On application of the statistical method for determination of atmospheric temperature profiles from satellites

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ABSTRACT

Natural orthogonal functions are applied to the problem of determining the vertical temperature profile from measurements of the Earth's outgoing radiation in the absorption band of CO_2. There are examples of solving this problem. Optimal systems of orthogonal functions agreed with statistics of errors of the measurement of the outgoing radiation and with the kernel of the integral equation. Dimension of subspace of solutions having a real sense at a given accuracy of measurements and a number of allowed temperature profiles are found.

1. Introduction

Kaplan (1959) put forward the idea of obtaining the vertical temperature distribution of the atmosphere $T(p)$ by measuring the outgoing radiation I_ν at various wave numbers of the 15 micron CO_2 band from satellites. This problem is reduced to the numerical solution of a Fredholm integral equation of the first type

$$I_\nu = B_\nu[T(p_0)]\,\tau_\nu(p_0) - \int_{\tau(0)}^{\tau(p0)} B_\nu[T(p)]\,d\tau_\nu(p), \quad (1)$$

where $B_\nu[T(p)]$ is the Planck function, $\tau_\nu(p)$ is the transmission function dependent on pressure p, p_0 is the surface pressure. In equation (1) I_ν is assumed to have been obtained from a local vertical and the surface is assumed to have been black. The different aspects of the indirect soundings of atmospheric temperature profiles from satellites have been recently reviewed by Wark & Fleming (1966) (see also Malkevich, 1964). One of the important questions is that of mathematical methods of stabilizing the solution of the inverse problem. It is well known that the inverse problems which are being considered are incorrect, i.e. that small errors in the measured quantities I_ν generally give large oscillating errors in the solutions $T(p)$ of the fundamental equation (1). In order to filter high frequency oscillations Malkevich & Tatarsky (1965) (see also Malkevich, 1964) suggested that the solutions be expressed by empirical orthogonal functions $\varphi_k(p)$ obtained from statistical investigations of temperature profiles which are considered as random functions of p. Namely $\varphi_k(p)$ are the eigenfunctions of the autocorrelation function for the deviation of the temperature profiles from the climatological one (if the temperature is given at discrete levels the eigen vectors of autocorrelation matrix should be used). The theory of the empirical orthogonal functions allowing for optimal parametrization of arbitrary random functions has been given by Oboukhov (1960). The orthogonal functions so determined allow the best approximation of any temperature profile by a minimum number φ_k as follows

$$T(p) = \bar{T}(p) + \sum_{k=1}^{n} c_k\,\varphi_k(p). \quad (2)$$

Here $\bar{T}(p)$ is the climatological profile for the given region and season. Examples of the empirical orthogonal vectors are given in Fig. 1.

2. Optimal parametrization of solution

Putting (2) into Eq. (1) and assuming that

$$B_\nu[T(p)] = B_\nu[\bar{T}(p)] + \frac{\partial B_\nu[\bar{T}(p)]}{\partial T} \sum_{k=1}^{n} c_k\,\varphi_k(p), \quad (3)$$

one can obtain the equations for determination of the unknown coefficients c_k. From (1) and (3)

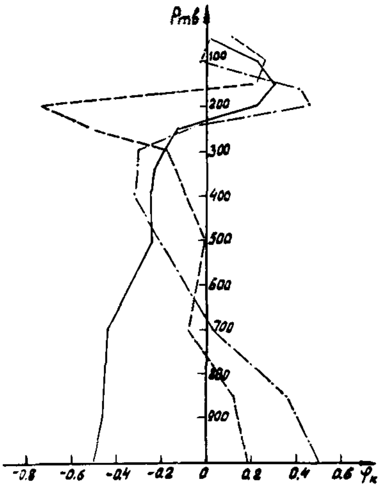

Fig. 1. The empirical orthogonal vectors computed as eigenvectors of autocorrelation matrix for temperature profiles according to Koprova & Malkevich (1965). Solid line, $\varphi_1(p)$; dashed line, $\varphi_2(p)$; dotted line, $\varphi_3(p)$.

$$\sum_{k=1}^{n} D_{\nu k} c_k = f_\nu \quad (\nu = 1, 2, \ldots, N \geqslant n), \tag{4}$$

where

$$D_{\nu k} = \frac{\partial B_\nu[\bar{T}(p_0)]}{\partial T} \varphi_e(p_0)\, \tau_\nu(p_0) - \int_{\tau(0)}^{\tau(p0)} \frac{\partial B_\nu[\bar{T}(p)]}{\partial T} \varphi_k(p)\, d\tau_\nu(p), \tag{5}$$

$$f_\nu = I_\nu - B_\nu[\bar{T})p_0)]\, \tau_\nu(p_0) + \int_{\tau(0)}^{\tau(p0)} B_\nu[\bar{T}(p)]\, d\tau_\nu(p). \tag{6}$$

