The transport of energy by internal waves

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ABSTRACT

Two different concepts of energy transport by waves are developed and compared. The "wave energy flux" is related to the transport of a quantity known as the wave energy, and has often been used in analyzing the characteristics of the waves themselves. The "correlation wave energy flux" is related to the transport of energy by a phenomenon called a wave, is analogous to turbulence energy transports, and is of use in total system energy budgets. In dissipative fluids and fluids with spatially varying mean flows, the two are not the same. In the course of the analysis, a comparison is made between the Stokes expansion commonly used in linear wave theory and the mean and deviation approach of turbulence.

1. Introduction

Consider the total energy density per unit mass, E, of a fluid. Since energy is conserved, the local rate of change of E may be related to the divergence of the total energy flux, $\mathbf{F}$, including convection, radiation, and also advection. If ϱ is the fluid density,

$$\frac{\partial}{\partial t}(\varrho E) = -\nabla \cdot \mathbf{F}. \tag{1.1}$$

This holds on an instantaneous basis.

In many cases, it is useful or necessary to average the various observable parameters of the fluid in time or space and deal with averaged energy densities and fluxes. For example, we take the average of the advection of energy by molecules and call it a conductive heat flux. At a grosser level, we average over elements of a turbulent fluid, and correlations between deviations from this average provide a turbulent heat flux. These averaged fluxes are useful because they can be combined with bulk or gross fluxes, such as mean energy advection, in computing the averaged values of $\mathbf{F}$ and E.

Similarly, it is useful to average over a hydrodynamic wave in a fluid and to assign rationally a portion of the averaged total energy flux to the wave as a wave energy flux comparable to conductive or turbulent energy fluxes. We shall refer to this as the "correlation wave energy flux."

The usual derivations of linear wave theory through the use of Stokes perturbation expansions lead to difficulties when considering the problems of energetics (Eckart, 1960). The total energy, E, contains terms of a higher order of the expansion than those used to derive the linear wave equations. Hence, it is convenient to *define* an energy density, called the "wave energy density," in terms of the first-order wave perturbations. This energy density is associated with a "wave energy flux," which is not, however, the same as the correlation wave energy flux alluded to above. The differences are important in fluids with dissipation and with spatially non-uniform motion.

In recent years several authors have considered the problem of wave energy propagation in fluids with non-uniform mean motions. Longuet-Higgins and Stewart (1960, 1961) and Whitham (1962) have dealt with the propagation of water waves. They have shown that the total energy flux consists of the advection of the mean kinetic and potential energies by the mean flow, the advection of external wave energy by the mean flow, the wave energy flux relative to the fluid, and, in addition, a term involving the product of mean fluid velocity and the wave radiation stress tensor. (The wave radiation stress tensor is the equivalent of the Reynolds stress tensor of turbulence theory.)

These authors have discussed at length whether this last term should be treated as part of the wave energy flux. An equation for the rate of change of wave energy density involves the divergence of a flux and also source and sink terms. The term in question does not appear in this equation, and these authors do not include it as part of the wave energy flux. In their equation for wave energy, however, the product of the radiation stress tensor and mean rate of strain of the fluid appears as a source term, and is interpreted as a source of wave energy at the expense of mean flow kinetic energy.

More recently Bretherton (1966) and Hines & Reddy (1967) have considered the energy propagation of acoustic and internal gravity waves in stratified fluids with shear. Bretherton considers the equation for wave energy and is not directly concerned with the flux term in question, though he recognizes the associated source term. Hines and Reddy, on the other hand, include (a component of) the mean velocity-radiation stress tensor product as a part of the (vertical) wave energy transport.

Whitham has pointed out the difficulty of dividing the total energy flux into "wave" and "mean" components. We feel that an additional difficulty arises from an ambiguity in the phrase "wave energy flux." Authors have not always clearly differentiated between the wave energy flux and the correlation wave energy flux.

Each of these fluxes has its own range of validity and usefulness. The objectives of this paper are to differentiate clearly between them and to develop and justify a sound definition of the total wave energy flux. The wave energy flux is reasonably well defined through the work of Longuet-Higgins & Stewart (1960, 1961), Whitham (1962), Bretherton (1966), and Eliassen & Palm (1954), for example, and is especially well presented by Eckart (1960).

