On the contribution of a limited region to the global energy budget

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ABSTRACT

Energy budget equations applicable to any limited atmospheric region are developed employing the concept of available potential energy. These equations contain, in addition to the customary generation, conversion, and dissipation terms, expressions for the boundary fluxes.

Utilizing Lorenz's definition of available potential energy, the energetics of the limited region are defined as the contribution that the region makes to the global energy budget. This allows one to examine energy processes in a limited region in terms of their relation to the energetics of the general circulation.

Introduction

Over the past decade investigations of energy processes have contributed substantially to the meterologist's understanding of the atmosphere's general circulation. Comprehensive lists of many of these studies and reviews of their results are contained in papers by Oort (1964) and Dutton & Johnson (1967). One of the major conclusions evolving from these research efforts is that on the average the mean flow is fed by the eddies, which in turn are maintained by the conversion of potential to kinetic energy (baroclinic instability). This is in contrast to the previous notion that the eddies result primarily from the redistribution of the kinetic energy in the mean flow (barotropic instability).

The importance of eddy motions, which obviously exhibit significant temporal and spatial variations, makes it apparent that different regions can be expected to influence the general circulation in different ways. The purpose of this paper is to formulate and discuss complete equations appropriate for studying the contribution of any limited region to the global energy budget.

Of primary importance in studies of atmospheric energetics has been the concept of available potential energy. Originally applied to the atmosphere by Margules (1903), it remained for Lorenz (1955*a*) to present the concept in a mathematical form which could be used in studies of the general circulation. Many investigators have used Lorenz's original equations, and their results have of course provided a great deal of information about the general circulation. However, these equations do possess certain mathematical approximations. According to Dutton and Johnson (1967), who reexamined the basic concepts of available potential energy, Lorenz's approximate forms of available potential energy and its generation may in some cases differ significantly from the exact forms. Lorenz (1955*b*) was the first to restate the available potential energy in its exact form and develop a more general expression for its generation. The equations developed here represent an extension of Lorenz's 1955*b* work to limited atmospheric regions.

Energy budget equations

Assuming hydrostatic balance the total potential energy, defined as the sum of the internal and gravitational potential energy, is given by

$$\pi = c_p \int_M \frac{\theta p^{\varkappa}}{p_{00}^{\varkappa}}\, dM \tag{1}$$

where c_p is the heat capacity at constant pressure, θ is the potential temperature, p is pressure, p_{00} is 1000 mb, $\varkappa$ is R/c_p, and M is the mass of the entire atmosphere. To arrive at an expression for available potential energy it is first necessary to specify the state of minimum total potential energy. According to Lorenz (1955*a*) this is achieved by the adiabatic redistribution of the entire atmosphere to a horizontal, hydrostatically stable state. In this reference state the pressure in (1) is given by its reference value $p_r(\theta)$, while the potential temperature remains unchanged. Hence the reference (minimum) total potential energy of the entire atmosphere is

$$\pi_r = c_p \int_M \frac{\theta p_r^{\varkappa}}{p_{00}^{\varkappa}} dM \tag{2}$$

Dutton and Johnson (1967) have shown that under the assumption of hydrostatic balance the reference pressure at any point is given by the global mean pressure of the potential temperature surface passing through that point. It can be expressed as

$$p_r(\theta, t) = \frac{1}{\sigma_e} \int_{\sigma_e} \int_{\sigma_T}^{\theta} \frac{\partial p}{\partial \theta} d\theta\, d\sigma \tag{3}$$

where σ_e is the area of earth and θ_T is the potential temperature at the top of the atmosphere. Subtracting (2) from (1) and applying Poisson's equation yields the expression for the available potential energy of the entire atmosphere

$$A = \pi - \pi_r = c_p \int_M \frac{p^{\varkappa} - p_r^{\varkappa}}{p^{\varkappa}} T\, dM \tag{4}$$

where T is temperature.

It should be pointed out that there exists some controversy over the definition of a realistic state of minimum total potential energy, and indeed other reference states have been proposed by van Mieghem (1956) and Pfeffer *et al.* (1966). However, as noted by Dutton & Johnson (1967), selection of an optimum reference state would be useful only if the resulting available potential energy would yield information on the state that the atmosphere attempts to reach and why particular modes of circulation are chosen. Unfortunately the concept of available potential energy does not yield this information, but instead is useful only in a diagnostic description of energy processes. Hence, no advantage is gained by using a more complicated definition of the reference state in the equations developed in this paper.

