g_j - \Delta^{(k-1)} g_i^*}{r_{ij}^3} dS}$$

$$t = \frac{r_0}{r_j} \approx 1.$$

General procedure. The residuals at the physical surface are used as corrections to the previous estimate of the reduced gravity anomaly.

$$\Delta^{(1)} g^* = \Delta g$$
$$\Delta^{(2)} g^* = \Delta^{(1)} g^* + \Delta^2 g$$
$$— — — — —$$
$$\Delta^{(k)} g^* = \Delta^{(k-1)} g^* + \Delta^k g$$

Note: This method is somewhat more tedious than the previous ones. Truncation of the integration is less critical than for the previous methods in models where Δg is rather constant around the actual point. *For large height differences methods 11 and 12 give smaller truncation errors!*

5.4. *Integration of residual differences*

Method: 14

$$\Delta g^* = \sum_{k=1}^{\infty} \Delta^k g = \Delta^1 g + \Delta^2 g + \Delta^3 g + \dots$$

where

$$\Delta^1 g_j = \Delta g_j$$

$$\Delta^2 g_j = \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{t^{-2}\Delta^1 g_j - \Delta^1 g_i}{r_{ij}^3} dS$$

$$\Delta^3 g_j = \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{t^{-2}\Delta^2 g_j - \Delta^2 g_i}{r_{ij}^3} dS$$

— — — — —

For $(k > 1)$

$$\boxed{\Delta^k g_j = \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{t^{-2}\Delta^{k-1} g_j - \Delta^{k-1} g_i}{r_{ij}^3} dS}$$

$$t = \frac{r_0}{r_j}.$$

General procedure. The residuals at the physical surface are used as corrections to the previous estimate of the reduced gravity anomaly.

$$\Delta^{(1)} g^* = \Delta g$$

$$\Delta^{(2)} g^* = \Delta^{(1)} g^* + \Delta^2 g$$

— — — — —

$$\Delta^{(k)} g^* = \Delta^{(k-1)} g^* + \Delta^k g.$$

Note: This method does not require repeated integration over the buffer zones and is favourable with respect to truncation of the integration.

5.5. *Integration of reduced gravity with simple reduction of residuals*

Method: 21

$$\Delta g^* = \sum_{k=1}^{\infty} \Delta^k g = \Delta^1 g + \Delta^2 g + \Delta^3 g + \dots$$

where

$$\Delta^1 g_j = \Delta g_j$$

$$\Delta^2 g_j = \frac{r_j^2}{r_0^2}\left[\Delta g_j - \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta^{(1)} g_i^*}{r_{ij}^3} dS\right]$$

$$\Delta^3 g_j = \frac{r_j^2}{r_0^2}\left[\Delta g_j - \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta^{(2)} g_i^*}{r_{ij}^3} dS\right]$$

— — — — —

For (k > 1)

$$\boxed{\Delta^k g_j = \frac{r_j^2}{r_0^2}\left[\Delta g_j - \frac{r_j^2 - r_0^2}{4\pi r_j}\iint \frac{\Delta^{(k-1)} g_i^*}{r_{ij}^3} dS\right]}$$

General procedure.

$$\begin{aligned} \Delta^{(1)} g^* &= \Delta g \\ \Delta^{(2)} g^* &= \Delta^{(1)} g^* + \Delta^2 g \\ &\text{— — — — —} \\ \Delta^{(k)} g^* &= \Delta^{(k-1)} g^* + \Delta^k g \end{aligned}$$

The residuals are reduced from the physical surface to the reference surface using a constant reduction. Cf. equation (4.1:6).

Note: This method is of prime interest for gravity reduction from very large altitudes (satellites). Cf. Chapter 9 Over-relaxation.

5.6. *Integration of reduced residuals*

Method: 22

$$\Delta g^* = \sum_{k=1}^{\infty} \Delta^k g = \Delta^1 g + \Delta^2 g + \Delta^3 g + \dots$$

where

$$\Delta^1 g_j = \Delta g_j$$

$$\Delta^2 g_j = \frac{r_j^2}{r_0^2}\left[\Delta^1 g_j - \frac{r_j^2 - r_0^2}{4\pi r_j}\iint \frac{\Delta^1 g_i}{r_{ij}^3} dS\right]$$

$$\Delta^3 g_j = \Delta^2 g_j - \frac{(r_j^2 - r_0^2) r_j}{4\pi r_0^2}\iint \frac{\Delta^2 g_i}{r_{ij}^3} dS$$

— — — — —

For ($k > 2$) (cf. Bjerhammar, 1965*a*)

$$\boxed{\Delta^k g_j = \Delta^{k-1} g_j - \frac{(r_j^2 - r_0^2) r_j}{4\pi r_0^2}\iint \frac{\Delta^{k-1} g_i}{r_{ij}^3} dS}$$

General procedure.

$$\begin{aligned} \Delta^{(1)} g^* &= \Delta g \\ \Delta^{(2)} g^* &= \Delta^{(1)} g^* + c_1 \\ &\text{— — — — —} \\ \Delta^{(k)} g^* &= \Delta^{(k-1)} g^* + c_{k-1}. \end{aligned}$$

Note: This method gives a reduction of the residuals and avoids repeated integration over the buffer zones.

5.7. *Integration of reduced gravity differences*

Method: 23

$$\Delta g^* = \sum_{k=1}^{\infty} \Delta^k g = \Delta^1 g + \Delta^2 g + \Delta^3 g + \ldots$$

where

$$\Delta^1 g_j = \Delta g_j$$

$$\Delta^2 g_j = \frac{(r_j^2 - r_0^2) r_j}{4\pi r_0^2} \iint \frac{\Delta g_j t^{-2} - \Delta^{(1)} g_i^*}{r_{ij}^3} dS$$

$$\Delta^3 g_j = \frac{(r_j^2 - r_0^2) r_j}{4\pi r_0^2} \iint \frac{\Delta g_j t^{-2} - \Delta^{(2)} g_i^*}{r_{ij}^3} dS$$

— — — — —

For $(k > 1)$

$$\boxed{\Delta^k g_j = \frac{(r_j^2 - r_0^2) r_j}{4\pi r_0^2} \iint \frac{\Delta g_j t^{-2} - \Delta^{(k-1)} g_i^*}{r_{ij}^3} dS}$$

General procedure.

$$\Delta^{(1)} g^* = \Delta g$$

$$\Delta^{(2)} g^* = \Delta^{(1)} g^* + c_1$$

— — — — —

$$\Delta^{(k)} g^* = \Delta^{(k-1)} g^* + c_{k-1}.$$

Note: This method uses integration of reduced gravity differences.

5.8. *Integration of reduced residual differences*

Method: 24

$$\Delta g^* = \sum_{k=1}^{\infty} \Delta^k g = \Delta^1 g + \Delta^2 g + \Delta^3 g + \ldots$$

where

$$\Delta^1 g_j = \Delta g_j$$

$$\Delta^2 g_j = \frac{(r_j^2 - r_0^2) r_j}{4\pi r_0^2} \iint \frac{\Delta^1 g_j t^{-2} - \Delta^1 g_i}{r_{ij}^3} dS$$

$$\Delta^3 g_j = \frac{(r_j^2 - r_0^2) r_j}{4\pi r_0^2} \iint \frac{\Delta^2 g_j - \Delta^2 g_i}{r_{ij}^3} dS$$

— — — — —

For $(k > 2)$

$$\boxed{\Delta^k g_j = \frac{(r_j^2 - r_0^2) r_j}{4\pi r_0^2} \iint \frac{\Delta^{k-1} g_j - \Delta^{k-1} g_i}{r_{ij}^3} dS}$$

General procedure.

$$\Delta^{(1)}g^* = \Delta g$$
$$\Delta^{(2)}g^* = \Delta^{(1)}g^* + c_1$$
$$— — — — —$$
$$\Delta^{(k)}g^* = \Delta^{(k-1)}g^* + c_{k-1}.$$

Note: This method uses integration of reduced residual differences. It is favourable with respect to truncation of the integration and buffer zones.

5.9. *Integration of reduced gravity and heights*

Method: 31

$$\Delta g^* = \sum_{k=1}^{\infty} \Delta^k g = \Delta^1 g + \Delta^2 g + \Delta^3 g + \dots$$

where
$$\Delta^1 g_j = \Delta g_j$$

$$\Delta^2 g_j = \frac{h_j\left(\Delta g_j - \dfrac{r_j^2 - r_0^2}{4\pi r_j}\displaystyle\iint \frac{\Delta^{(1)} g_i^*}{r_{ij}^3}\,dS\right)}{\dfrac{r_j^2 - r_0^2}{4\pi r_j}\displaystyle\iint \frac{h_i}{r_{ij}^3}\,dS}$$

$$\Delta^3 g_j = \frac{h_j\left(\Delta g_j - \dfrac{r_j^2 - r_0^2}{4\pi r_j}\displaystyle\iint \frac{\Delta^{(2)} g_i^*}{r_{ij}^3}\,dS\right)}{\dfrac{r_j^2 - r_0^2}{4\pi r_j}\displaystyle\iint \frac{h_i}{r_{ij}^3}\,dS}$$

$$— — — — —$$

For $(k > 1)$ (cf. Bjerhammar, 1963*c*)

$$\boxed{\Delta^k g_j = \frac{h_j\left(\Delta g_j - \dfrac{r_j^2 - r_0^2}{4\pi r_j}\displaystyle\iint \frac{\Delta^{(k-1)} g_i^*}{r_{ij}^3}\,dS\right)}{\dfrac{r_j^2 - r_0^2}{4\pi r_j}\displaystyle\iint \frac{h_i}{r_{ij}^3}\,dS}}$$

General procedure.

$$\Delta^{(1)}g^* = \Delta g$$
$$\Delta^{(2)}g^* = \Delta^{(1)}g^* + c_1 h$$
$$— — — — —$$
$$\Delta^{k}\ g^* = \Delta^{(k-1)}g^* + c_{k-1} h$$

The residuals are reduced from the physical surface to the reference surface using a constant derivative of first order for each individual fixed point. Cf. equation (4.1:13).

Note: This method makes use of a more sophisticated reduction of the residuals.

5.10. *Integration of reduced residuals and heights*

Method: 32

$$\Delta g^* = \sum_{k=1}^{\infty} \Delta^k g = \Delta^1 g + \Delta^2 g + \Delta^3 g + \dots$$

where

$$\Delta^1 g_j = \Delta g_j$$

$$\Delta^2 g_j = \frac{h_j\left(\Delta g_j - \dfrac{r_j^2 - r_0^2}{4\pi r_j}\displaystyle\iint \frac{\Delta g_i}{r_{ij}^3}\,dS\right)}{\dfrac{r_j^2 - r_0^2}{4\pi r_j}\displaystyle\iint \frac{h_i}{r_{ij}^3}\,dS}$$

$$\Delta^3 g_j = \Delta^2 g_j - \frac{h_j \displaystyle\iint \frac{\Delta^2 g_i}{r_{ij}^3}\,dS}{\displaystyle\iint \frac{h_i}{r_{ij}^3}\,dS}$$

— — — — —

For $(k > 2)$

$$\boxed{\Delta^k g_j = \Delta^{k-1} g_j - \frac{h_j \displaystyle\iint \frac{\Delta^{k-1} g_i}{r_{ij}^3}\,dS}{\displaystyle\iint \frac{h_i}{r_{ij}^3}\,dS}}$$

General procedure.

$$\Delta^{(1)} g^* = \Delta g$$

$$\Delta^{(2)} g^* = \Delta^{(1)} g^* + c_1 h$$

— — — — —

$$\Delta^{(k)} g^* = \Delta^{(k-1)} g^* + c_{k-1} h.$$

Note: A sophisticated reduction of the residuals without using repeated integration over the buffer zones.