Our approach to the total wave energy flux is along lines already well developed in the theory of turbulence. While this approach is neither particularly original nor formally necessary, it is illuminating. It is developed in Section 3, after a discussion of the classical approach to waves and the wave energy flux in Section 2. The analysis shows that the first-order Stokesian perturbations are adequate to derive the correlation energy flux to second order.

The physical significance of the correlation wave energy flux is examined in Section 4, and its component parts are considered in Sections 5–7. Arguments are developed for the inclusion of the fluid velocity-radiation stress tensor term as a part of the total wave energy flux.

Our analyses are made for fairly general fluids in which non-isentropic processes may occur. We omit the case of fluids with free surfaces, however, and our discussion is thus more oriented to atmospheric than oceanic problems.

2. Wave energy flux

Consider the total energy density per unit mass of a fluid, E. We may write an equation for the local rate of change of this energy, performing suitable manipulations on the equations of motion and mass conservation and on the first law of thermodynamics. Since total energy is conserved, the equation takes the form

$$\frac{\partial}{\partial t}(\varrho E) + \nabla \cdot \mathbf{F} = 0, \tag{2.1}$$

where ϱ is the fluid density and $\mathbf{F}$ is the total energy flux. The exact form of E and $\mathbf{F}$ depends on the particular fluid being considered. If a wave is present, we would like to be able to assign a portion of the energy density and energy flux to the wave.

The standard approach to wave problems is to expand all relevant parameters in a perturbation series. For example,

$$\begin{aligned} p &= p^{(0)} + p^{(1)} + p^{(2)} \\ u &= \mathbf{u}^{(0)} + \mathbf{u}^{(1)} + \mathbf{u}^{(2)}, \end{aligned} \tag{2.2}$$

where p is fluid pressure and $\mathbf{u}$ is fluid velocity. The terms are taken to be ranked in an increasing order of smallness of some relevant parameter. The zero-order terms are the state of the fluid in the absence of the wave. The first-order terms can then be obtained, and are the periodic linear wave perturbations. If needed, higher-order terms can be computed.

Expansions similar to equation (2.2) can be substituted into the proper form of the energy equation (2.1). If the linear wave solutions are assumed to be periodic and a time average of the resulting equation is taken, the lowest-order terms related to the wave are of second order. Unfortunately, they contain not only products of first-order terms, but also second-order terms and products of second-order and zero-order terms of the expansions (Eckart, 1960). Thus, to obtain second-order energy densities and fluxes, we must obtain second-order wave perturbation solutions.

As Eckart (1960) and also Eliassen and Palm (1954) have pointed out, this problem can be circumvented by *defining* a second-order quantity having the dimensions of energy density, but involving only products of first-order perturbations. We shall refer to this quantity, E_w, as the wave energy density. The time behavior of this quantity can also be obtained through manipulation of the first-order approximations to the basic equations to yield an equation of the form

$$\frac{\partial}{\partial t}(\varrho^{(0)} E_w) + \nabla \cdot \mathbf{F}_w = S. \tag{2.3}$$

The flux term, $\mathbf{F}_w$, which one obtains by such manipulations is *defined* as the wave energy flux. Since E_w is not a total energy density, it is not conserved, and there may be source or sink terms, S, representing transformations between external wave energy and other forms of energy.

If the fluid has non-uniform zero-order motions, S contains as a component

$$-\varrho^{(0)} \overline{\mathbf{u}^{(1)} \mathbf{u}^{(1)}} \cdot \nabla \cdot \mathbf{u}^{(0)}, \tag{2.4}$$

where a mean over a time or space cycle of the wave is denoted by an overbar. This is simply the product of the wave radiation stress tensor,

$$\tau_w = -\varrho^{(0)} \overline{\mathbf{u}^{(1)} \mathbf{u}^{(1)}}, \tag{2.5}$$

and the zero-order rate of strain of the fluid. The radiation stress tensor may also be thought of as a flux or transport of momentum, and it is equivalent to the Reynolds stress tensor of turbulence theory.