It is clear that the form (2) permits a linearization of the integral equation (1) without any assumption of the spectral interval of the band under consideration (in contradistiction to the method of Yamamoto, 1961). To test the method of optimal parametrization the radiances I_ν in eight spectral intervals were computed for a set of temperature profiles $T(p_i)$ and for a climatological profile $\bar{T}(p_i)$ obtained by averaging $T(p_i)$ at every level. Then considering I_ν as the known data of measurements and using the empirical orthogonal vectors, the profiles were obtained from equations (4)–(6)

$$T'(p_i) = T(p_i) - \bar{T}(p_i). \tag{7}$$

An example of the solution is given in Fig. 2. For comparison the solution of the equation (1) obtained by Yamamoto's polynomial approximation of the unknown function is represented in Fig. 2. It is seen also in Fig. 2 that the empirical orthogonal vectors permit a determination of the solution to equation (1) with the same error as the direct approximation of the known vector $T'(p_i)$ by three empirical orthogonal eigenvectors $\varphi_k(p_i)$ ($i = 1, 2, \ldots, 10$ are numbers of the levels where the profile $T'(p_i)$ is given). The success in this approximation has a simple explanation: the first empirical vector has the same vertical structure as the typical profile $T'(p_i)$. This is clearly seen from Figs. 1 and 2. The described method was also used for estimating the vertical distribution of temperature by means of ballon flight data of I_ν in 5 cm^{-1} spectral intervals centered at 677.5, 691, 697, 703 and 709 cm^{-1}. Values of the radiance in the 899 cm^{-1} window are also received during these flights. Hilleary, Wark & James (1965) pointed out that the data were accurate—better than 1 per cent of the signals.[1] All the spectral intervals are located near the center of 15 μ CO_2 band where absorption is very strong. It allows us to linearize the integral equation in the way proposed by Yamamoto (1961), however, in this case the measurements have not the reliable information about the temperature distribution in the lower tropo-

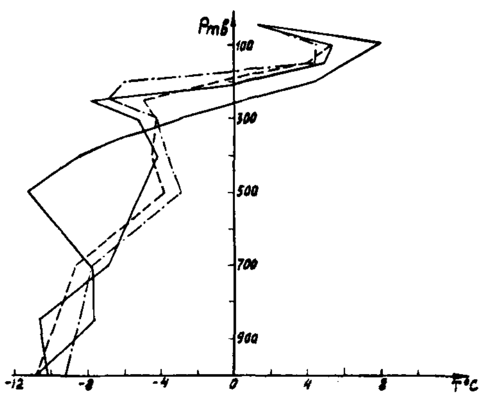

Fig. 2. The deviation of the temperature profile from the average one obtained by using the method of optimal approximation for solution of the equation (1). The real profile (thin solid line), the straight optimal approximation of the known vectors $T'(p_i)$ (dashed line), the optimal solution of (1) (dotted line), and the solution by Yamamoto's method (thick solid line).

sphere. Therefore the large variations of the real temperature profile in the 1000–700 mb layer will change the Earth radiance in the mentioned intervals of the 15 μ band less than 1 per cent. Generally it is impossible to determine the temperature of the lower troposphere from measurements only near the center of the band and the additional information is required. Wark *et al.* (1965) probably used the window radiances 899 cm^{-1} and the favourable circumstance that the slope of the temperature profile in the troposphere is nearly constant. In the statistical method the correlation between the temperature deviations at different levels is used as an additional information. The temperature profile estimated by the statistical method is given in Fig. 3. The real profile and the results of computations derived by Wark *et al.* (1965) are also shown here. Since the functions $\varphi_k(p)$ had previously been computed only for the troposphere we could not obtain the temperature for high levels. We had no climatological profile for the region of ballon flight and, therefore, we used $\bar{T}(p)$ for a different region. It happened so that the real temperature deviations in the lower troposphere close to zero do not correlate with the large and negative deviations in the stratosphere. Nevertheless the statistical method gives a satisfactory solution.