5.11. *Integration of reduced gravity differences and heights*

Method: 33

$$\Delta g^* = \sum_{k=1}^{\infty} \Delta^k g = \Delta^1 g + \Delta^2 g + \Delta^3 g + \dots$$

where

$$\Delta^1 g_j = \Delta g_j$$

$$\Delta^2 g_j = \frac{h_j \displaystyle\iint \frac{t^{-2}\Delta g_j - \Delta^{(1)} g_i^*}{r_{ij}^3}\,dS}{\displaystyle\iint \frac{h_i}{r_{ij}^3}\,dS}$$

$$\Delta^3 g_j = \frac{h_j \iint \dfrac{t^{-2}\Delta g_j - \Delta^{(2)} g_i^*}{r_{ij}^3}\, dS}{\iint \dfrac{h_i}{r_{ij}^3}\, dS}$$

— — — — —

For (k > 1)

$$\boxed{\Delta^k g_j = \frac{h_j \iint \dfrac{t^{-2}\Delta g_j - \Delta^{(k-1)} g_i^*}{r_{ij}^3}\, dS}{\iint \dfrac{h_i}{r_{ij}^3}\, dS}}$$

General procedure.

$$\Delta^{(1)} g^* = \Delta g$$

$$\Delta^{(2)} g^* = \Delta^{(1)} g^* + c_1 h$$

— — — — —

$$\Delta^{(k)} g^* = \Delta^{(k-1)} g^* + c_{k-1} h.$$

5.12. *Least squares solution with heights*

Method: 51

$$\Delta g^* = \sum_{k=1}^{\infty} \Delta^k g = \Delta^1 g + \Delta^2 g + \Delta^3 g + \dots$$

where
$$\Delta^1 g_j = \delta g_j + c_{10} + c_{11} h + c_{12} h^2 + \dots \quad (\delta g = \Delta g)$$

$$\delta^2 g_j = \delta g_j - \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta^1 g_i}{r_{ij}^3}\, dS$$

$$\Delta^2 g_j = \delta^2 g_j + c_{20} + c_{21} h + c_{22} h^2 + \dots$$

$$\delta^3 g_j = \delta^2 g_j - \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta^2 g_i}{r_{ij}^3}\, dS$$

— — — — —

For $k > 0$

$$\Delta^k g_j = \delta^k g_j + c_{k0} + c_{k1} h + c_{k2} h^2 + \dots$$

$$\delta^{k+1} g_j = \delta^k g_j - \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta^k g_i}{r_{ij}^3}\, dS.$$

Procedure. The unknown c-coefficients are determined by least squares using the relation

$$\delta^k g_j = \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta^k g_i}{r_{ij}^3} dS.$$

Note: The method of least squares is used for eliminating the residuals with heights as parameters.

5.13. *Least squares solution with residuals*

Method: 61

$$\Delta g^* = \sum_{k=1}^{\infty} \Delta^k g = \Delta^1 g + \Delta^2 g + \Delta^3 g + \dots$$

where

$$\Delta^1 g_j = c_{11}\delta g_j + c_{12}(\delta g_j)^2 + c_{13}(\delta g_j)^3 + \dots \quad (\delta g = \Delta g)$$

$$\delta^2 g_j = \delta g_j - \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta^1 g_i}{r_{ij}^3} dS$$

$$\Delta^2 g_j = c_{21}\delta^2 g_j + c_{22}(\delta^2 g_j)^2 + c_{23}(\delta^2 g_j)^3 + \dots$$

$$\delta^3 g_j = \delta^2 g_j - \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta^2 g_i}{r_{ij}^3} dS$$

— — — — —

For $k > 0$

$$\Delta^k g_j = c_{k1}\delta^k g_j + c_{k2}(\delta^k g_j)^2 + c_{k3}(\delta^k g_j)^3 + \dots$$

$$\delta^{k+1} g_j = \delta^k g_j - \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta^k g_i}{r_{ij}^3} dS$$

$$r_j > r_0.$$

Procedure. The unknown c-coefficients are determined by least squares using the relation

$$\delta^k g_j = \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta^k g_i}{r_{ij}^3} dS.$$

Note: The method of least squares is used for eliminating the residuals. The method is somewhat controversial. (Points at sea-level have to be included in buffer zones.)

6. Iterative solution of the gravity reduction with systems of linear equations

Here we consider the solution of the integral equation for gravity reduction when the reduced gravity is constant over uniquely defined surface element. Then our integral equation can be treated as systems of linear equations.

Then we have a system of linear equations and for every discrete boundary value we obtain

$$\Delta g_j = \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta g_1^*}{r_{ij}^3} dS_1 + \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta g_2^*}{r_{ij}^3} dS_2 + \dots \tag{1}$$

This system of equations will include several thousand unknowns in most geodetic applications. Matrix inversion will be useful for simpler models and theoretical studies but will normally be too cumbersome in practical applications.

The convergence of the iterative solution will be discussed after transcribing our system of equations using matrix notations

$$\boxed{\underset{n1}{\Delta \mathbf{g}} = \underset{nn}{\mathbf{K}}\, \underset{n1}{\Delta \mathbf{g}^*}} \tag{2}$$

Using method 12 we obtain (all other methods can be discussed in a similar way)

$$\Delta^k \mathbf{g} = \Delta^{k-1}\mathbf{g} - \mathbf{K}\Delta^{k-1}\mathbf{g} = (\mathbf{I} - \mathbf{K})\,\Delta^{k-1}\mathbf{g} \tag{3}$$

and

$$\Delta^{(k)}\mathbf{g}^* = \Delta\mathbf{g} + (\mathbf{I} - \mathbf{K})\,\Delta^{(k-1)}\mathbf{g}^*. \tag{4}$$

Here we put

$$\underset{i}{\text{maximum}}\ |\Delta g_i| = \|\Delta \mathbf{g}\| \tag{5}$$

$$\underset{i}{\text{maximum}} \sum_j |(\delta_{ij} - k_{ij})| = \|\mathbf{I} - \mathbf{K}\| \quad (\delta_{ij} = \text{Kronecker delta}). \tag{6}$$

Then we have

$$\|\Delta^k \mathbf{g}\| \leq \|\mathbf{I} - \mathbf{K}\| \cdot \|\Delta^{k-1}\mathbf{g}\|. \tag{7}$$

Furthermore we have

$$\frac{r_j^2 - r_0^2}{4\pi r_j} \iint_S \frac{1}{r_{ij}^3} dS = \frac{r_0^2}{r_j^2} \leq 1. \tag{8}$$

For a surface element "concentric" with P_j we obtain

$$\frac{r_j^2 - r_0^2}{4\pi r_j} \iint_{\Delta S} \frac{1}{r_{ij}^3} dS = \frac{r_j + r_0}{2 r_j^2}\left[1 - \frac{h}{\varrho}\right] r_0 = k_{ii} \tag{9}$$

where

$$h = r_j - r_0$$

$$\varrho = r_{ij}\,\text{max}.$$

Thus we obtain the following *sufficient condition* for convergence

$$\|\mathbf{I} - \mathbf{K}\| < 1 \tag{10}$$

or

$$\max_i \sum_j |\delta_{ij} - k_{ij}| < 1. \tag{11}$$

Here we have

$$|\delta_{ii}-k_{ii}|=1-k_{ii} \tag{12}$$

$$|\delta_{ij}-k_{ij}|=0+k_{ij} \quad (i\neq j) \tag{13}$$

or

$$k_{ii}>\sum_{j(\neq i)} k_{ij}. \tag{14}$$

This condition is rewritten

$$\frac{(r_j+r_0)r_0}{2r_j^2}\left[1-\frac{h}{\varrho}\right]>\frac{r_0^2}{r_j^2}-\frac{(r_j+r_0)r_0}{2r_j^2}\left[1-\frac{h}{\varrho}\right] \tag{15}$$

or

$$\left[2+\frac{h}{r_0}\right]\left[1-\frac{h}{\varrho}\right]>1 \tag{16}$$

$$\boxed{\varrho>\frac{h(2r_0+h)}{r_0+h}} \tag{17}$$

We can replace this condition by another *sufficient* condition

$$\boxed{\varrho>2h} \tag{18}$$

This means that the surface element of the fixed point has to be inclosed by a cone with the side larger than twice the height of the fixed point. Using this condition we can expect a very good convergence. For a sharper convergence condition from latent roots cf. chapter 9.

7. Convergence with square surface element

The preceding analysis was made for a circular surface element and we are here adding a corresponding study for a square surface element.

We have for a rectangular surface element with the fixed point above the origin of the coordinate system and with the x- and y-axes coinciding with two of the sides of the rectangle

$$\|\mathbf{I}-\mathbf{K}\|<1$$

as a sufficient condition for convergence where

$$k_{ii}=\frac{r_j^2-r_0^2}{4\pi r_j}\int_0^{y_1}\int_0^{x_1}\frac{1}{(x^2+y^2+h^2)^{1.5}}dx\,dy \quad \text{(Plane reference surface).} \tag{1}$$

After integration we have

$$k_{ii}=\frac{r_j^2-r_0^2}{4\pi r_j}\cdot\frac{1}{h}\operatorname{arctg}\frac{y_1x_1}{h\sqrt{x_1^2+y_1^2+h^2}}. \tag{2}$$

For small heights we use the approximation

$$k_{ii} = \frac{1}{2\pi} \operatorname{arctg} \frac{y_1 x_1}{h\sqrt{x_1^2 + y_1^2 + h^2}}. \tag{3}$$

With the fixed point in the centre of a square surface element with the side 2ϱ we obtain

$$k_{ii} = \frac{2}{\pi} \operatorname{arctg} \frac{\varrho^2}{h\sqrt{2\varrho^2 + h^2}}. \tag{4}$$

For further analysis we compute the value of the sum of all elements from the spherical case. Then we obtain for small heights the sufficient condition for convergence

$$\frac{2}{\pi} \operatorname{arctg} \frac{\varrho^2}{h\sqrt{2\varrho^2 + h^2}} > 1 - \frac{2}{\pi} \operatorname{arctg} \frac{\varrho^2}{h\sqrt{2\varrho^2 + h^2}}. \tag{5}$$

or

$$\frac{4}{\pi} \operatorname{arctg} \frac{\varrho^2}{h\sqrt{2\varrho^2 + h^2}} > 1. \tag{6}$$

Furthermore we have

$$\frac{\varrho^2}{h\sqrt{2\varrho^2 + h^2}} > 1. \tag{7}$$

This relation gives the sufficient condition

$$\varrho > h\sqrt{1 + \sqrt{2}}. \tag{8}$$

In practical models acceptable convergence can be found for surface elements of much smaller size. (The convergence can be improved after adding the corrections in succesive order according to Gauss.)

8. Error analysis

For the iterative process we have

$$\Delta^k \mathbf{g} = (\mathbf{I} - \mathbf{K})\Delta^{k-1}\mathbf{g}. \tag{1}$$

Further corrections are obtained in the following way

$$\Delta^{k+1}\mathbf{g} = (\mathbf{I} - \mathbf{K})\Delta^k \mathbf{g} \tag{2}$$

$$\Delta^{k+2}\mathbf{g} = (\mathbf{I} - \mathbf{K})^2 \Delta^k \mathbf{g} \tag{3}$$

$$\Delta^{k+3}\mathbf{g} = (\mathbf{I} - \mathbf{K})^3 \Delta^k \mathbf{g} \tag{4}$$

— — — — —

$$\Delta^{\infty}\mathbf{g} = (\mathbf{I} - \mathbf{K})^{\infty} \Delta^k \mathbf{g}. \tag{5}$$

We consider $\Delta^k \mathbf{g}$ as our last correction. Then the omitted corrections $\boldsymbol{\epsilon}^{(k)}$ can be written

$$\boldsymbol{\epsilon}^{(k)} = \sum_{u=1}^{\infty} (\mathbf{I}-\mathbf{K})^u \Delta^k \mathbf{g}. \tag{6}$$

The limiting value of this infinite series is then

$$\boxed{\|\boldsymbol{\epsilon}^{(k)}\| \leqslant \frac{\|\mathbf{I}-\mathbf{K}\|}{1-\|\mathbf{I}-\mathbf{K}\|}\|\Delta^k \mathbf{g}\|} \tag{7}$$

For a "concentric" surface element of the fixed point we obtain

$$\|\mathbf{I}-\mathbf{K}\| = 1 + \frac{r_0^2}{r_j^2} - \frac{(r_j+r_0)r_0}{r_j^2}\left[1-\frac{h}{\varrho}\right] \tag{8}$$

$$1-\|\mathbf{I}-\mathbf{K}\| = \frac{(r_j+r_0)r_0}{r_j^2}\left[1-\frac{h}{\varrho}\right] - \frac{r_0^2}{r_j^2}. \tag{9}$$

With the reference surface at mean sea-level we use for terrestrial points the approximation

$$\|\mathbf{I}-\mathbf{K}\| = 2-2\left[1-\frac{h}{\varrho}\right] = \frac{2h}{\varrho} \tag{10}$$

$$1-\|\mathbf{I}-\mathbf{K}\| = 2\left[1-\frac{h}{\varrho}\right] - 1 = 1-\frac{2h}{\varrho}. \tag{11}$$

Then we obtain

$$\|\boldsymbol{\epsilon}^{(k)}\| \leqslant \frac{2h}{\varrho-2h}\|\Delta^k \mathbf{g}\|. \tag{12}$$

For $\varrho > 2h$ we have

$$\boxed{\|\boldsymbol{\epsilon}^{(k)}\| \leqslant \infty} \tag{13}$$

For $\varrho > 3h$ we have

$$\|\boldsymbol{\epsilon}^{(k)}\| \leqslant 2\|\Delta^k \mathbf{g}\|. \tag{14}$$

For $\varrho > 4h$ we have

$$\|\boldsymbol{\epsilon}^{(k)}\| \leqslant 1\|\Delta^k \mathbf{g}\|. \tag{15}$$

For a sharper error limit from latent roots cf. chapter 9.