The wave energy density and flux are useful in obtaining approximations to the wave behavior in fluids with non-uniform propagation characteristics. If there are no source terms, and the propagation characteristics of the wave are spatially and temporally uniform, then the mean value of external wave energy flux may be written

$$\bar{\mathbf{F}}_w = (\mathbf{u}^{(0)} + \mathbf{c}_g) \varrho^{(0)} \bar{E}_w, \tag{2.6}$$

where $\mathbf{c}_g$ is the wave group velocity relative to the fluid. While group velocity is formally defined in terms of the propagation of a wave packet, in this case it is also the velocity of propagation of mean external wave energy *relative* to the fluid.

If the fluid has wave sources and sinks, or non-uniform propagation characteristics, the local group velocity, as defined by wave packet propagation, and the velocity of propagation of wave energy are only approximately the same. To the extent that this approximation holds, and to the extent that the wave is truly periodic, the group velocity can be used to obtain higher-order estimates of the wave character.

Initially one takes the first-order WKB approximation to the linear first-order wave equations. Coefficients of these equations are assumed locally constant. From these, local values of $\mathbf{c}_g$ and $\bar{S}$ may be estimated. If equation (2.3) is averaged, one can find

$$\nabla \cdot [(\mathbf{u}^{(0)} + \mathbf{c}_g) \varrho^{(0)} \bar{E}_w] = \bar{S} \tag{2.7}$$

and then use equation (2.5) to find the spatial variation of $\bar{E}_w$. Improved approximations to the first-order wave perturbations can then be recovered. This is essentially the procedure used by Longuet-Higgins and Stewart, and by Bretherton.

3. Total wave energy flux

There is an alternative to the Stokes expansion: one may follow usage in the theory of turbulence and divide parameters into suitably defined means and instantaneous local deviations from the means. If such quantities are substituted into expressions for E and $\mathbf{F}$ in equation (2.1) and a mean value obtained, the result contains both parameter means and correlations between deviations.

We shall *define* the mean energy transport attributable to the wave (correlation wave energy flux) as being the sum of terms involving such correlations. We shall defer defense of this definition, except to note that it is in line with the theory of turbulent energy transport.

Consider a rather general fluid with viscosity and thermal conductivity. For the moment, questions of compressibility and thermal structure are irrelevant. The fluid is taken to be located in a gravitational (or other) potential field, and possibly in a rotating system. The following additional notation will be used:

T: temperature;
e: internal energy per unit mass;
h: enthalpy per unit mass;
$\mathbf{W}$: non-mechanical heat flux (conductive, radiative);
$\mathfrak{F}$: friction stress tensor;
φ: gravitational potential.

Other external body forces can be included as components of φ if necessary.

Two different averaging processes will be used. The linear mean of a quantity q will be denoted by

$$\bar{q} \equiv \frac{1}{U}\int_{-U/2}^{U/2} q\,dU, \qquad (3.1)$$

where U is a time and/or space scale which is large compared to the scale of the wave, but small compared to the scale of the mean fluid parameters. We implicitly assume that such a scale can be found. (A similar assumption is implicit in the usual use of the Stokes expansion, where the zero-order terms are taken to be time-invariant and the first-order terms to be periodic.)

The deviation from the mean is defined by

$$q' \equiv q - \bar{q}. \qquad (3.2)$$

It follows that

$$\bar{q'} = 0. \qquad (3.3)$$

We shall also use a density-weighted mean,

$$\bar{\varrho}\hat{q} \equiv \frac{1}{U}\int_{-U/2}^{U/2} \varrho q\,dU, \qquad (3.4)$$

with a deviation,

$$q'' \equiv q - \hat{q}, \qquad (3.5)$$

from which it also follows that

$$\hat{q''} = 0, \qquad (3.6)$$

and $$\overline{\varrho q''} = 0 = \bar{\varrho}\,\overline{q''} + \overline{\varrho' q''}. \qquad (3.7)$$

This average was first applied by Reynolds (1895) to incompressible fluids and by Hesselberg (1926) to compressible fluids.