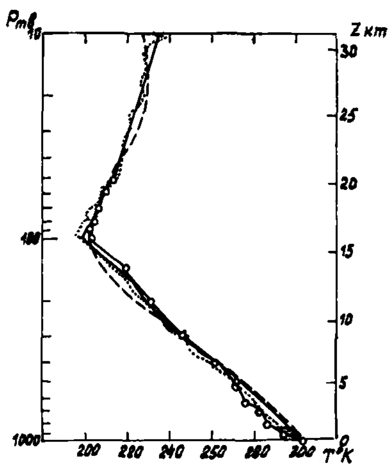

Fig. 3. Temperature profiles estimated from ballon radiation data in no clouds condition by different methods: segment series solution (solid line) and trigonometric series solution (dashed line) obtained by Wark *et al.* (1965); empirical functions solution (circle line), and real temperature profile from radiosonde (dotted line).

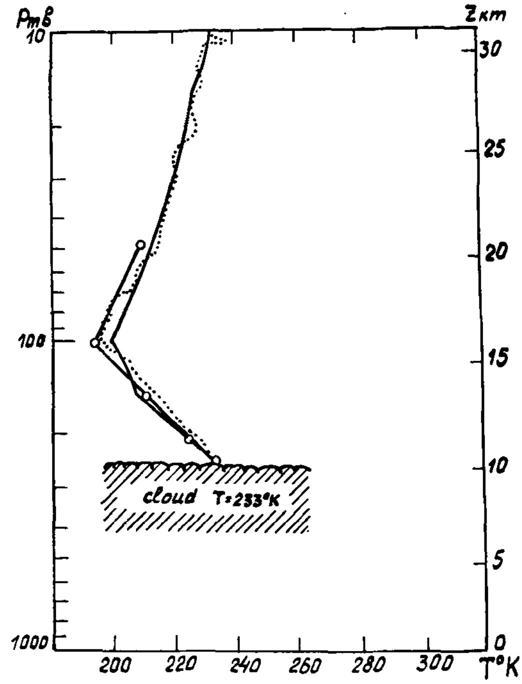

Fig. 4. Temperature profiles estimated from radiation data in cloudy conditions (the symbols are the same as in Fig. 3).

The same result is obtained for the cloud case as is shown in Fig. 4.

It is clear, that two examples are not sufficient to show an advantage or disadvantage of one or the other method of solving the inverse problem. Thus, we do not draw any conclusions with regard to the practical application of the statistical method. Moreover, the very important role of the transmission function which usually is not exactly known in the real measurements should be investigated. For example the test of the transmission used above by comparing the measured and computed values of I_ν showed the difference of 4–6% for intervals 577–697 cm^{-1} and 1% for 703 and 709 cm^{-1}. These differences can result in a significant error in the temperature profile obtained from the inverse problem. For practical application it is necessary to consider the influence of errors of the experimental data on the solution of the inverse problem. This question is very closely connected with the problem of regularizing the integral equation (1) considered by Tykhonov (1963) and Twomey (1963).

Stabilizing the solution

The optimal parametrization of the temperature profiles by empirical orthogonal function permits a determination of the minimum numbers of parameters describing an unknown profile with a given accuracy of approximation and thus to minimize the instability of the system (4). However, the optimal procedure gives no answer of how many real parameters c_k one can deduce from the system (4) if the errors of measured data I_ν are given. To make this question clear it is necessary to consider the structure of the useful information obtained in the real experiment. It is necessary to take into account the statistical properties of random errors of measurements. Kozlov (1966) suggested that the general method be used for the analysis of the similar problems based on geometrical properties of Fisher information.

Let us consider the space of solutions R_n which is parametrized by the coefficients $c_1, \ldots, c_n$ of the expansion (1). The number n is determined only by a given accuracy of approximation and thus one can assume that all unknown temperature profiles belong to R_n. It is clear, however, that R_n contains functions corresponding to the real profiles as well as physically meaningless functions. The part of R_n which corresponds to real profiles will be denoted by D. The distance between the two functions $T_1(p)$ and $T_2(p)$ will be determined from the following expression

$$\varrho(T_1, T_2) = \left\{ \sum_{i,k=1}^{n} g_{ik}[c_i^{(1)} - c_i^{(2)}]\,[c_k^{(1)} - c_k^{(2)}] \right\}^{1/2}, \quad (8)$$

where g_{ik} are the components of Fisher's information matrix corresponding to the given method of measurements; and $c_i^{(1)}$, $c_i^{(2)}$ are the coefficients of the expansion of the functions $T_1(p)$ and $T_2(p)$ accordingly. If one assumes that I_ν at chosen spectral intervals are measured independently and the errors of measurements are normally distributed with the dispersion σ^2 then the quantities g_{ik} are connected with the coefficients $D_{\nu k}$ of the system (4) by the following expression

$$g_{ik} = \frac{1}{\sigma^2} \sum_{\nu=1}^{N} D_{\nu i} D_{\nu k} \quad (i, k = 1, 2, \ldots, n). \quad (9)$$