9. Over-relaxation

A more powerful iterative approach for linear systems of equations is obtained according to Richardson when using overrelaxation. Also the method of Gauss–Seidel can be adopted for over-relaxation.

Our system of linear equations is transcribed in the following way

$$\mathbf{K}\Delta\mathbf{g}^* = \Delta\mathbf{g} \Leftrightarrow (\mathbf{L}+\mathbf{D}+\mathbf{R})\Delta\mathbf{g}^* = \Delta\mathbf{g} \tag{1}$$

where

$d_{ii} = k_{ii} \quad d_{ij} = 0 \quad \text{for} \quad i \neq j$

$l_{ij} = k_{ij} \quad \text{for} \quad i > j \quad l_{ij} = 0 \quad \text{for} \quad i < j$

$r_{ij} = k_{ij} \quad \text{for} \quad i < j \quad r_{ij} = 0 \quad \text{for} \quad i > j.$

We now multiply our iterative correction by a relaxation coefficient (ω) according to the formulas

$$\Delta^{(k)}\mathbf{g}^* = \Delta^{(k-1)}\mathbf{g}^* + \omega\mathbf{D}^{-1}(\Delta\mathbf{g} - \mathbf{K}\Delta^{(k-1)}\mathbf{g}^*) \tag{2}$$

$$\Delta^{k}\mathbf{g} = \omega\mathbf{D}^{-1}(\Delta\mathbf{g} - \mathbf{K}\Delta^{(k-1)}\mathbf{g}^*) \tag{3}$$

$$\boldsymbol{\epsilon}^{(k)} = \Delta\mathbf{g}^* - \Delta^{(k)}\mathbf{g}^* = (\mathbf{I} - \omega\mathbf{D}^{-1}\mathbf{K})\boldsymbol{\epsilon}^{(k-1)} \tag{4}$$

$$\boxed{\boldsymbol{\epsilon}^{(k)} = (\mathbf{I} - \omega\mathbf{D}^{-1}\mathbf{K})^{(k)}\boldsymbol{\epsilon}^{(0)}} \tag{5}$$

Convergence condition: If and only if the latent roots (λ) fulfil the condition

$$|\lambda_{\max}| \quad \text{of} \quad (\mathbf{I} - \omega\mathbf{D}^{-1}\mathbf{K}) < 1 \tag{6}$$

then the iterative solution is convergent for a system with a unique solution. This is the iterative method of Richardson and for $\omega = 1$ the method is identical with the method of Jacobi.

The method of Gauss–Seidel lends itself to over-relaxation in a convenient approach (succesive over-relaxation)

$$\Delta^{(k)}\mathbf{g}^* = \Delta^{(k-1)}\mathbf{g}^* + \omega\mathbf{D}^{-1}[\Delta\mathbf{g} - \mathbf{L}\Delta^{k}\mathbf{g}^* - (\mathbf{D}+\mathbf{R})\Delta^{(k-1)}\mathbf{g}^*] \tag{7}$$

$$\boldsymbol{\epsilon}^{(k)} = \Delta\mathbf{g}^* - \Delta^{(k)}\mathbf{g}^* = [\mathbf{I} - \omega\mathbf{D}^{-1}\mathbf{L}]^{-1}[\mathbf{I} - \omega\mathbf{D}^{-1}(\mathbf{D}+\mathbf{R})]\boldsymbol{\epsilon}^{(k-1)}) \tag{8}$$

$$\boxed{\boldsymbol{\epsilon}^{(k)} = \{[\mathbf{I} + \omega\mathbf{D}^{-1}\mathbf{L}]^{-1}[\mathbf{I} - \omega\mathbf{D}^{-1}(\mathbf{D}+\mathbf{R})]\}^{k}\boldsymbol{\epsilon}^{(0)}} \tag{9}$$

Convergence condition:

$$|\lambda_{\max}| \quad \text{of} \quad \{[\mathbf{I} + \omega\mathbf{D}^{-1}\mathbf{L}]^{-1}[\mathbf{I} - \omega\mathbf{D}^{-1}(\mathbf{D}+\mathbf{R})]\} < 1 \tag{10}$$

for a system with unique solution. Ostrowski (1954) showed that this method is always convergent for a positive definite system that is symmetrical provided

$$0 < \omega < 2. \tag{11}$$

Our primary system equations can be made symmetrical in the following way

$$\mathbf{K}^*\mathbf{K}\Delta\mathbf{g}^* = \mathbf{K}^*\Delta\mathbf{g} \tag{12}$$

and further
$$\mathbf{NY} = \mathbf{D}^{-\frac{1}{2}}\mathbf{K}^*\Delta\mathbf{g} \tag{13}$$
where
$$\mathbf{N} = \mathbf{D}^{-\frac{1}{2}}(\mathbf{K}^*\mathbf{K})\,\mathbf{D}^{-\frac{1}{2}} = \text{symmetrical} \tag{14}$$
$$\mathbf{Y} = \mathbf{D}^{+\frac{1}{2}}\Delta\mathbf{g}^*. \tag{15}$$

In this approach we have a convergent procedure of our iterative solution for any practical application with equal number of unknowns and observations. (Such a system will be positive definite for the gravity reduction.) The size of the surface element is not limited by equations (6:17) or (7:7) when a slower convergence is accepted.

10. Truncations

In our original study from 1962 we obtained the following *integral* equation for gravity reduction (equation 3.1:1 above)

$$\Delta g = \frac{1}{4\pi r_j^2}\iint \Delta g^*\left[\frac{r_j}{r_{ij}} - 3\frac{r_0}{r_j}\cos\omega - \frac{2\left[\left(\frac{r_0}{r_j}\right)^2 - \left(\frac{r_0}{r_j}\right)\cos\omega\right]r_j^3}{r_{ij}^3} - 1\right]dS. \tag{1}$$

This equation is transcribed

$$\Delta g = \frac{[r_j^2 - r_0^2]}{4\pi r_j}\iint \Delta g^*\left[\frac{1}{r_{ij}^3} - \frac{(3r_0\cos\omega + r_j)}{r_j^2(r_j^2 - r_0^2)}\right]dS. \tag{2}$$

Equation (2) has been derived using the generalized Stokes' formula with elimination of the zero and first order harmonics.

After omitting the two terms of lowest order (0 and 1) it was obtained

$$\Delta g = \frac{r_j^2 - r_0^2}{4\pi r_j}\iint \frac{\Delta g^*}{r_{ij}^3}\,dS. \tag{3}$$

This equation can also be obtained from Green's integral equation

$$U_j = \frac{1}{4\pi}\iint\left(U_i\frac{\partial(1/r)}{\partial n} - \frac{1}{r}\frac{\partial U_i}{\partial n}\right)dS \tag{4}$$

or Poisson's integral equation

$$U_j = \frac{r_j^2 - r_0^2}{4\pi r_0}\iint \frac{U_i}{r_{ij}^3}\,dS \tag{5}$$

after introducing the harmonic variables

$$U_j = r_j\Delta g_j \tag{6a}$$
$$U_i = r_0\Delta g_i^*. \tag{6b}$$

There is a slight difference between the two formulas which has to be explained. (Cf. equations (2) and (3).)

A solution of the boundary value problem is impossible if the gravity anomalies include first order harmonics. If we consider a purely analytical solution of our gravity reduction problem when zero and first order harmonics are missing, then the results will be identical. Using a regression analysis for the solution we will have a slight conversion of the information towards zero and first order harmonics and there will be a small falsification of the results in eq. 3.

If the gravity measurement really contains zero and first order harmonics, then these two harmonics will still be included in the reduced gravity data. Under some circumstances we can consider this an advantage and then we have to choose equation 3.

Specially if we are working with a gravity field which includes a large low order harmonic we have to consider this problem. In such a case any truncation of the integration can interfere with the final results in an inconvenient way. This problem has carefully been studied by de Witte.

In our study we have rather favourable conditions because the fixed point is always situated at the physical surface. Thus we only have to consider the reduction down to the reference ellipsoid.

For a surface element concentric with the fixed point and with a constant value of Δg^* we have

$$\delta(\Delta g_0) = \frac{1}{4\pi r_j^3} \iint \Delta g_0^* (3r_0 \cos\omega + r_j)\, dS = \left[\tfrac{3}{4}\left(\frac{r_0}{r_j}\right)^3 \sin^2\omega + \left(\frac{r_0}{r_j}\right)^2 \sin^2\frac{\omega}{2}\right] \Delta g_0^* \qquad (7)$$

where

Δg_0^* = constant value of Δg^*

ω = geocentric angle (aperture) of the surface element

$\delta(\Delta g_0)$ = integrated value of the difference between equation (2) and (3).

Here we have for $r_0 = r_j$

ω	$\delta(\Delta g_0)$
0	0
1°	0.0003 Δg_0^*
10°	0.0302 Δg_0^*
90°	1.25 Δg_0^*
180°	1.00 Δg_0^*.

In most applications we can restrict the gravity reduction to an area of a few degrees. The differences caused by the choice of degenerations or smoothing will then be larger than the truncation errors.

Furthermore, we have to consider that the two solutions have to satisfy the boundary values in all given points. Therefore, the truncation errors will be without any influence on the final results for the given boundary values. It is further of importance to note that our gravity reduction mainly interacts on very high harmonics which are quite local.

If we expand our study from the analytical integration with constant value of Δg_0^* to the actual case with varying Δg^* values in a discrete approach we have

$$\Delta \mathbf{g} = \mathbf{K}_1 \Delta \mathbf{g}_1^* \tag{8}$$

and

$$\Delta \mathbf{g} = \mathbf{K}_2 \Delta \mathbf{g}_2^* \tag{9}$$

where

$\mathbf{K}_1$ = integral operator from equation 2
$\mathbf{K}_2$ = integral operator from equation 3.

For the non-singular case we will have

$$\Delta \mathbf{g}_1^* = \mathbf{K}_1^{-1} \Delta \mathbf{g} \tag{10}$$

and

$$\mathbf{K}_1 \Delta \mathbf{g}_1^* = \mathbf{K}_1 \mathbf{K}_1^{-1} \Delta \mathbf{g} = \Delta \mathbf{g}. \tag{11}$$

In the same way

$$\Delta \mathbf{g}_2^* = \mathbf{K}_2^{-1} \Delta \mathbf{g} \tag{12}$$

$$\mathbf{K}_2 \Delta \mathbf{g}_2^* = \mathbf{K}_2 \mathbf{K}_2^{-1} \Delta \mathbf{g} = \Delta \mathbf{g}. \tag{13}$$

Thus we have proved that there is no effect on the given boundary values.

There is still a satisfactory justification for choosing the simpler formula (3) when this equation is used as an *integral* equation and not only as an upward continuation formula.

In the solution of our integral equation for gravity reduction we have to exclude the remote zones and we have to select an adequate truncation. With a circular area ΔS_1 concentric with the fixed point we obtain $(\Delta S_1 + \Delta S_2 = S)$

$$\Delta g = \frac{r_j^2 - r_0^2}{4\pi r_j} \iint_{\Delta S_1} \frac{\Delta g_1^*}{r_{ij}^3} dS_1 + \frac{r_j^2 - r_0^2}{4\pi r_j} \iint_{\Delta S_2} \frac{\Delta g_2^*}{r_{ij}^3} dS_2. \tag{14}$$

With $\Delta g_1^* = \Delta g_2^* = \Delta g^*$ we have

$$\Delta g = \frac{(r_j + r_0) r_0}{2 r_j^2} \left[1 - \frac{h}{\varrho}\right] \Delta g^* + \varepsilon \tag{15}$$

where

ϱ = distance between the fixed point and the zone boundary.