In our case, we assume that the deviations arise from the presence of a wave and are in all suitable senses of small order compared to the mean quantities.

The total energy density is

$$E = \tfrac{1}{2}\mathbf{u}\cdot\mathbf{u} + e + \varphi. \qquad (3.8)$$

If we take a density-weighted average of E, we find that

$$\bar{\varrho}\hat{E} = (\tfrac{1}{2}\bar{\varrho}\hat{\mathbf{u}}\cdot\hat{\mathbf{u}} + \tfrac{1}{2}\overline{\varrho\mathbf{u}''\cdot\mathbf{u}''} + \bar{\varrho}\hat{e} + \bar{\varrho}\varphi). \qquad (3.9)$$

By suitable operations on the equations of motion and mass conservation, and on the first law of thermodynamics, one can show that

$$\frac{\partial}{\partial t}\bar{\varrho}\hat{E} + \nabla\cdot(\hat{\mathbf{u}}\bar{\varrho}\hat{E}) + \nabla\cdot(\bar{p}\hat{\mathbf{u}} - \bar{\mathfrak{F}}\cdot\hat{\mathbf{u}}) + \nabla\cdot\overline{\mathbf{W}}$$
$$+\nabla\cdot(\overline{p\mathbf{u}''} - \overline{\mathfrak{F}\cdot u''} + \hat{\mathbf{u}}\cdot\overline{\varrho\mathbf{u}''\mathbf{u}''}$$
$$+\tfrac{1}{2}\overline{\varrho\mathbf{u}''\mathbf{u}''\cdot\mathbf{u}''} + \overline{\varrho e''\mathbf{u}''}) = 0. \qquad (3.10)$$

The derivation is given in Eliassen and Kleinschmidt (1957) and will not be repeated here. The first four terms would be present in the absence of wave perturbations. They represent the rate of change of total energy per unit volume, the advection of total energy by the mean flow, the work done by mean pressure and stresses, and the divergence of the mean non-mechanical heat flux. The fifth term may be subjected to further manipulation.

If h is the fluid enthalpy per unit mass,

$$h = \frac{p}{\varrho} + e, \qquad (3.11)$$

and $$\overline{\mathbf{u}''\varrho h} = \overline{p\mathbf{u}''} + \overline{\varrho e\mathbf{u}''}, \qquad (3.12)$$

which is equivalent to

$$\overline{\varrho\mathbf{u}''h''} = \overline{p\mathbf{u}''} + \overline{\varrho e''\mathbf{u}''}. \qquad (3.13)$$

If we define the operator

$$\frac{\hat{D}}{Dt} \equiv \frac{\partial}{\partial t} + \hat{\mathbf{u}}\cdot\nabla, \qquad (3.14)$$

and use the averaged equation of mass conservation,

$$\frac{\partial\bar{\varrho}}{\partial t} + \nabla\cdot\bar{\varrho}\hat{\mathbf{u}} = 0, \qquad (3.15)$$

we can substitute equations (3.12), (3.13) and (3.14) into (3.9) to obtain

$$\bar{\varrho}\frac{\hat{D}\hat{E}}{Dt} + \nabla\cdot(\bar{p}\hat{\mathbf{u}} - \bar{\mathfrak{J}}\cdot\hat{\mathbf{u}}) + \nabla\cdot\overline{\mathbf{W}} + \nabla\cdot\bar{\mathbf{J}} = 0, \qquad (3.16)$$

where

$$\bar{\mathbf{J}} = \overline{\varrho h'' u''} + \overline{\varrho \mathbf{u}''\mathbf{u}''\cdot(\hat{\mathbf{u}} + \tfrac{1}{2}\mathbf{u}'')} - \overline{\mathfrak{J}\cdot\mathbf{u}''}. \qquad (3.17)$$

The first and second terms on the right-hand side of equation (3.17) were considered for turbulence by Montgomery (1948), who analyzed the problem in terms of the thermodynamics of an open system. He identified the first term as the "eddy flux of heat per unit area." The second is an eddy flux of kinetic energy. In the present context, we define $\bar{\mathbf{J}}$ as the correlation wave energy flux.