Rewriting (4) as the sum of integrals over limited regions of the atmosphere yields

$$A = \sum_j A_j = \sum_j \left(c_p \int_{M_j} \frac{p^{\varkappa} - p_r^{\varkappa}}{p^{\varkappa}} T\, dM \right) \tag{5}$$

where M_j specifies the mass contained in any arbitrary limited region and $\Sigma M_j = M$. From (5) the contribution of a limited region (j) to the available potential energy of the entire atmosphere is defined as

$$A_j = c_p \int_{M_j} \frac{p^{\varkappa} - p_r^{\varkappa}}{p^{\varkappa}} T\, dM$$
$$= \frac{c_p}{g} \int_{\sigma} \int_{p_2}^{p_1} \frac{p^{\varkappa} - p_r^{\varkappa}}{p^{\varkappa}} T\, dp\, d\sigma \tag{6}$$

where σ, p_1 and p_2 are the area and pressure intervals corresponding to M_j.

It is essential to keep in mind the word "contribution" in the equations and discussions that follow. No attempt is made here to define the available potential energy of a limited region. Instead the contribution represents a quantity which is by necessity interrelated with global energy processes. To the extent that energetics can be used to diagnose conditions in the atmosphere, this approach allows one to examine processes in a limited region in terms of their relationship to the general circulation.

The final Eulerian budget equation for A_j is formulated by taking the local time derivative of (6). Assuming the horizontal limits of the region remain constant, this becomes

$$\frac{\partial A_j}{\partial t} = \frac{c_p}{g} \int_{\sigma} \int_{p_2}^{p_1} \left(\frac{\partial T}{\partial t} - \frac{p_r^{\varkappa}}{p_{00}^{\varkappa}} \frac{\partial \theta}{\partial t} - \frac{\theta}{p_{00}^{\varkappa}} \frac{\partial p_r^{\varkappa}}{\partial t} \right) dp\, d\sigma$$
$$+ \frac{c_p}{g} \int_{\sigma} \left(\frac{p_1^{\varkappa} - p_{r_1}^{\varkappa}}{p_1^{\varkappa}} T_1 \frac{\partial p_1}{\partial t} - \frac{p_2^{\varkappa} - p_{r_2}^{\varkappa}}{p_2^{\varkappa}} T_2 \frac{\partial p_2}{\partial t} \right) d\sigma \tag{7}$$

The first integral of (7) contains implicitly the source (diabatic heating), sink (conversion to kinetic energy), and boundary flux terms required in the final budget equation of A_j. To express these terms explicitly, it is necessary to rewrite the local time derivatives contained in this integral as the difference between the total time derivatives and the advection terms. Thus, the first integral of (7) becomes

$$\frac{c_p}{g}\int_\sigma\int_{p_2}^{p_1}\left\{\left(\frac{dT}{dt} - w\cdot\nabla_p T - \omega\frac{\partial T}{\partial p}\right)\right.$$
$$- \frac{p_r^\varkappa}{p_{00}^\varkappa}\left(\frac{d\theta}{dt} - w\cdot\nabla_p\theta - \omega\frac{\partial\theta}{\partial p}\right)$$
$$\left.- \frac{\theta}{p_{00}^\varkappa}\left(\frac{dp_r^\varkappa}{dt} - w\cdot\nabla_p p_r^\varkappa - \omega\frac{\partial p_r^\varkappa}{\partial p}\right)\right\}dp\,d\sigma \tag{8}$$

where ∇_p is the del operator on a pressure surface, ω is dp/dt, and w is the horizontal wind.

Application of the first law of thermodynamics, the continuity equation, and Poisson's equation transforms (8) into

$$\frac{1}{g}\int_\sigma\int_{p_2}^{p_1}\omega\alpha\,dp\,d\sigma + \frac{1}{g}\int_\sigma\int_{p_2}^{p_1}\frac{p^\varkappa - p_r^\varkappa}{p^\varkappa}Q\,dp\,d\sigma$$
$$- \frac{c_p}{g}\int_\sigma\int_{p_2}^{p_1}\frac{p^\varkappa - p_r^\varkappa}{p^\varkappa}\nabla_p\cdot Tw\,dp\,d\sigma$$
$$+ \frac{c_p}{g}\int_\sigma\int_{p_2}^{p_1}\frac{p_r^\varkappa}{p_{00}^\varkappa}\frac{\partial\omega\theta}{\partial p} - \frac{\partial\omega T}{\partial p}dp\,d\sigma$$
$$- \frac{c_p}{g}\int_\sigma\int_{p_2}^{p_1}\frac{T}{p^\varkappa}\left(\frac{dp_r^\varkappa}{dt} - w\cdot\nabla_p p_r^\varkappa - \omega\frac{\partial p_r^\varkappa}{\partial p}\right)dp\,d\sigma \tag{9}$$