For $r_j = r_0$ we have

$$\varepsilon \leqslant \frac{h\,\Delta g^*}{\varrho}. \tag{16}$$

A non-constant Δg gives

$$|\varepsilon| < \frac{h}{\varrho} |\Delta g^* \max| \simeq \frac{h}{\varrho} |\Delta g \max|. \tag{17}$$

11. Auxiliary reference surface

We have found that the truncation errors increase with the reduction height. Earlier it has been verified that the errors of the iterative process also increase with the reduction height. Therefore, we sometimes will find it necessary to use different reference surfaces in order to improve the convergence.

The general residual at the physical surface can be defined in the following way for an arbitrary step of the iteration

$$\overset{\bullet}{\Delta^k G} = \Delta g - \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta^{(k)} g^*}{r_{ij}^3} \, dS. \tag{1}$$

When using a detailed analysis in a steep terrain we can expect large residuals and a slow convergence. This problem is of special importance for a determination of the vertical deflection. Here our solution is fulfilled in steps using two different references spheres.

(1) A primary reference sphere coinciding with the reference ellipsoid at the centre of the actual area. This reference sphere has been found suitable for surface elements down to size $5' \times 5'$. (Four *iterative* steps were sufficient to reduce the residual R.M.S. below ± 0.5 milligal in the West Alps.) From theoretical point of view we have to extend this gravity reduction all over the earth but mostly $5° \times 5°$ will be sufficient for a local study.

(2) An auxiliary reference sphere through the actual point is used for a refined reduction of the *residuals* in a local area. This additional reduction is needed for the computation of vertical deflections. Then the residuals are first computed at all gravity stations of the local area using the primary reference sphere. Inside this local area the residuals are reduced to an auxiliary reference surface which intersects the actual point P_A.

The reduction of the gravity residuals to the auxiliary reference sphere is from theoretical point of view more complex but only an approximate solution is needed for reducing our residuals which are considered small (cf. Fig. 8).

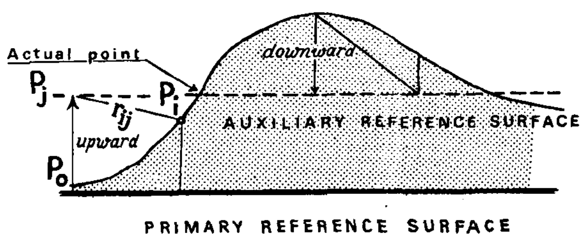


Fig. 8. Solution with an auxiliary reference surface. The vertical deflection will be computed for the actual point.

Because the auxiliary reference sphere *intersects* the physical surface we have to make upward as well as downward gravity reduction of all residuals to the new reference sphere. If we temporarily consider the physical surface to be a "sphere" we have for *upward* continuation *without* using integral equations

$$\Delta G_j^* = \frac{r_j^2 - r_0^2}{4\pi r_j} \iint_S \frac{\Delta G_i}{r_{ij}^3} \, dS. \tag{2}$$

After adding and subtracting G_j we obtain with an obvious approximation

$$\Delta G_j^* \simeq \Delta G_j + \frac{1}{2\pi}\iint_S \frac{(\Delta G_i - \Delta G_j)(r_j - r_i)}{r_{ij}^3}\, dS. \tag{3}$$

We introduce the following *modified* definitions for *upward* and *downward* continuation from an *arbitrary* surface

ΔG_j = gravity *residual* of the surface of the earth in the vertical projections of the fixed point
ΔG_j^* = gravity *residual* at the fixed point of the new reference sphere
ΔG_i = gravity *residual* at the moving point of the surface of the earth
r_j = geocentric distance of the fixed point at the new reference surface
r_i = geocentric distance of the moving point at the surface of the earth
r_{ij} = distance between the fixed point at the new reference sphere and the moving point at the surface of the earth
S = new reference sphere.

As an alternative method we can use the derivative from equation (4.1:12)

$$\Delta G_j^* = \Delta G_j - \frac{h_{j0}}{2\pi}\iint \frac{\Delta G_i - \Delta G_j}{\sigma_{ij}^3}\, dS \tag{4}$$

where

h_{j0} = height of the fixed point above the new reference sphere
σ_{ij} = distance between the fixed and the moving point when both are projected on the reference sphere.

A more sophisticated expression is obtained from equation (4.1:13)

$$\Delta G_j^* = \Delta G_j - h_{j0}\,\frac{\dfrac{r_j^2 - r_0^2}{4\pi r_j}\displaystyle\iint \frac{\Delta G_i - \Delta G_j}{r_{ij}^3}\, dS}{\dfrac{r_j^2 - r_0^2}{4\pi r_j}\displaystyle\iint \frac{h_i}{r_{ij}^3}\, dS}\,. \tag{5}$$

Here we compute the derivatives using the primary reference sphere. Then we use these derivative for a reduction from the fixed points to the *new* reference sphere.

In a very simple approach we put

$$\boxed{\Delta G = \Delta G^*} \tag{6}$$

For many applications this method will be sufficient.

After the introduction of an auxilliary references surface intersecting the physical surface no further iterative steps can be added without using a rather complicated reduction technique which will not be considered here.

A computation of the vertical deflection requires a local gravity reduction of the residuals to an auxiliary reference sphere through the actual point where the vertical deflection will be computed. Sometimes it will be convenient to include an upward continuation of Δg^* to the new reference surface in order to have all reduced gravity data in a common system.

Instead of reducing the residuals to an auxiliary reference surface we can eliminate them completely with "isometric anomalies" and then use a Newtonian computation for the final study. (Cf. "Isometric Anomalies" below.)

12. Free air anomalies

In the present study we have generally accepted free air anomalies for our solution of the boundary value problem.

We define the free air anomaly (Δg) in the following way

$$\Delta g = g - \gamma_z \tag{1}$$

where

g = actual gravity of the fixed point
γ_z = theoretical gravity of the reference point.

The potential at the reference point in the theoretical gravity field has to be identical with the potential at the fixed point in the true gravity field when the two points are located in the same vertical. This strict definition of the free air anomaly has been given by Molodensky. There are considerable difficulties in a practical application of this definition because the potentials are not well determined until we have a complete solution of the boundary value problem. Therefore, an advanced solution will require an *iterative approach also with respect to the computation of the free air anomalies*. However, there seems to be no justification for these time consuming operations when using gravity data now available for geodetic studies. In actual applications we have to accept gravity anomalies where the theoretical gravity has been computed from the best available heights.

The free air anomaly is normally computed according to the formula

$$\Delta g = g - \left(\gamma_0 + z \frac{\partial \gamma}{\partial z}\right) \tag{2}$$

where

$\frac{\partial \gamma}{\partial z} = -0.3086$ milligal/m

z = height of the reference point above the reference sphere (ellipsoid)
γ_0 = theoretical gravity at the sphere (ellipsoid).

Everywhere in our study the free air anomaly is referred to the physical surface and not to the geoid.

In classical geodesy the gravity reduction was introduced in order to give a direct solution of the gravimetric boundary value problem. Our gravity reduction requires an *antireduction* up to the surface of the earth in order to give a solution of the boundary value problem. This antireduction is included in our formulas for a computation of the potential and its derivatives in space. Without this antireduction we compute a theoretical geoid "the space geoid"

1.
$$N=\frac{1}{4\pi r_0\gamma_0}\iint \Delta g^* \sum_{n=2}^{\infty}\frac{2n+1}{n-1}P_n(\cos\omega)\,dS \quad \text{(Cf. Stokes, 1849)} \tag{3}$$

where

N = geoidal height,
γ_0 = theoretical gravity.

This geoid is obtained after translocating all external mass inside the reference sphere. The "space geoid" is compatible with the satellite geoid.

2. *Potential*

After proper antireduction out to any point in space at the geocentric distance r_j we obtain the disturbance potential

$$\begin{aligned} T&=\frac{1}{4\pi r_j}\iint \Delta g^* \sum_{n=2}^{\infty}\left(\frac{r_0}{r_j}\right)^n\frac{2n+1}{n-1}P_n(\cos\omega)\,dS \\ &=\frac{1}{4\pi r_j}\iint \Delta g^* S(\omega)\,dS \quad \text{(Cf. Pizetti, 1911)} \end{aligned} \tag{4}$$

3. *Gravity*

Gravity anywhere in space is obtained by the formula

$$\Delta g=\frac{r_j^2-r_0^2}{4\pi r_j}\iint \frac{\Delta g^*}{r_{ij}^3}\,dS \quad \text{(Cf. Poisson, Green)} \tag{5}$$

4. *Vertical deflection*

Vertical deflection anywhere in space is obtained by the formula

$$\left.\begin{aligned} \xi&=-\frac{1}{\gamma_j}\frac{\partial T}{\partial x}=-\frac{1}{4\pi\gamma_j r_j}\iint \frac{\partial S(\omega)}{\partial\omega}\frac{\partial\omega}{\partial\omega_x}\frac{\partial\omega_x}{\partial x}\,dS \quad \text{(North-South)} \\ \eta&=-\frac{1}{\gamma_j}\frac{\partial T}{\partial y}=-\frac{1}{4\pi\gamma_j r_j}\iint \frac{\partial S(\omega)}{\partial\omega}\frac{\partial\omega}{\partial\omega_y}\frac{\partial\omega_y}{\partial y}\,dS \quad \text{(East-West)} \end{aligned}\right\} \tag{6}$$

(Sign convention according to Bomford)

$$\left.\begin{aligned} \xi&=\frac{1}{4\pi\gamma_j r_j^2}\iint \Delta g^*\frac{\partial S(\omega)}{\partial\omega}\cos\alpha\,dS \\ \eta&=\frac{1}{4\pi\gamma_j r_j^2}\iint \Delta g^*\frac{\partial S(\omega)}{\partial\omega}\sin\alpha\,dS \end{aligned}\right\} \tag{7}$$

$\partial \omega_x$ = angular displacement of the fixed point (positive north, geocentric)
$\partial \omega_y$ = angular displacement of the fixed point (positive east, geocentric)

$$S(\omega) = \left(\frac{2}{r} - 3r + 1 - 5t \cos \omega - (3t \cos \omega) \ln \frac{1 - t \cos \omega + r}{2}\right)$$

$$\frac{\partial S(\omega)}{\partial \omega} = -t \sin \omega \left(\frac{2}{r^3} - 8 + \frac{3(r+1)^2}{2r\phi} - 3 \ln \phi\right)$$

$$\frac{\partial \omega}{\partial \omega_x} = -\cos \alpha \quad \frac{\partial \omega}{\partial \omega_y} = -\sin \alpha \quad \frac{\partial \omega_x}{\partial x} = \frac{\partial \omega_y}{\partial y} = \frac{1}{r}$$

$$\phi = \frac{(r+1)^2 - t^2}{4} \quad t = \frac{r_0}{r_j} \quad r = r_{ij}/r_j \quad \alpha = \text{azimuth}.$$

We note that an "*antireduction*" is included in the computation of potentials, gravity and vertical deflections at the surface of the earth and in space. In a correct solution this antireduction compensates exactly for the previous downward reduction when solving for Δg^*.

13. Bouguer anomalies

The present study has everywhere been made using free air anomalies which are convenient for this type of analysis. It is obvious that any gravity anomaly can be used if all relevant considerations are made. The Bouguer anomaly is probably the most interesting alternative and we make a comparative study.

First we define the "refined" Bouguer anomaly (Δg_{B*}).

$$\Delta g_{B*} = g - \gamma_z + G \iiint_V \frac{\zeta \varrho}{d_{ij}^3} dV \tag{1}$$

where (cf. Fig. 9)

ζ = height of the moving point above an auxiliary plane through the fixed point P_j normal to corresponding radius vector of the earth
ϱ = density at the moving point
d_{ij} = distance between the fixed point and the moving point
G = constant of gravitation
V = the volume between the reference sphere and the physical surface.

The physical surface is unknown and therefore the integration is more correctly referred to the volume between the geoid and the physical surface. In this Bouguer anomaly all mass outside the geoid has been removed for a known density (normally for $\varrho = 2.67$).

In some applications a slightly different formula can be used.

$$\Delta g_B = g - \left(\gamma_0 + z\frac{\partial \gamma}{\partial z}\right) - 2\pi G\varrho z \tag{2}$$

where

$2\pi G\varrho z$ = attraction of an infinite cylinder of height z.

Here we have

$$-z\frac{\partial \gamma}{\partial z} = +0.3086\ z \text{ milligal} \quad \text{(Free air reduction)}$$

$$-2\pi G\varrho z = -0.1119\ z \text{ milligal} \quad \text{(Bouguer plate)}$$

$$-z\frac{\partial \gamma}{\partial z} - 2\pi G\varrho z = +0.1967\ z \text{ milligal}$$

The introduction of an infinite cylinder is not without complications and should be avoided here.

Two different applications of Bouguer anomalies will be demonstrated here.