If we now drop all components of $\bar{\mathbf{J}}$ of higher than second order, we find to this order of accuracy:

$$\bar{\mathbf{J}} \simeq \bar{\varrho}\overline{h'' u''} + \bar{\varrho}\hat{\mathbf{u}}\cdot\overline{\mathbf{u}''\mathbf{u}''} - \overline{\mathfrak{J} u''}. \qquad (3.18)$$

It is also possible to write $\bar{\mathbf{J}}$ in terms of deviations from linear rather than weighted means. To second order, we obtain

$$\mathbf{u}'' = \mathbf{u}' - \frac{1}{\bar{\varrho}}\overline{\mathbf{u}'\varrho'}, \qquad (3.19)$$

and

$$h'' = h' - \frac{1}{\bar{\varrho}}\overline{h'\varrho'},$$

so that to second-order accuracy,

$$\bar{\mathbf{J}} \simeq \overline{\bar{\varrho} h'\mathbf{u}'} + \bar{\varrho}\hat{\mathbf{u}}\cdot\overline{\mathbf{u}'\mathbf{u}'} + \frac{\bar{\mathfrak{J}}}{\bar{\varrho}}\overline{\varrho'\mathbf{u}'} - \overline{\mathfrak{J}'\mathbf{u}'}. \qquad (3.20)$$

Finally, we may compare the mean and deviation quantities with the more normally used Stokes expansion for periodic waves (equations 2.2). If $u^{(1)}$ is periodic in the time or space sense in which averages are taken,

$$\overline{\mathbf{u}^{(1)}} = 0, \qquad (3.21)$$

and

$$\bar{\mathbf{u}} = \mathbf{u}^{(0)} + \overline{\mathbf{u}^{(2)}}, \qquad (3.22)$$

and hence

$$\mathbf{u}' = \mathbf{u}^{(1)} + (\mathbf{u}^{(2)} - \overline{\mathbf{u}^{(2)}}), \qquad (3.23)$$

and

$$\mathbf{u}'' = \mathbf{u}^{(1)} + (\mathbf{u}^{(2)} - \overline{\mathbf{u}^{(2)}}) - \frac{1}{\bar{\varrho}}\overline{u'\varrho'}. \qquad (3.24)$$

The means and deviations thus differ from the corresponding zero- and first-order wave approximations by quantities of second order; and to this order,

$$\bar{\mathbf{J}} \simeq \varrho^{(0)}\overline{h^{(1)}\mathbf{u}^{(1)}} + \varrho^{(0)}\mathbf{u}^{(0)}\cdot\overline{\mathbf{u}^{(1)}\mathbf{u}^{(1)}} + \frac{\mathfrak{J}^{(0)}}{\varrho^{(0)}}\cdot\overline{\varrho^{(1)}\mathbf{u}^{(1)}} - \overline{\mathfrak{J}^{(1)}\cdot u^{(1)}}. \qquad (3.25)$$

However, the products of means cannot be similarly treated. To second order:

$$\bar{p}\hat{\mathbf{u}} = p^{(0)}\mathbf{u}^{(0)} + \overline{p^{(2)}}\mathbf{u}^{(0)} + p^{(0)}\overline{\mathbf{u}^{(2)}} \neq p^{(0)}\mathbf{u}^{(0)}. \qquad (3.26)$$

4. A physical point of view

Equation (3.16) may be interpreted as describing the energetics of a macroscopic fluid "pseudo-particle" moving with the mean mass flow. This is what we would observe if we were to follow the fluid with sensors too gross in time or space resolution to detect the wave itself, but sufficiently sensitive to include the perturbation motions as a contribution to the fluid energy (in the same sense that molecular motions are included as internal energy).

The rate of change of the mean energy of the pseudo-particle could be measured, and the work done by mean stresses and the divergence of the mean non-mechanical heat flux either measured or inferred. The presence of the correlation wave energy flux $\bar{\mathbf{J}}$ could not be detected directly; but its presence would be known by its effect on the system energetics (just as a molecular heat flux is not sensed directly, but measured calorimetrically).