The distance (8) is in agreement with the kernel of the fundamental equation (2) and with the errors of measurements. It gives the limited accuracy of the determination of the parameters of the temperature profile from experimental data. The distance (8) in some direction is expressed in terms of a mean square error of deducing the coordinate along this direction. The case $\varrho(T_1, T_2)$ means that two profiles $T_1(p)$ and $T_2(p)$ are not distinguishable in the error limit. Kozlov (1966) has shown that this is connected with the following fact: in some directions the projection of the distance between these functions to the directions is less than unity. It means that the error of determining unknown functions in these directions is larger than any possible change of a corresponding coordinate. In other words the experimental data have no useful information on the projection of the unknown function $T(p)$ to these directions. Therefore one must find that part of the space D where any $\varrho > 1$. The method for constructing this part is obtained from the geometrical sense of the problem; thus, vectors

$$u_1 = (u_1^{(1)}, \ldots, u_1^{(n)}), \ldots, \qquad u_2 = (u_r^{(1)}, \ldots, u_r^{(n)}) \quad (r \leqslant n) \quad (10)$$

should be found in space R_n; for each of these the distance $\varrho > 1$. In a very important case when D is given by

$$d_{ik} = \overline{c_i c_k}. \quad (11)$$

$u_1, u_2, \ldots, u_r$ are the eigenvectors of the following form

$$\sum_{i,k=1}^{n} g_{ik} d_{il} u_k = \lambda u_l \quad (l = 1, \ldots, n) \quad (12)$$

and the corresponding eigenvalues $\lambda_1, \ldots, \lambda_r$ are larger than unity.

In (11) the bar denotes averaging over all realisations of random functions $T(p)$. It follows from Oboukhov's theory (1960) that d_{ik} in this case is a diagonal matrix and the elements of its principle diagonal are equal to the eigenvalues of an autocorrelation matrix.

Using the vectors $u_1, \ldots, u_r$ one can find a new basis $\psi_m(p)$ in the space R_n which allows expansion of every profile $T(p)$ in the following way

$$T(p) = \bar{T}(p) + \sum_{m=1}^{r} b_m \Psi_m(p). \quad (13)$$

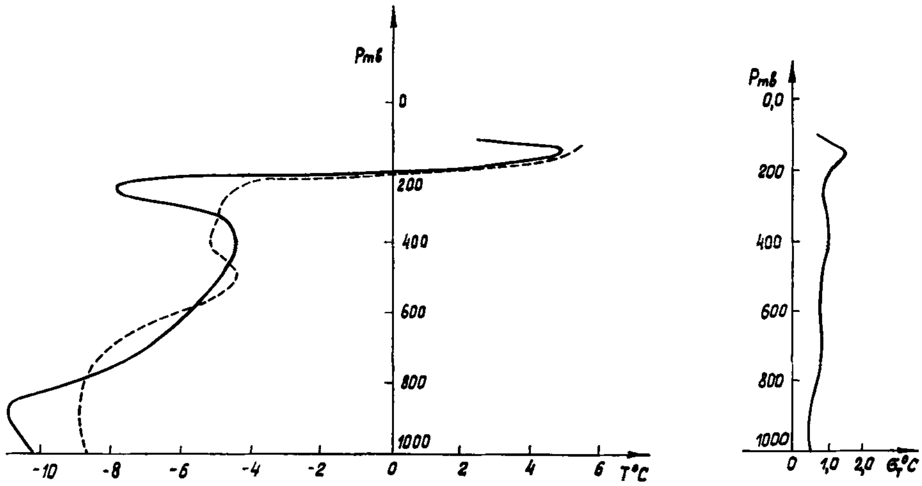

Fig. 5. The solution of equation (1) using the statistical structure of errors of measurements and the properties of the kernel. The solid curve—the actual profile (the same as in Fig. 2), the dashed curve—the deduced profile, and the right curve—the means square error due to the random errors of measurements.