1. *Interpolation of free air anomalies*

Bouguer anomalies can be used as a convenient auxiliary tool for interpolation (and extrapolation) of free air anomalies. With an adequate choice of density for the Bouguer reduction we can expect a rather smooth behaviour of the reduced gravity anomaly and this will simplify any interpolation (or extrapolation). At least in areas with very large height differences such an intermediate reduction will be helpful. After completed auxiliary operations we can convert the results to free air anomalies for the final solutions.

2. *Direct solution with Bouguer anomalies*

The classical Bouguer anomaly should be referred to the physical surface. This anomaly we reduce to the reference sphere using the relation

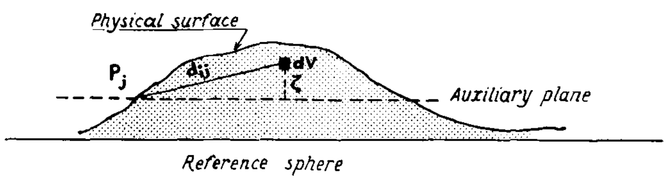


Fig. 9. Relations for the Bouguer anomaly.

$$\Delta g_{B*} = \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta g^*_{B*}}{r_{ij}^3}\, dS \tag{3}$$

where

Δg_{B*} = classical Bouguer anomaly for the physical surface
Δg^*_{B*} = "new Bouguer" anomaly for the reference surface.

Here we have replaced the original Bouguer anomaly (including terrain correction) with a new more sophisticated "new Bouguer" anomaly which is valid for the reference sphere. After this operation we have removed all mass outside the reference surface to infinity in a useful way and we can make a direct application of our closed formulas for the reference sphere. In this way we can compute the potential anywhere outside the physical surface

$$T = \frac{1}{4\pi r_j} \iint_S \Delta g_B^* \sum_{n=2}^{\infty} \left(\frac{r_0}{r_j}\right)^n \frac{2n+1}{n-1} P_n(\cos \omega)\, dS + G \iiint_V \frac{\varrho}{d_{ij}}\, dV, \tag{4}$$

where

V = volume between the reference surface and the physical surface,
d_{ij} = distance between the fixed and the moving point.

Derivatives of T give the vertical deflection and gravity.

14. Isostatic anomalies

The isostatic hypothesis has been used in the classical methods for refined estimates of the unknown densities of the crust of the earth. The Airy-Heiskanen system is based on the hypothesis that the mountains have a constant density

$$\varrho_0 = 2.67 \text{ gr/cm}^3$$

and are floating on an underlayer of density (cf. Fig. 10)

$$\varrho_1 = 3.27 \text{ gr/cm}^3.$$

According to the hypothesis of equilibrium a mountain of the height z must have a root of the depth d, where

$$d(\varrho_1 - \varrho_0) = z\varrho_0. \tag{1}$$

Thus we have for the mountains

$$d = \frac{\varrho_0}{\varrho_1 - \varrho_0} \cdot z = 4.45\, z. \tag{2}$$

For the oceans we obtain

$$d'(\varrho_1 - \varrho_0) = z'(\varrho_0 - \varrho'), \tag{3}$$

where

z' = depth of the ocean,
ϱ' = density of the ocean.

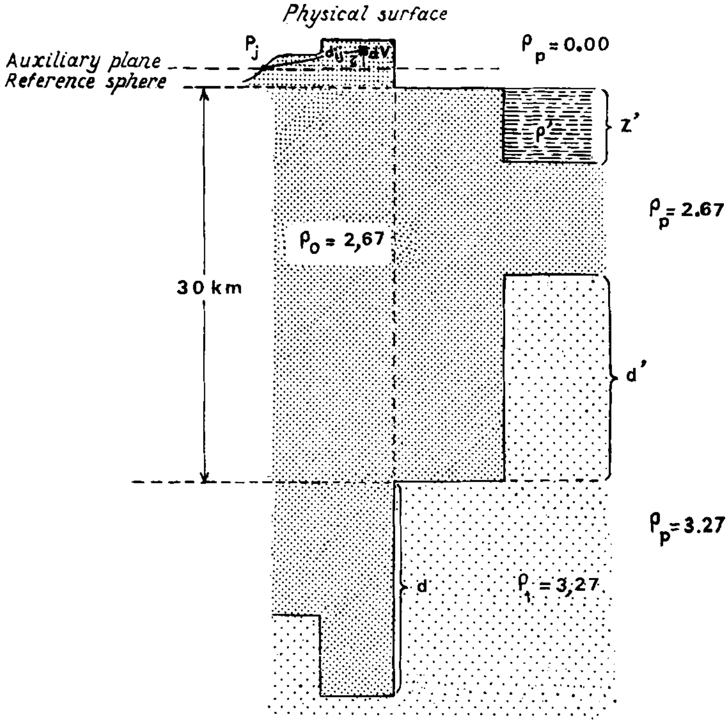


Fig. 10. Relations for the isostatic anomaly.

Thus we have for the given values

$$d' = \frac{z'(\varrho_0 - \varrho')}{(\varrho_1 - \varrho_0)} = 2.73\ z'. \tag{4}$$

The normal thickness of the crust is assumed to be 30 km. Cf. figure below.

The isostatic reduction is normally composed of the following sections: (1) Free air reduction, (2) Bouguer plate reduction, (3) terrain correction for mass outside the geoid, (4) terrain correction for mass disturbances inside the geoid.

All these corrections can be included in the following formula

$$\Delta g_I = \Delta g + G \iiint_V \frac{(\varrho_a - \varrho_p)\zeta}{d_{ij}^3} dV, \tag{5}$$

where

ϱ_a = estimated density before mass transportation ($\varrho_0, \varrho_1, \varrho'$),
ϱ_p = estimated density after mass transportation,
V = volume of the earth.
(*Note.* $\varrho_p = 0$ above crust, $\varrho_p = 2.67$ in crust, $\varrho_p = 3.27$ below crust.)

These anomalies are very smooth and allow excellent interpolation. Also here two different applications are of interest.

1. *Interpolation of free air anomalies*

The isostatic anomalies are still smoother than the Bouguer anomalies. They form an excellent tool for interpolation. After completed interpolation free air anomalies can be computed and the solution of the boundary value problem can be fulfilled with our new methods. (Integral equations).

2. *Direct solution with isostatic anomalies*

The classical isostatic anomaly is referred to the physical surface and we make a reduction to the reference sphere

$$\Delta g_I = \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta g_I^*}{r_{ij}^3} dS. \tag{6}$$

Then we have the potential in space

$$T = \frac{1}{4\pi r_j} \iint \Delta g_I^* \sum_{n=2}^{\infty} \left(\frac{r_0}{r_j}\right)^n \frac{2n+1}{n-1} P_n(\cos\omega)\, dS + G \iiint_V \frac{\varrho_a - \varrho_v}{d_{ij}} dV. \tag{7}$$

Vertical deflection and gravity is obtained after derivations of T. The convergence of the iterative solution is independent of the anomalies. Cf. chapter 6. However, Bouguer anomalies and isostatic anomalies are smoother than the free air anomalies and give a reduction of the general size of the anomaly. In some cases this will make it possible to use just one iterative step.

In some applications insufficient gravity data make it necessary to use certain estimate of densities for the local zone. Then mass computations according to Newton are needed.

15. Rudzki anomalies

In classical geodesy it was an essential problem to find the correct physical geoid all over the earth. Therefore, it was natural for Rudzki (1905) to define an anomaly giving the true potential of the earth. The new anomaly was obtained by reflecting the topography below the geoid when making a specified mass correction.

For a geoid which is a plane we easily verify that a pure reflexion gives exact potential without any mass correction.

In general we have the potential of an infinitesimal mass element dm outside the reference sphere (cf. Fig. 11)

$$dW_1 = \frac{G\, dm}{\sqrt{r_j^2 + r_0^2 - 2r_j r_0 \cos\omega}} \tag{1}$$

where

r_j = geocentric distance of the mass element.

For the reflected mass point dm' we have

$$dW_2 = \frac{G\, dm'}{\sqrt{r_k^2 + r_0^2 - 2r_0 r_k \cos\omega}} \tag{2}$$

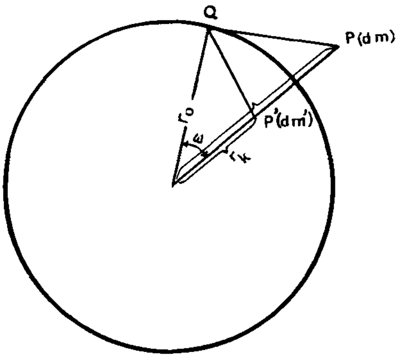


Fig. 11. Relations for the Rudzki anomaly.

where

r_k = geocentric distance of the mass element.

If $$dm' = \frac{r_0}{r_j} dm$$

and $$r_k = r_0 \left(\frac{r_0}{r_j}\right)$$

$$\boxed{dW_1 = dW_2} \tag{3}$$

There is no solution of the problem if the mass distribution and the geoid are unknown. Only if the mass distribution is known all over the earth (also below the geoid) can we solve this problem. Disregarding the practical difficulties we still can consider an application of the Rudzki anomaly for a solution of the boundary value problem.

There is an interesting similarity between the Rudzki reduction and our own gravity reduction. Both methods translocate all masses outside the reference surface in new positions below this surface.

In our own method this translocation is made without changing the potential or gravity at the physical surface or in space. Rudzki changes the potential and gravity all over in space.

It is obvious that we cannot use directly our downward continuation for reducing the gravity values from the surface of the earth to the geoid (or references sphere). This will create a mass distribution that is completely contradictional to the Rudzki method. However, if we first translocate all external mass inside the geoid according to Rudzki, then we can reduce the external Rudzki anomalies to the reference sphere using the formula

$$\Delta g_R = \Delta g + G \iiint_V \frac{(\varrho_a - \varrho_p)\zeta}{d_{ij}^3} dV \tag{4}$$

and
$$\Delta g_R = \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta g_R^*}{r_{ij}^3} dS \tag{5}$$

where

Δg_R = Rudzki anomaly at the physical surface
Δg_R^* = Rudzki anomaly at the reference sphere.

The potential outside the earth can now be obtained in the following way

$$T = \frac{1}{4\pi r_j} \iint_S \Delta g_R^* \sum_{n=2}^{\infty} \left(\frac{r_0}{r_j}\right)^n \frac{2n+1}{n-1} P_n(\cos\omega) dS + G \iiint_V \frac{\varrho_a - \varrho_p}{d_{ij}} dV \tag{6}$$

It is obvious that this answer is identical with our original solution for any finite value of ϱ. For most applications $\varrho_a = 2.67$ gr/cm³ is used for the continents and $\varrho_p = 5.34$ gr/cm³ inside the reflected topography.

The most simple solution is obtained for $\varrho = 0$. For any value of $\varrho \neq 0$ we add considerable computational difficulties without changing the answer. The potential of the geoid is obtained for $r_0 = r_j$.

Then we have

$$T = \frac{1}{4\pi r_j} \iint \Delta g_R^* \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n(\cos\omega) dS \tag{7}$$

With a correct choice of ϱ we obtain the physical geoid. However, there is no method that permits a mathematical determination of the internal densities. Any guessed values of ϱ will give a new geoid and we have no unique solution for the geoid. The Rudzki anomaly is considered inferior to the Bouguer anomaly for interpolation purposes. However Tengström has found several useful applications of this reduction method.

16. Isometric anomalies

In classical geodesy different gravity anomalies have been used for a solution of the gravimetric boundary value problem. Of special interest have been the anomalies according to Bouguer and Rudzki but mostly the free air anomaly has been used. Some geodesists have preferred the more complicated isostatic anomaly. We are here going to use the free air anomaly in order to obtain an estimate of the density of disturbing mass. Later this density estimate is used for a direct solution of our gravimetric boundary value problem.

We make use of the following integral equation

$$\Delta g_j = G \iiint_V \frac{\zeta \varrho}{d_{ij}^3} dV, \tag{1}$$

where

Δg_j = free air gravity anomaly at the actual point P_j

G = constant of gravitation

ζ = height of the moving point P_i above an auxiliary plane through P_j normal to the radius vector of the earth

d_{ij} = distance between the actual point and the moving point

ϱ = density at the moving point

V = volume between the physical surface and the reference surface.

Here we consider ϱ as an unknown quantity. Any solution of this equation will locate all disturbing mass between the "physical" surface and the reference surface. However it should be noted that the height of the physical surface above the reference surface can only be given from "theorethical heights" until the disturbance potentials have been determined. This means that the so-called "telluroid" and the reference surface enclose our disturbing masses.