In fact, the analogy with molecular energy transport is quite strong. As already pointed out, the term $\overline{\varrho h''\mathbf{u}''}$ is simply the "eddy" advection of enthalpy or heat content across a surface, moving with the mean mass flow; while $\overline{\varrho\hat{\mathbf{u}}\cdot\mathbf{u}''\mathbf{u}''}$ is the "eddy" advection of kinetic energy.

The mean energy flux,

$$\bar{p}\hat{\mathbf{u}} - \bar{\mathfrak{J}}\cdot\hat{\mathbf{u}} + \overline{\mathbf{W}}, \qquad (4.1)$$

is not the same as the zero-order energy flux that is obtained by using the zero-order Stokes expansion terms. As shown, for example, in equation (3.25), they differ by terms involving the mean values of second-order expansion terms now buried as part of expression (4.1).

Although Whitham takes a contrary view, we think that this is the proper place for such quantities. They should be regarded as a second-order reaction of the mean state of the fluid to the presence of the wave, rather than as a part of the energy transport by the wave. There are several reasons for this point of view:

1. It is in harmony with the macroscopic point of view and with our way of thinking about other types of energy fluxes. For example, electromagnetic radiation absorbed by a fluid or a conductive heat flux may give rise to secondary circulations; but we do not consider these circulations to be a part of the electromagnetic or thermal energy flux. This viewpoint also prevails in the theory of acoustic streaming (Nyborg, 1965).

2. The components of the mean energy flux and mean total wave energy flux are basically defined in terms of physically measurable quantities, the quantities which we would use if we mistook the wave for turbulence. On the other hand, quantities such as $p^{(0)}$, which would prevail in the absence of the wave, cannot be measured in its presence.

3. As a practical matter, the correlation wave energy flux is expressible in terms of first-order linear wave theory. Linear theory can thus tell us how much energy a wave transports from the troposphere to the thermosphere, for example, so long as we do not ask how it alters the mean state of the atmosphere.

5. The enthalpy flux

There are several ways in which the wave-induced enthalpy flux may be written. For example,

$$\bar{\varrho}\overline{h''\mathbf{u}''} = \bar{\varrho}C_p\overline{T''\mathbf{u}''} = \bar{\varrho}\hat{T}\overline{S''u''} + \overline{p'\mathbf{u}''}, \quad (5.1)$$

where C_p is the specific heat at constant pressure, and S is specific entropy.

In a case where the fluid is a gas with potential temperature θ,

$$S = C_p \ln \theta + \text{constant}, \quad (5.2)$$

and hence

$$\bar{\varrho}\overline{h''\mathbf{u}''} = \frac{\bar{\varrho}\hat{T}}{\hat{\theta}} C_p\overline{\theta''u''} + \overline{p'\mathbf{u}''}. \quad (5.3)$$

If the fluid is homentropic and the wave processes are isentropic, $S'' = 0$. This is true for waves in a gravitationless isothermal gas, for example. In this simple case,

$$\bar{\varrho}\overline{h''\mathbf{u}''} = \overline{p'\mathbf{u}''} \simeq \overline{p^{(1)}\mathbf{u}^{(1)}}. \quad (5.4)$$

If the fluid has no mean motion (or a reaction mean motion of second order only) and $\mathfrak{J} = 0$, this is the correlation wave energy transport to second order and is identical with the mean value of the wave energy flux.

If the processes are isentropic, but there is a gradient in the mean entropy of the fluid, there will be entropy fluctuations. Let $\boldsymbol{\xi}$ be the displacement of a fluid particle and $\boldsymbol{\xi}''$ its deviation displacement. Then,

$$S'' = \boldsymbol{\xi}'' \cdot \nabla\hat{S}, \quad (5.5)$$

and

$$\bar{\varrho}\overline{h''\mathbf{u}''} = \bar{\varrho}\hat{T}\overline{\mathbf{u}''\xi''} \cdot \nabla\hat{S} + \overline{p'\mathbf{u}''}$$
$$\simeq \varrho^{(0)}T^{(0)}\overline{\mathbf{u}^{(1)}\boldsymbol{\xi}^{(1)}} \cdot \nabla S^{(0)} + \overline{p^{(1)}\mathbf{u}^{(1)}}. \quad (5.6)$$

If the wave is periodic, $\overline{\mathbf{u}^{(1)} \cdot \boldsymbol{\xi}^{(1)}} = 0$, since displacement and velocity in the same direction are in phase quadrature. Thus there can be no entropy flux along the mean entropy gradient. If the particle orbits of first-order wave theory are elliptical or circular, then the first term on the right-hand side of equation (5.6) contributes an entropy flux *transverse* to the mean entropy gradient.