where α is the specific volume and Q is the rate of heating per unit mass. Again using Poisson's equation and combining the fourth and fifth integrals of (9) yields

$$- \frac{c_p}{g}\int_\sigma\int_{p_2}^{p_1}\frac{\partial}{\partial p}\left(\frac{p^\varkappa - p_r^\varkappa}{p^\varkappa}\omega T\right)dp\,d\sigma$$
$$- \frac{c_p}{g}\int_\sigma\int_{p_2}^{p_1}\frac{T}{p^\varkappa}\left(\frac{dp_r^\varkappa}{dt} - w\cdot\nabla_p p_r^\varkappa\right)dp\,d\sigma \tag{10}$$

Use of (10) and the relation

$$- \left(\frac{p^\varkappa - p_r^\varkappa}{p^\varkappa}\right)\nabla_p\cdot Tw + \frac{T}{p^\varkappa}w\cdot\nabla_p p_r^\varkappa =$$
$$- \nabla_p\cdot\frac{p^\varkappa - p_r^\varkappa}{p^\varkappa}Tw$$

permits (9) to be expressed as

$$\frac{1}{g}\int_\sigma\int_{p_2}^{p_1}\omega\alpha\,dp\,d\sigma + \frac{1}{g}\int_\sigma\int_{p_2}^{p_1}\frac{p^\varkappa - p_r^\varkappa}{p^\varkappa}Q\,dp\,d\sigma$$
$$- \frac{c_p}{g}\int_\sigma\int_{p_2}^{p_1}\nabla_p\cdot\frac{p^\varkappa - p_r^\varkappa}{p^\varkappa}Tw\,dp\,d\sigma$$
$$- \frac{c_p}{g}\int_\sigma\int_{p_2}^{p_1}\frac{\partial}{\partial p}\left(\frac{p^\varkappa - p_r^\varkappa}{p^\varkappa}T\omega\right)dp\,d\sigma$$
$$- \frac{c_p}{g}\int_\sigma\int_{p_2}^{p_1}\frac{T}{p^\varkappa}\frac{dp_r^\varkappa}{dt}dp\,d\sigma \tag{11}$$

Using Leibnitz's rule the time derivative of (3) can be written as

$$\frac{dp_r}{dt} = \int_{\sigma_e}\int_{\theta_T}^{\theta}\left\{\frac{\partial}{\partial t}\frac{\partial p}{\partial\theta} + \nabla_\theta\cdot\frac{\partial p}{\partial\theta}w + \frac{\partial}{\partial\theta}\left(\frac{\partial p}{\partial\theta}\frac{d\theta}{dt}\right)\right\}d\theta\,d\sigma$$

The integrand of this expression is the continuity equation for nonadiabatic flow in (x, y, θ) coordinates and in this form is equal to zero. Hence $dp_r/dt = 0$. This result and substitution of (11) into (7) yield the final budget equation for A_j

$$\frac{\partial A_j}{\partial t} = \overbrace{\frac{1}{g}\int_\sigma\int_{p_2}^{p_1}\frac{p^\varkappa - p_r^\varkappa}{p^\varkappa}Q\,dp\,d\sigma}^{\text{a}}$$
$$+ \overbrace{\frac{1}{g}\int_\sigma\int_{p_2}^{p_1}\omega\alpha\,dp\,d\sigma}^{\text{b}}$$
$$\overbrace{- \frac{1}{g}\int_\sigma\int_{p_2}^{p_1}\nabla_p\cdot\left(c_p\frac{p^\varkappa - p_r^\varkappa}{p^\varkappa}Tw\right)dp\,d\sigma}^{\text{c}}$$
$$\overbrace{- \frac{1}{g}\int_\sigma\int_{p_2}^{p_1}\frac{\partial}{\partial p}\left(c_p\frac{p^\varkappa - p_r^\varkappa}{p^\varkappa}T\omega\right)dp\,d\sigma}^{\text{d}}$$
$$\overbrace{+ \frac{1}{g}\int_\sigma\left(c_p\frac{p_1^\varkappa - p_{r_1}^\varkappa}{p_1^\varkappa}T_1\frac{\partial p_1}{\partial t} - c_p\frac{p_2^\varkappa - p_{r_2}^\varkappa}{p_2^\varkappa}T_2\frac{\partial p_2}{\partial t}\right)d\sigma}^{\text{e}} \tag{I}$$

The corresponding budget equation for the kinetic energy is obtained by taking the local time derivative of the relation

$$K_j = \frac{1}{g}\int_\sigma \int_{p_2}^{p_1} \frac{w \cdot w}{2}\, dp\, d\sigma \qquad (12)$$

and applying the horizontal equation of kinetic energy per unit mass in the form

$$\frac{\partial}{\partial t}\frac{w \cdot w}{2} = -w \cdot \nabla_p \phi + w \cdot F - \nabla_p \cdot \left(\frac{w \cdot w}{2} w\right) - \frac{\partial}{\partial p}\left(\omega \frac{w \cdot w}{2}\right) \qquad (13)$$

where ϕ is geopotential.