We note that for the free air anomaly g is determined at the surface of the earth and γ at the "theoretical height". This somewhat artificial anomaly is in our application transferred to corresponding density anomalies. After recomputation back to the boundary values we obtain our original free air anomalies.

We can consider our relation as an integral equation with respect to ϱ. It is well known that this equation will have no unique solution in the general case.

In practical applications we only know Δg_j for a set of discrete points at the surface of the earth. Any solution of the gravimetric boundary value problem has to include some estimate of the density if gravity is not known all over the boundary surface.

In order to obtain a unique solution we introduce the following conditions: (1) the density ϱ is kept constant along the vertical, (2) density variations with respect to ϱ_0 (2.67 gr/cm^3) are minimized according to the method of least squares.

With these two additional conditions we obtain a unique solution of ϱ.

This solution gives a useful prediction of gravity in unsurveyed areas for most applications.

Practical procedure

With only two-dimensional variation of ϱ_i we obtain the following system of linear equations

$$\Delta g_j = G\left[\varrho_1 \iiint_{\Delta V_1} \frac{\zeta}{d_{ij}^3}\, dV_1 + \varrho_2 \iiint_{\Delta V_2} \frac{\zeta}{d_{ij}^3}\, dV_2 + \dots\right] \tag{2}$$

Auxiliary condition $\Sigma(\varrho - \varrho_0)^2 = \text{minimum}$

In some applications it will be natural to keep the density constant over blocks that are defined by a system of grids.

Here the solution is unique when the number of c-coefficients is equal to the number of gravity anomalies.

With the two given equations we use all information available to determine gravity for the unsurveyed points between our gravity stations. It is obvious that the two methods will not give identical results for the extrapolated points. However, the two solutions are equivalent with respect to the given points.

Solution of the boundary value problem

In the present approach we have translocated all mass disturbances to the volume between the reference surface and the physical surface.

With known densities for the external mass we obtain the disturbance potential

$$T = G \iiint_V \frac{\varrho}{d_{ij}} dV. \tag{3}$$

The vertical deflection is obtained using the derivations

$$\frac{\partial T}{\partial x} = -G \iiint_V \frac{\varrho}{d_{ij}^2} \cos(x, d_{ij}) dV \tag{4}$$

$$\frac{\partial T}{\partial y} = -G \iiint_V \frac{\varrho}{d_{ij}^2} \cos(y, d_{ij}) dV \tag{5}$$

where

x = direction north (perpendicular to radius vector),
y = direction east (perpendicular to radius vector),

$$\xi = -\frac{1}{\gamma} \cdot \frac{\partial T}{\partial x} \tag{6}$$

$$\eta = -\frac{1}{\gamma} \cdot \frac{\partial T}{\partial y}. \tag{7}$$

In this approach the solution of the boundary value problem is completed with pure Newtonian operations. Among the merits of the method we can mention that the gravity prediction is closely linked with the method. The method will probably be useful also for exploration purposes. Any large density anomalies will indicate some important change of the geological parameters.

In the interesting case when gravity is given as analytical function all along the physical surface of the earth our solution will still not give the exact densities. However, our solution of the gravimetric boundary value problem will be "correct" at the surface of the earth and in space.

It should be noted that we should apply the value $\varrho_0 = 2.67$ gr/cm³ only when unreduced free air gravity anomalies are used.

It is normally accepted that Bouguer anomalies are smoother than the free air anomaly and also more convenient for interpolation purposes.

We are now using our previous density estimate to define a new anomaly Δg_{0j} which will be completely smooth when applied successfully

$$\Delta g_{0j} = \Delta g_j - G \iiint_V \frac{\zeta \varrho}{d_{ij}^3} dV. \tag{8}$$

According to equation (1) this anomaly is equal to *zero* all over the actual surface. This anomaly is called the *isometric* anomaly.

For a local study it can be convenient to eliminate the gravity *residuals* according to equation (8). Then it will be unnecessary to fulfil the iteration to very high orders and large corrections are avoided. Instead an additional correction of the disturbance potential (and its derivatives) has to be made in accordance with equation (3). For applied geophysical interpretations it will be convenient to plot corresponding "isometric density anomalies" ϱ. It should be noted that these density anomalies are not the true anomalies of the actual body. They are just the mathematical anomalies that satisfy the given boundary values (Δg_j) and the hypothesis of minimum density variation in relation to the fixed density ($\varrho_0 = 2.67$) and the chosen reference surface.

17. A solution of the boundary value problem without gravity reduction

A solution of the boundary value problem can be made using a gravity reduction from the topographical surface down to the spherical or ellipsoidal reference surface. This gravity reduction gives a geoid where all mass outside the reference surface has been translocated inside in such a way that the gravity values at the surface of the earth are kept unchanged. Our gravity reduction has been defined in equation (3.1:1).

After a strict application of this gravity reduction we can use the classical formulas for the sphere (Stokes' formula and Vening–Meinesz formula etc.). This gravity reduction works satisfactorily for all practical models but is unstable for test models with heavy point masses between the reference surface and the topographical surface. This instability only refers to the reduced gravity anomalies at the reference surface but not to the topographical surface or points in space.

The disturbance potential is a much smoother function than the gravity anomaly. A derivation of the corresponding integral equation follows:

The potential is transcribed using Poisson's integral

$$T_j = \frac{r_j^2 - r_0^2}{4\pi r_0} \iint \frac{T^*}{r_{ij}^3} dS \tag{1}$$

where

T_j = disturbance potential at the topographical surface

T^* = disturbance potential at the reference sphere.

T^* represents the potential when all mass outside the reference surface has been translocated inside in such a way that the gravity at the topographical surface is unchanged. (For the solution of this integral equation cf. Bjerhammar 1963 *c*, 1964).

According to Brun we have

$$\frac{\partial T_j}{\partial r_j}+\frac{2T_j}{r_j}=-\Delta g_j. \tag{2}$$

Here we obtain the derivative

$$\frac{\partial T_j}{\partial r_j}=\frac{1}{8\pi r_0 r_j}\iint\frac{4r_j^2 r_{ij}^3-3r_{ij}(r_j^2-r_0^2+r_{ij}^2)(r_j^2-r_0^2)}{r_{ij}^6}T^*dS. \tag{3}$$

Equations (2) and (3) give

$$\frac{1}{8\pi r_0 r_j}\iint\left[\frac{4r_j^2}{r_{ij}^3}-\frac{3(r_j^2-r_0^2)^2}{r_{ij}^5}\right]T^*dS+\frac{r_j^2-r_0^2}{8\pi r_0 r_j}\iint_S\frac{T^*}{r_{ij}^3}dS=-\Delta g_j. \tag{4}$$

In order to simplify the application of this integral equation we introduce the parameter T_0^* for the reduced potential at the reference surface corresponding to T_j.

Then we obtain

$$\frac{1}{8\pi r_0 r_j}\iint\frac{4r_j^2}{r_{ij}^3}T_0^*dS=\frac{2r_0T_0^*}{r_j^2-r_0^2} \tag{5}$$

$$-\frac{3(r_j^2-r_0^2)^2}{8\pi r_0 r_j}\iint\frac{1}{r_{ij}^5}T_0^*dS=-\frac{r_0(3r_j^2+r_0^2)}{2r_j^2(r_j^2-r_0^2)}T_0^* \tag{6}$$

$$\frac{r_j^2-r_0^2}{8\pi r_0 r_j}\iint\frac{T_0^*}{r_{ij}^3}dS=\frac{r_0T_0^*}{2r_j^2}. \tag{7}$$

Equations (4), (5), (6) and (7) give

$$\boxed{T_0^*=-\frac{r_j^2\Delta g_j}{r_0}-\frac{r_j}{8\pi r_0^2}\iint\left[\frac{5r_j^2-r_0^2}{r_{ij}^3}-\frac{3(r_j^2-r_0^2)^2}{r_{ij}^5}\right](T^*-T_0^*)\,dS} \tag{8}$$

This integral equation can be used for an iterative determination of the "space geoid" (Bjerhammar, 1964) without using any gravity reduction. (Iterated values of T^* are used under the integral sign.)

When using a preceding gravity reduction to the reference surface we obtain from equation (1) the approximate planar solution

$$\Delta g^*\simeq\Delta g_j+\frac{r_j^2-r_0^2}{4\pi r_j}\iint\frac{\Delta g_j-\Delta g}{r_{ij}^3}dS \tag{9}$$

and from equation (8) the disturbance potential for a spherical reference surface

$$\boxed{T_0^* = -r_0\,\Delta g^* - \frac{r_0}{2\pi}\iint \frac{T^* - T_0^*}{r_{ij}^3}\,dS} \tag{10}$$

(Cf. Molodensky, 1962)

This equation corresponds directly to Stokes' formula. For low order solutions we can omit the gravity reduction. If gravity anomalies were available all over the earth we prefer to use Stokes' formula or spherical harmonics for a direct determination of the disturbance potential.

A high order solution of the gravity reduction integral equations gives some numerical difficulties for models with point masses between the reference surface and the topographical surface. However, there is no need to know the actual values of Δg^* for a solution of the gravimetric boundary value problem. An iterative solution of a corresponding integral equation for T can be made using the Stokes' formula and equations (4) or (8). The disturbance potential as well as derivatives of the disturbance potential can later be computed for points at the surface of the earth and in space. Cf. equation (1). This type of solution *requires no gravity reduction* and is simple to perform.

Approximations

Equations (4) can be rewritten

$$\frac{r_j}{2\pi r_0}\iint \frac{T^*}{r_{ij}^3}\,dS + \frac{r_j^2 - r_0^2}{8\pi r_0 r_j}\iint \left[\frac{1}{r_{ij}^3} - \frac{3(r_j^2 - r_0^2)}{r_{ij}^5}\right] T^* dS = -\Delta g_j \tag{20}$$

or

$$\frac{r_j}{2\pi r_0}\iint \frac{T^*}{r_{ij}^3}\,dS + \frac{r_j^2 - r_0^2}{8\pi r_0 r_j}\iint \frac{r_{ij}^2 - 3r_j^2 + 3r_0^2}{r_{ij}^5}\, T^* dS = -\Delta g_j. \tag{21}$$

For $r_j \doteq r_0$ we have the approximation

$$\frac{r_j}{2\pi r_0}\iint \frac{T^*}{r_{ij}^3}\,dS + \frac{r_0 T_0^*}{2r_j^2} - \frac{r_0(3r_j^2 + r_0^2)}{2r_j^2(r_j^2 - r_0^2)}\,T_0^* \doteq -\Delta g_j \tag{22}$$

or

$$\frac{r_j}{2\pi r_0}\iint \frac{T^*}{r_{ij}^3}\,dS - \frac{r_0\,(r_j^2 + r_0^2)}{r_j^2\,(r_j^2 - r_0^2)}\,T_0^* \doteq -\Delta g_j \tag{23}$$

or

$$T_0^* - \frac{(r_j^2 - r_0^2)\,r_j^3}{2\pi r_0^2(r_j^2 + r_0^2)}\iint \frac{T^*}{r_{ij}^3}\,dS \doteq \Delta g_j\,\frac{r_j^2(r_j^2 - r_0^2)}{r_0(r_j^2 + r_0^2)}. \tag{24}$$

This expression is further approximated (low order solution)

$$T_0^* - \frac{h}{2\pi}\iint \frac{T^*}{r_{ij}^3}\,dS = \Delta g_j\,h \tag{25}$$

where $$r_j - r_0 = h.$$

We also note the following derivations

$$\frac{r_j^2-r_0^2}{4\pi r_j}\iint\frac{T^*}{r_{ij}^3}dS=T_0^*+h\frac{\partial T_0^*}{\partial h}+h^2\tfrac{1}{2}\frac{\partial^2 T_0^*}{\partial h^2}+\dots \tag{26}$$

The boundary value gives

$$\frac{\partial T^*}{\partial r_j}-\frac{T^*}{\gamma}\frac{\partial\gamma}{\partial n}=-\Delta g^* \tag{27}$$

$$\frac{\partial T^*}{\partial r_j}=-\Delta g^*-\frac{2T^*}{r_j}. \tag{28}$$

Thus we have after omitting non-linear coefficients of the integral equation

$$\frac{r_j^2-r_0^2}{4\pi r_0}\iint\frac{T^*}{r_{ij}^3}dS=T_0^*-h\,\Delta g_0^*-\frac{2hT^*}{r_j}$$

or

$$\boxed{T_0^*\left(1-\frac{2h}{r_j}\right)-\frac{r_j^2-r_0^2}{4\pi r_0}\iint\frac{T^*}{r_{ij}^3}dS=h\,\Delta g_0^*} \tag{29}$$

Approximative values of T^* can be found from Stokes' formula, equations (10), (25) or (29). The final iterative solution is made with equation (8).