The functions $\Psi_m(p)$ are determined from the following expression

$$\Psi_m(p) = \frac{\sigma \sum_{i=1}^{m} u_m^{(i)} \varphi_i(p)}{\left[\sum_{i,k=1}^{n} \sum_{\nu=1}^{N} D_{\nu k} D_{\nu i} u_m^{(k)} u_m^{(i)}\right]^{1/2}} \quad (m = 1, \ldots, r) \qquad (14)$$

and the coeffeicients b_m are computed from the experimental data

$$b_m = \sum_{\nu=1}^{N} f_\nu B_{\nu m}, \qquad (15)$$

where

$$B_{\nu m} = \frac{1}{\sigma} \frac{\sum_{k=1}^{n} D_{\nu k} u_m^{(k)}}{\left[\sum_{i,k=1}^{n} \sum_{\nu=1}^{N} D_{\nu k} D_{\nu i} u_m^{(k)} u_m^{(i)}\right]^{1/2}}. \qquad (16)$$

Thus equations (11)–(16) makes it possible to determine the temperature profiles from spectral radiance data with a maximum use of the information that is actually contained in the measurements. The parametrization (13) is optimal in an information sense: exclusion of any term from the expansion (13) leads to a loss of useful information, and inclusion of an extra term leads to unrealistic profiles.

The eigenvalues $\lambda_1, \ldots, \lambda_r$ allow us to estimate the value of the information obtained by this method of measurement. The order of the quantity

$$V = \left[\prod_{m=1}\right] \qquad (17)$$

is equal to the number of distinguishable temperature profiles when the radiance is measured with a given accuracy. The example of value of information has been computed with the assumption that the mean square error of measurements is equal to $\sigma = 0.4°$ (or about 1% of I_ν). The eigenvalues $\lambda_1, \ldots, \lambda_r$ were computed for eight spectral intervals of the CO_2 band 15 μ used by Malkevich & Tatarsky (1965). The temperature profiles were expanded by 5 empirical orthogonal functions $\varphi_n(p)$. In this case

$$\lambda_1 = 488,\ \lambda_2 = 8.1,\ \lambda_3 = 2.5,\ \lambda_4 + \lambda_5 < 1,\ V = 95.$$

These results mean that only the three first eigenvalues are larger than unity. Therefore the optimal expansion (13) contains only 3 terms and for a given error of measurements about 100 profiles can be distinguishable. Fig. 5 represents an example of determination of the temperature profile from the expressions (13)–(16) and also the mean square error of the solution $T(p)$

$$\sigma_T(p) = \left[\sum_{m=1}^{3} \Psi_m^2(p)\right]^{1/2} \qquad (18)$$

due to random errors in the measurements of I_ν.

It is seen from Fig. 5 that $\sigma_T(p)$ increases to about 3° near the tropopause.

Conclusions

The principal conclusion follows from the discussion above: the procedure described permits a determination of stable temperature profiles with minimum information on the Earth radiance. It is due to the following reasons:

1. The empirical orthogonal functions used in the statistical method describe the inherent properties of the unknown functions and therefore allow an optimal solution of the fundamental equation i.e. the reproduce the profile with the best approximation.
2. The statistical structure of random errors of measurements on the basis of the information geometry gives the optimization of the solution under radiance spectra i.e. provides the most effective use of measured data.
3. The agreement of the structures of unknown functions and errors of measurements with the kernel of the integral equation allow the determination of an optimal set of temperature profiles which are actually contained in the initial information.

Besides the mathematical method which has been discussed above there are a number of physical problems related to the properties of the radiance of the real atmosphere. Some of them were indicated by Wark & Fleming (1966). The main difficulties arise due to the influence of thin clouds or partial cloudy conditions. Indirect temperature soundings of the lower atmosphere requires consideration of the water vapor transmittance and a determination of the vertical profile of water vapor. The solution of this problem is similar to that of the temperature profile. The empirical orthogonal functions should be used for solution of the problem of vapor profiles.

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О ПРИМЕНЕНИЕ СТАТИСТИУЕСКОГО МЕТОДА ДЛЯ ОПРЕДЕЛЕНИЯ ПРОФИЛЕЙ АТМОСФЕРНОЙ ТЕМПЕРАТУРЫ СО СПУТНИКОВ

Естественные ортогональные функции применяются в задаче определения вертикального профиля температуры по измерениям уходящего излучения Земли в полосе поглощения углекислого газа. Приводятся примеры решения этой задачи. Определяются оптимальные системы ортогональных функций, согласованные со статистикой ошибок измерения уходящего излучения и с ядром интегрального уравнения. Устанавливаются размерность подпространства решений, имеющих реальный смысл при заданной точности измерений, и число разрешаемых профилей температуры.