The resulting budget equation for K_j is

$$\frac{\partial K_j}{\partial t} = \overbrace{\frac{1}{g}\int_\sigma \int_{p_2}^{p_1} w \cdot F\, dp\, d\sigma}^{\text{a}} \overbrace{- \frac{1}{g}\int_\sigma \int_{p_2}^{p_1} w \cdot \nabla_p \phi\, dp\, d\sigma}^{\text{b}} \overbrace{- \frac{1}{g}\int_\sigma \int_{p_2}^{p_1} \nabla_p \cdot \left(\frac{w \cdot w}{2} w\right) dp\, d\sigma}^{\text{c}} \overbrace{- \frac{1}{g}\int_\sigma \int_{p_2}^{p_1} \frac{\partial}{\partial p}\left(\omega \frac{w \cdot w}{2}\right) dp\, d\sigma}^{\text{d}} \overbrace{+ \frac{1}{g}\int_\sigma \left(\frac{w_1 \cdot w_1}{2}\frac{\partial p_1}{\partial t} - \frac{w_2 \cdot w_2}{2}\frac{\partial p_2}{\partial t}\right) d\sigma}^{\text{e}} \qquad \text{(II)}$$

Note that although the development is performed in (x, y, p) coordinates, terms Ie and IIe still allow the possibility of variable vertical boundaries (e.g., surface and tropopause level). However, if constant pressure levels are assumed as the vertical boundaries, terms Ie and IIe will vanish.

Discussion of budget terms

The reader is reminded that in the following discussion each of the terms represents the contribution of a limited region to the corresponding global value; however, for simplicity the word contribution will usually not be included.

Term Ia describes the generation of available potential energy by diabatic processes. Although this term appears quite different from Lorenz's (1955*a*) approximate generation expression, it is identical to that obtained in later work by Lorenz (1955*b*) and very similar to the expression obtained more recently by Dutton & Johnson (1967). The factor $E = (p^\times - p_r^\times)/p^\times$ has been called the efficiency factor by Dutton & Johnson.[1] This factor specifies where the addition (or subtraction) of heat is most efficient for generating available potential energy. The strongest positive generation will occur when centers of positive and negative efficiency coincide, respectively, with centers of diabatic heating and cooling.

Term Ib represents the change in A_j due to vertical redistribution of mass within the region, and in nearly all of the published papers on atmospheric energetics this term is the basic expression from which conversion of energy has been estimated. However, term IIb, which represents the change of K_j produced by cross-isobaric flow, also represents conversion. From the expression

$$w \cdot \nabla_p \phi = \nabla_p \cdot \phi w - \phi \nabla_p \cdot w$$

and the continuity equation, Ib and IIb are related by

$$\frac{1}{g}\iint \omega\alpha\, dp\, d\sigma - \frac{1}{g}\iint w \cdot \nabla_p \phi\, dp\, d\sigma = -\frac{1}{g}\iint \nabla_p \cdot \phi w\, dp\, d\sigma - \frac{1}{g}\iint \frac{\partial}{\partial p}\omega\phi\, dp\, d\sigma \qquad (14)$$

The right-hand terms of (14) represent the work of the pressure gradient forces on the boundaries of the limited region. When these terms are

[1] Actually, they defined the efficiency factor as $\frac{p^\times - p_r^\times}{p^\times}\frac{\partial z}{\partial \theta}$; however, this difference does not alter the physical interpretation of the expression.

zero, a change in A_j as given by Ib will be accompanied by an equal but opposite change in K_j as given in IIb. This is the sense in which conversion is usually considered. However, as pointed out by Pfeffer (1957), when the pressure work terms are not zero, neither Ib nor IIb can be thought of as conversion processes in the usual sense. This suggests that the usual concept of conversion is inappropriate when discussing any arbitrary limited atmospheric region. In order to retain the label "conversion" the author suggests that the concept be generalized to include the possibility of production of one form of energy within a region at the expense of this same form of energy and/or another form of energy outside of the region. For example, the depletion of A_j as given by a negative value of Ib, might be accompanied by a corresponding increase of K outside, rather than within, the region. With this broader concept of conversion in mind the author suggests that terms Ib and IIb be referred to as *A-conversion* and *K-conversion*, respectively.