17.1. *Vertical deflection*

With the disturbance potential given by T^* we have to find the corresponding expressions of the vertical deflection and the gravity at the topographical surface and in space.

We define the vertical deflection by the two equations

$$\xi_j=-\frac{1}{\gamma_j}\cdot\frac{\partial T_j}{\partial x}=-\frac{1}{\gamma_j}\frac{\partial T_j}{\partial\omega_x}\cdot\frac{\partial\omega_x}{\partial x} \tag{1}$$

$$\eta_j=-\frac{1}{\gamma_j}\cdot\frac{\partial T_j}{\partial y}=-\frac{1}{\gamma_j}\cdot\frac{\partial T_j}{\partial\omega_y}\cdot\frac{\partial\omega_y}{\partial y} \tag{2}$$

where

ξ_j = the vertical deflection in the meridian
η_j = the vertical deflection perpendicular to the meridian
x = rectangular coordinate positive north
y = rectangular coordinate positive east
ω = the geocentric angle between the fixed and the moving point
$\partial\omega_x$ = angular displacement of the fixed point (positive north, geocentric)
$\partial\omega_y$ = angular displacement of the fixed point (positive east, geocentric).

Furthermore we have

$$T_j=\frac{r_j^2-r_0^2}{4\pi r_0}\iint\frac{T^*}{(r_j^2+r_0^2-2r_jr_0\cos\omega)^{\frac{3}{2}}}dS. \tag{3}$$

Here we introduce the parameter U

$$U = \frac{T^*}{(r_j^2 + r_0^2 - 2r_j r_0 \cos\omega)^{\frac{3}{2}}}. \tag{4}$$

Thus we have

$$\xi_j = -\frac{1}{\gamma_j} \cdot \frac{r_j^2 - r_0^2}{4\pi r_0} \iint \frac{\partial U}{\partial \omega} \cdot \frac{\partial \omega}{\partial \omega_x} \cdot \frac{\partial \omega_x}{\partial x} dS \tag{5}$$

and

$$\eta_j = -\frac{1}{\gamma_j} \cdot \frac{r_j^2 - r_0^2}{4\pi r_0} \iint \frac{\partial U}{\partial \omega} \cdot \frac{\partial \omega}{\partial \omega_y} \cdot \frac{\partial \omega_y}{\partial y} dS \tag{6}$$

where

x = rectangular coordinate positive north
y = rectangular coordinate positive east.

After differentiation we obtain

$$\frac{\partial \omega}{\partial \omega_x} = -\cos\alpha \tag{7}$$

$$\frac{\partial \omega}{\partial \omega_y} = -\sin\alpha \tag{8}$$

α = azimuth clockwise from north

$$\frac{\partial \omega_x}{\partial x} = \frac{1}{r_j} \tag{9}$$

$$\frac{\partial \omega_y}{\partial y} = \frac{1}{r_j}. \tag{10}$$

Here α is the azimuth between the fixed and the moving point.

$$\frac{\partial U}{\partial \omega} = -3r_j r_0 \cdot \frac{T^* \sin\omega}{(r_j^2 + r_0^2 - 2r_j r_0 \cos\omega)^{\frac{5}{2}}}. \tag{11}$$

Finally we have

$$\phi = \phi_A - \xi \tag{12}$$

$$\lambda = \lambda_A - \eta \sec\phi \tag{13}$$

where

ϕ = the true latitude
ϕ_A = the observed latitude
λ = the true longitude
λ_A = the observed longitude

and

$$\boxed{\xi_j = -\frac{3(r_j^2 - r_0^2)}{\gamma_j \cdot 4\pi} \iint \frac{T^* \cos\alpha \sin\omega}{r_{ij}^5} dS} \tag{14}$$

$$\boxed{\eta_j = -\frac{3(r_j^2 - r_0^2)}{\gamma_j \cdot 4\pi} \iint \frac{T^* \sin\alpha \sin\omega}{r_{ij}^5} dS} \tag{15}$$

These integrals are somewhat simpler than corresponding derivations for Δg^*. Both integrals have a discontinuity for $r_{ij} = 0$. With the reference surface below the topographical surface all discontinuities are avoided.

A successful application of these formulas for the vertical deflection will require a dense network of T^* values around the actual point.

In the application of the formulas for the vertical deflections we have to note that the deflections in most cases are related to a reference ellipsoid and not a reference sphere. This difference can be taken care of by the gravity anomaly.

In the application of this technique we note that values of T^* can always be found that satisfy the Δg-values of all discrete measurements.

18. Reciprocal solution

In the practical application of the gravity reduction according to (3.1:2) we can expect rather small corrections when the side of the surface element and the distance to the reference sphere are of the same size. When using small surface elements extremely large corrections are hard to avoid for large distances between the physical surface and the reference surface. This problem is of special importance around the actual point where the vertical deflection has to be determined. In critical models corrections of 1000 milligal or more can be found. Then the integration will be hard to execute without special precaution. All these difficulties can be eliminated by using a "resolvent" procedure in the following way.

The vertical deflection is defined by the formula (cf. 12:7)

$$\xi = -\frac{t}{4\pi \gamma_j r_j^2} \iint \Delta g^* \sin\omega \cos\alpha \, k' dS. \tag{1}$$

This relation is transcribed using matrix notations

$$\underset{11}{\xi} = \underset{1r}{\mathbf{f}} \, \underset{r1}{\Delta \mathbf{g}^*} = \sum_{i=1}^{r} f_i \Delta g_i^* \tag{2}$$

where $$f_i = -\frac{t}{4\pi \gamma_j r_j^2} \iint_{\Delta S_i} \sin\omega \cos\alpha \, k' dS. \tag{3}$$

Furthermore we have from (6:2)

$$\underset{r}{\Delta \mathbf{g}^*} = \underset{rr}{\mathbf{K}^{-1}} \underset{r1}{\Delta \mathbf{g}}. \tag{4}$$

Equations (2) and (4) give

$$\underset{11}{\xi} = \underset{1r}{\mathbf{f}} \underset{rr}{\mathbf{K}^{-1}} \underset{r1}{\Delta \mathbf{g}}. \tag{5}$$

We introduce a new parameter **u**

$$\underset{1r}{\mathbf{u}} = \underset{1r}{\mathbf{f}} \underset{rr}{\mathbf{K}^{-1}}.$$

For a non-singular **K** we obtain

$$\underset{1r}{\mathbf{u}} \underset{rr}{\mathbf{K}} = \underset{1r}{\mathbf{f}} \tag{6}$$

or after transposition

$$\boxed{\underset{rr}{\mathbf{K}^*} \underset{r1}{\mathbf{u}^*} = \underset{r1}{\mathbf{f}^*}} \tag{7}$$

Then we have the final solution

$$\boxed{\underset{11}{\xi} = \underset{1r}{\mathbf{u}} \underset{r1}{\Delta \mathbf{g}}} \tag{8}$$

In this approach we have avoided all gravity reduction. This procedure has some important *advantages.*

1. The solution of equation (7) can be made without knowledge of the gravity values.

2. Incorrect measurements will not influence the final solution of our system of equations (7).

3. New measurements can be added without resolving the whole system of equations. It is only necessary to maintain the original surface elements.

4. The new parameter u can be expected to be much "smoother" than Δg^* for small surface elements.

This approach has earlier been described somewhat modified (Bjerhammar, 1959). It has a practical disadvantage because it is necessary to make independent solutions for T, ξ and η. Therefore the most practical approach will be to use the following procedure:

1. The local zone ($5' \times 5'$) is computed with *reciprocal* methods.
2. Outside the local zone *gravity reduction* is made for an *intermediate* zone ($5° \times 5°$).
3. Outside the intermediate zone we put $\Delta g^* = \Delta g$.

The matrix solution presented above is considered more appropriate than the direct approach from an integral equation

$$-\frac{\sin \omega \cos \alpha k'}{4\pi \gamma_j r_j} = \frac{1}{4\pi} \iint \frac{(r_i^2 - r_0^2) u_i}{r_{ji}^3 r_i} dS \tag{9}$$

with

$\xi = \int\int u \Delta g dS$

r_{ft} = distance between the moving point (P_t) at the surface of the earth and the fixed point (P_f) at the reference surface

r_t = geocentric distance of the moving point

S = reference surface.

In a similar way T or η can be computed.

19. Molodensky's method

Molodensky makes use of a single layer potential

$$T = \int\int_S \frac{\phi}{r} dS \tag{1}$$

where

T = the disturbing potential

ϕ = the density of the single layer

r = the distance between the moving point and the fixed point at the physical surface

S = surface of the physical earth.

The derivative of this potential in the direction of the surface normal n, is the wellknown expression

$$\frac{\partial T}{\partial n} = -\int\int_S \frac{\phi}{r^2} \frac{\partial r}{\partial n} dS - 2\pi\phi_0 \quad \text{(Cf. Goursat.)} \tag{2}$$

The corresponding derivative in the direction of the surface tangent, t, is given by

$$\frac{\partial T}{\partial t} = -\int\int_S \frac{\phi}{r^2} \frac{\partial r}{\partial t} dS + O. \tag{3}$$

Therefore, the derivative in an arbitrary direction, m, is given by

$$\frac{\partial T}{\partial m} = \frac{\partial T}{\partial n} \cos(n, m) + \frac{\partial T}{\partial t} \cos(m, t) \tag{4}$$

or

$$\frac{\partial T}{\partial m} = -\int\int_S \frac{\phi}{r^2} \left[\frac{\partial r}{\partial n} \cos(n, m) + \frac{\partial r}{\partial t} \cos(m, t)\right] ds - 2\pi\phi_0 \cos(n, m) \tag{5}$$

where

$$\frac{\partial r}{\partial n} = \cos(r, n) \tag{6}$$

$$\frac{\partial r}{\partial t} = \cos(r, t). \tag{7}$$

Thus we have

$$\frac{\partial T}{\partial m} = -\iint_S \frac{\phi}{r^2}\cos(r, m)\, dS - 2\pi\phi_0 \cos(n, m). \tag{8}$$

Finally we obtain from the boundary condition

$$\Delta g = 2\pi\phi_0 \cos(n, m) + \iint_S \frac{\phi}{r^2}\cos(r, m)\, dS + \frac{1}{\gamma}\left(\frac{\partial \gamma}{\partial m}\right)\iint_S \frac{\phi}{r}\, dS \tag{9}$$

or for the sphere

$$\Delta g = 2\pi\phi_0 - \frac{3}{2r_0}\iint \frac{\phi}{r}\, dS \tag{10}$$

where

r_0 = radius of the earth.

Here we check the iterative convergence in accordance with chapter 6.

We have for a sphere and an infinitesimal surface element

$$\sum k_{ij} = \frac{3}{2r_0}\iint \frac{1}{r}\, dS = 6\pi \tag{11}$$

$$k_{ii} = 2\pi < \sum k_{ij}. \tag{12}$$

With a surface element of radius 5° we obtain

$$\sum k_{ij} = 6\pi - 0{,}5 \tag{13}$$

$$k_{ii} = 2\pi - 0{,}5 < \sum k_{ij}. \tag{14}$$

This means that the iterative approach is not fulfilling the "hard" convergence condition (6:14) in a stationary process. Molodensky therefore used a non-stationary process where the first iterative step is given by the relation

$$T = \frac{1}{4\pi r_j}\iint (\Delta g + G_1)\sum_{n=2}^{\infty}\frac{2n+1}{n-1}P_n(\cos\omega)\, dS \tag{15}$$

where

$$G_1 = \frac{1}{2\pi}\iint \frac{h - h_j}{r^3}\Delta g\, dS \tag{16}$$

r = distance between the fixed and the moving point at the reference sphere

$$\xi = \frac{1}{4\pi\gamma_0 r_j^2}\iint (\Delta g + G_1)k' \cos\alpha\, dS - \frac{\Delta g}{\gamma_0}\tan(n, m), \tag{17}$$

where

k' = cf. equation (12:7)

(n, m) = angle between the surface normals of the physical surface and the reference surface

α = azimuth.