Terms Ic, Id, IIc, IId, represent the horizontal and vertical transport of mass which can change the contribution that the region makes to the available potential or kinetic energy of the entire atmosphere. Of course, when the entire atmosphere is considered, the change of energy by the flux of mass is zero. However, in investigations over limited regions these terms may be quite significant.

Term IIa is a deceivingly simple expression for the complicated processes involved in the dissipation of kinetic energy. Although long recognized as a significant feature of motions in the boundary layers, Holopainen (1963) and Kung (1966*a*, *b*, 1967) have provided evidence that it may also be important in the free atmosphere.

As mentioned in the previous section the remaining terms, Ie and IIe, reflect changes in the vertical boundaries of the limited region and will be zero when fixed pressure levels are used as boundaries. In pressure coordinates these terms represent the changes in A_j and K_j that occur when the total mass under consideration changes.

Summary

It has been the author's intention to utilize already well-developed concepts describing the energetics of the general circulation to formulate equations of use in studying the energetics of limited regions. The need for limited region studies and for equations of the type developed here seems quite evident. Some regions of the earth's atmosphere respond—or contribute—very differently to the global energy budget than others. Certainly, if one is to understand the energetics of individual storms and their interaction with larger scales, a limited region study is essential. While alternate methods of attacking this problem are surely conceivable, the approach used here contains the unique feature of examining energy processes in a limited region in terms of their contribution and relationship to global energy processes.

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BIBLIOGRAPHY

Dutton, J. A., and Johnson, D. R., 1967. The Theory of Available Potential Energy and a Variational Approach to Atmospheric Energetics. *Advances in Geophysics 12*, New York, Academic Press, 333–436.

Holopainen, E. O., 1963. On the Dissipation of Kinetic Energy in the Atmosphere. *Tellus 15*, No. 1, 26–32.

Kung, E. C., 1966*a*. Kinetic Energy Generation and Dissipation in the Large-scale Atmospheric Circulation. *Monthly Weather Review 94*, No. 2, 67–82.

Kung, E. C., 1966*b*. Large-scale Balance of Kinetic Energy in the Atmosphere. *Monthly Weather Review 94*, No. 11, 627–640.

Kung, E. C., 1967. Diurnal and Long—Term Variations of the Kinetic Energy Generation and Dissipation for a Five-Year Period. *Monthly Weather Review 95*, No. 9, 593–606.

Lorenz, E. N., 1955*a*. Available Potential Energy and the Maintenance of the General Circulation. *Tellus 7*, No. 2, 157–167.

Lorenz, E. N., 1955*b*. Generation of Available Potential Energy and the Intensity of the General Circulation. *Large Scale Synoptic Processes*, University of California (Los Angeles), Department of Meteorology, Final Report, J. Bjerknes, Project Director.

Margules, M., 1903. Über die Energie der Stürme. *Jahrb. Zentralanst. Meterol.* 1–26. (Translation by C. Abbe (1910) in *Smithsonian Institute Miscellaneous Collection 51*, 553–595.)

Oort, A. H., 1964. On Estimates of the Atmospheric Energy Cycle. *Monthly Weather Review 92*, No. 11, 483–493.

Pfeffer, R. L., 1957. On the Physical Significance of Energy Transformation Functions. *Studies of the Atmospheric General Circulation II*, Massachusetts Institute of Technology, Department of Meteorology, Final Report, General Circulation Project, V. P. Starr, Director.

Pfeffer, R. L., Mardon, D., Sterbenz, P., and Fowlis, W., 1966. A New Concept of Available Potential Energy. Report No. 66-1, Florida State University, Department of Meteorology.

van Mieghem, J., 1956. The Energy Available in the Atmosphere for Conversion into Kinetic Energy. *Beitrage zur Physik der Atmosphäre 29*, 129–142.

О ВКЛАДЕ ОГРАНИЧЕННОЙ ОБЛАСТИ В ГЛОБАЛЬНЫЙ БАЛАНС ЭНЕРГИИ

Рассматриваются уравнения баланса энергии, применимые к любой ограниченной области атмосферы, используя концепцию доступной потенциальной энергии. Эти уравнения содержат в дополнение к обычным членам, описывающим генерацию, преобразо- вание и диссипацию энергии, выражения для потоков на границе. Используя лоренцовское определение доступной потенциальной энергии, энергетика ограниченной области определяется как вклад, который эта область делает в глобальный баланс энергии. Это позволяет рассматривать энергетические процессы в ограниченной области во взаимосвязи с энергетикой общей циркуляции.