We note that the *homogeneous* equation corresponding to equation (10)

$$O = 2\pi\phi_0 - \frac{3}{2r_0}\iint \frac{\phi}{r}\,dS \tag{18}$$

has infinite solutions. In the same way we can expect that we have infinite solutions to the homogeneous equation formed from equation (9). Therefore any iterative solution will be *critical. There will only be solutions for specific data sets which always give infinite answers.*

20. Appendix

A complete program has been developed in Fortran IV (CD 3600 and CD 6600). This program accepts discrete data in a random distribution as well as grid data and makes a gravity reduction for grid data in four iterative steps. The discrete data receive one iterative correction in the complete program but only the iterative solution for grid values is included here.

Test model

An iterative solution is applied for a section in the European West-Alps. Heights and free air anomalies that have been used are shown in graphs together with the corresponding results from method 12. Alltogether 33 profiles of constant latitude are presented. Furthermore 5 selected profiles with the corrections of each iterative step are displayed in separate graphs. Finally the residual variances of the iterative steps are presented. All graphs are referred to method 12.

Methods employed for the graphs

Iterative method: Method 12, Chapter 5.2.

Computations: All computations have been made by B.-G. Reit.

Surface elements

The following surface elements have been used in the test model:

1. A *primary* "rectangular" zone ($5° \times 5°$) is subdivided in 60×60 smaller surface elements ($5' \times 5'$) which all receive gravity reduction.

2. Outside the primary zone a *buffer* zone is established. The measurement of the buffer zone will remain fixed during the gravity reduction and only contribute to the reduced values inside the primary zone. (A buffer zone of $15° \times 15°$ has been used in some applications.)

3. All values outside the buffer zone will contribute to the computation of T and the vertical deflection inside the zone.

Test area for gravity reduction (*West Alps*)

— 18 — — 5 — Sub Region no 2 Buffer zone $\Delta g \equiv \Delta g^*$ Number of elements $= 5 \times 18 = 90$ Size of elements $\simeq 20' \times 30'$		
— 8 — Sub region no 4 Buffer zone $\Delta g \equiv \Delta g^*$ Number of elements $= 8 \times 5 = 40$ Size of elements $\simeq 20' \times 30'$ — 5 —	— 33 — Sub region no 1 Principal zone $\Delta g \neq \Delta g^*$ Size of elements $= 5' \times 5'$ Number of elements $= 33 \times 50 = 1650$ — 50 —	Sub region no 5 Buffer zone $\Delta g \equiv \Delta g^*$ Number of elements $= 8 \times 5 = 40$ Size of elements $\simeq 20' \times 30'$ — 5 —
— 5 — Sub region nr 3 Buffer zone $\Delta g \equiv \Delta g^*$. Number of elements $= 5 \times 18 = 90$ Size of elements $\simeq 20' \times 30'$ — 18 —		

All buffer zones are used for integrations in the principal zones when computing the gravity reduction.

Comments

In the present study we have made an analysis of the boundary value problem using discrete and semi-analytical (or semi-discrete) methods. It should be pointed out that there is some confusion in the geodetic literature concerning the convergence when gravity is reduced from the physical surface of the earth down to an internal spherical reference surface. Studies by Molodensky and Moritz in 1960 and 1961 indicated that any such attempt was going to fail because there was no general way of solving the reduction problem, when mass was situated between the physical surface and the reference sphere. However, these studies were restricted to a *fully analytical* approach and the results are not valid for a discrete or semi-analytical solution. Using generalized matrix calculus we find the general solution for our gravity reduction. Perhaps this type of solution seems artificial in comparison with the fully analytical solution. If we carefully study the boundary value problem, we find that gravity has never been presented to the geodesists in analytical functions. In fact, terrestrial gravity data are only available at discrete points or as "grid values". Consequently, an analytical solution is representing an academic problem without any immediate geodetic application.

For the numerical solution of large systems with several thousand unknowns, we have to use *iterative* methods. Our first approach with iterative methods was described in the following way. (Cf. Bjerhammar, 1962 and method 11 above):

"In a practical application of the method one can use equation (7) as a method for the determination of iterative values of Δg^*. The values of Δg^* are just inserted as approximations for the first check. If the Δg_j now computed shows any difference with respect to the known Δg_j, the procedure is repeated with the discrepancy used

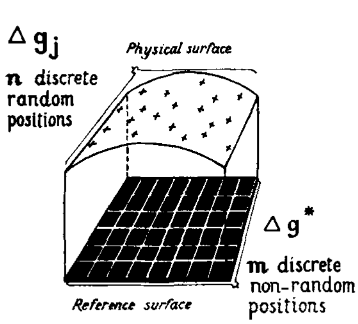


Fig. 12

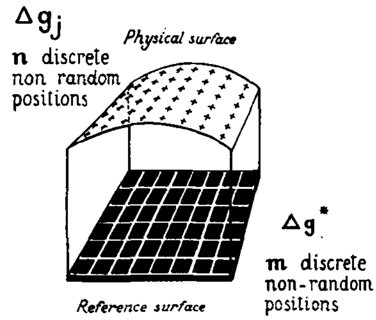


Fig. 13

Fig. 12. Fully discrete approach with random discrete boundary values. *Case 1.* $n = m$: Unique solution for rank $= m$; *Case 2.* $n > m$: degrees of freedom $\leqslant n - m$ for rank $= m$; *Case 3.* $n < m$: degenerations $\geqslant m - n$. Infinite exact solutions. (Convergence: poor.)

Fig. 13. Fully discrete approach with non-random discrete boundary values. *Case 1.* $n = m$: (Cf. fig.) Unique solution for rank $= m$; *Case 2.* $n > m$: degrees of freedom $\leqslant n - m$ for rank $= m$; *Case 3.* $n < m$: degenerations $\geqslant m - n$. Infinite exact solutions. (Convergence: good.)

as a correction." This method still seems to be the simplest one and has a good convergence. (Method 11 above.)

In our final computer program we have used a refined method where the iterative process is *restricted* to the *residuals* instead of the whole reduced gravity anomaly. (Cf. Bjerhammar, 1965 and method 12 above.) This improvement is of great importance when a buffer zone is used for the integrations. *Then integration is only*

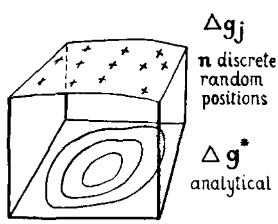


Fig. 14

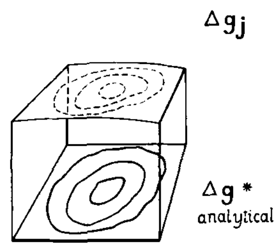
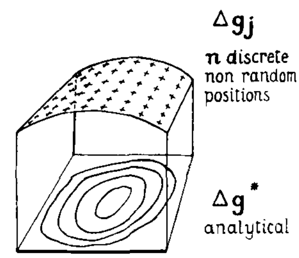


Fig. 15



Fig, 16

Fig. 14. Semi analytical approach with random discrete boundary values. A least squares estimate gives. *Case 1:* Degree of freedom $\leqslant n - m$; where $m =$ rank of the transformation matrix. (Convergence: poor.)

Fig. 15. Fully analytical approach. *Case 1:* Unique solution only available for specific models. (Spherical harmonics etc.) *Case 2:* Generalized least squares solution defined thus

$$\delta(\Delta g) = (\Delta g) - \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{f(\phi, \lambda)}{r_{ij}^3} dS$$

where $\iint p[\delta(\Delta g)]^2 dS = \min$; $p =$ weight; $f(\phi, \lambda) =$ unknown, analytical function.

Fig. 16. Semi analytical approach with non-random discrete boundary values. A least squares estimate gives. *Case 1:* Degrees of freedom $= n - m$; where $m =$ rank of the transformation matrix. (Convergence: fair.)

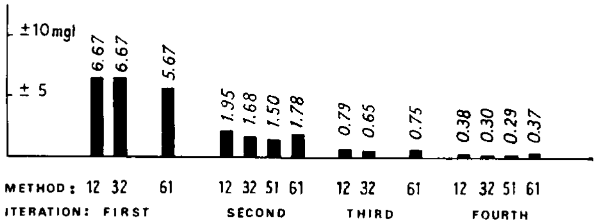


Fig. 17. RMS residuals from West-Alp test with four iterative steps and 1600 unknowns.

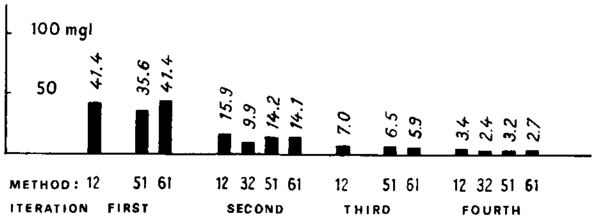


Fig. 18. Maximum corrections from West-Alp test with four iterative steps and 1600 unknowns.

needed in the first iterative step of the buffer zone! Therefore "method 12" was selected for the final solution. Method 13 is identical with method 11 but the integration is made over gravity differences and therefore harder truncation of the integration area can be used in a smooth topography. However integration over the buffer zone is still needed for each iterative step. In method 14 integration is made over residual differences and therefore this method can be somewhat more favourable than method 12 with respect to the truncation problem. This difference is probably of no practical importance and we have considered method 12 superior with respect to computer programming.

A comparison between the residual variance after each iterative step for four selected methods is displayed below. (Mgal R.M.S.)

Iteration	Method: 12	Method: 32	Method: 51	Method: 61
1	±6.67	±6.67	—	±5.67
2	±1.95	±1.68	±1.50	±1.78
3	±0.79	±0.65	—	±0.75
4	±0.38	±0.30	±0.29	±0.37

Method: 51 $c_{11} \neq 0$, $c_{ij} = 0$ for $i \neq 1$ and $j \neq 1$.

(Without using least squares technique.)

Method: 61 $c_{11} \neq 0$, $c_{ij} = 0$ for $i \neq 1$ and $j \neq 1$.

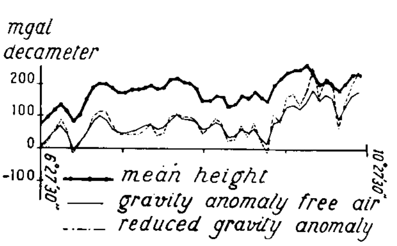


Fig. 19. Profile from the West-Alp test. *Longitude:* 6°27′30″–10°27′30″; *Latitude:* 46°22′30″.

The best performance was obtained for the methods 32 and 51 wich are more complex than the two remaining methods. The test area was taken from the West Alps and still the differences are small.

Our method 32 is approximately 25% better than method 12 after four iterative steps. With larger height differences we could expect still more pronounced differences. Method 32 requires somewhat more core memory and therefore we decided to use method 12 as our final computer program. A conversion to method 32 is simple and has been contemplated.

There is a formal relationship between the different methods that can be presented in the following way:

Integration	Methods
Reduced gravity anomaly	11, 21, 31
Gravity anomaly residual	12, 22, 32
Reduced gravity anomaly difference	13, 23, 33
Gravity anomaly residual difference	14, 24

Methods 11, 12 and 13 use the residuals as corrections.

Methods 21, 22 and 23 reduce the residual from the physical surface to the reference surface in a simple way in order to obtain the corrections.

Methods 31, 32 and 33 use a more refined reduction of the residuals.

All methods 51–61 include an application of the method of least squares. With only one unknown we can exclude this section of the method. These methods have not yet been fully explored, and we note that almost unlimited combinations of similar types can be made.

The size of the surface elements has been given some considerations. For the West Alps test we have made use of a grid system of $5' \times 5'$ surface elements. With decreasing size of the surface elements we can expect a slower convergence. Discrete gravity data in a completely random distribution can give serious difficulties for iterative methods and should be avoided. A successful approach will normally require gravity data in a grid system. Such discrete grid data can of course be computed from grid data in random distribution.

Finally we note that there is a free choice of the radius of the reference sphere. For example one can use a reference sphere that intersects the actual point of interest. For a vertical deflection this method seems justified. However with this method we obtain different reduced gravity anomalies "everywhere on the earth" for each new fixed point. This will cause considerable difficulties in a practical application and the method has therefore been rejected.

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О ГРАНИЧНОЙ ЗАДАЧЕ ФИЗИЧЕСКОЙ ГЕОДЕЗИИ

Даётся решение основной граничной задачи физической геодезии с использованием сферических гармоник в глобальном подходе и редукцией силы тяжести с помощью интегрального уравнения в локальных исследованиях, причём всё рассматривается для совместной системы уравнений. Эта процедура должна быть единственной. В обычных приложениях сила тяжести приводится к отсчётной сфере (эллипсоиду), которая совпадает со средним уровнем моря. Для вычисления вертикального отклонения производится окончательная редукция силы тяжести к отсчётной поверхности в реальной точке. Вводится «изометрическая» аномалия силы тяжести.