It is a little difficult at first to see how isentropic fluid motions back and forth across a fluid surface can transport entropy across that surface. Consider figure 1, which shows the

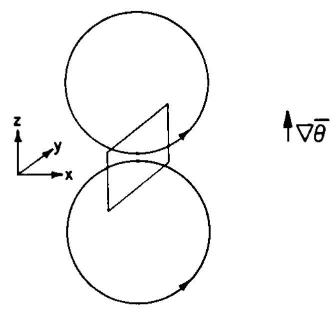

Fig. 1. Particle orbits through a surface dA. If there is a gradient of $\bar{\theta}$ in the z-direction, particles moving through the surface in the upper orbit have a greater value of θ than those in the lower orbit, and there is a net flux of θ in the x-direction.

orbits or fluid particles passing through a surface element, dA, moving with the mean flow. Assume that these particles carry some conserved quantity, say entropy, in an isentropic flow. Then the value of S for each particle is that of its mean position. Particles moving from left to right transport S characteristic of a level above dA, and *vice versa*. Thus, even though there is no net mass flow across dA, there may be a transport of entropy.

On the other hand, if we consider a closed surface, again moving with the mean flow, we can see physically that the flux through a surface element dA must be balanced by return fluxes elsewhere. If the mass in the enclosed volume is in the mean made up of the same fluid elements (as is the case in first-order wave theory), then quantities which are conserved cannot in the mean be advected out of the volume. Thus, for isentropic waves,

$$\nabla \cdot (\bar{\varrho}\, \overline{T''\mathbf{u}''\xi} \cdot \nabla \hat{S}) = 0. \tag{5.7}$$

The transverse flux arising from the gradient of mean entropy is non-divergent and does not contribute to the mean energetics of the fluid. In principle it is an experimentally observable component of $\bar{\mathbf{J}}$, however.

Entropy fluctuations will arise when diabatic processes are involved. If a fluid particle gives up an increment of heat, δQ, while residing in the pseudo-particle, it will leave with a decrease,

$$\delta S = \frac{\delta Q}{\hat{T}}, \tag{5.8}$$

in entropy. Thus, $\nabla \cdot (\bar{\varrho}\overline{\hat{T}S''\mathbf{u}''})$ arises as a result of advecting fluid particles transferring heat to or from the interior of the pseudo-particle Similarly, $\nabla \cdot \overline{p'\mathbf{u}''}$ may be interpreted in terms of advecting fluid particles doing work on the fluid interior of the pseudo-particle during their residence.

6. The kinetic energy flux

As Hines & Reddy have pointed out, the quantity

$$\bar{\varrho}\,\hat{\mathbf{u}} \cdot \overline{\mathbf{u}''\mathbf{u}''} \tag{6.1}$$

can be interpreted as an advection of a first-order perturbation in kinetic energy $\bar{\varrho}\,\hat{\mathbf{u}} \cdot \mathbf{u}''$ by a first-order perturbation in velocity. It may also be interpreted (Longuet-Higgins & Stewart, 1960, 1961) as the product of a radiation stress tensor, $\bar{\mathfrak{J}}$, with the mean flow velocity,

$$\bar{\mathfrak{J}} \cdot \hat{\mathbf{u}}, \tag{6.2}$$

where $$\bar{\mathfrak{J}} = \bar{\varrho}\,\overline{\mathbf{u}''\mathbf{u}''}. \tag{6.3}$$

The divergence of expression (6.2) may be interpreted variously as the divergence of an energy source, or as the sum of two energy source terms:

$$\nabla \cdot (\bar{\varrho}\,\hat{\mathbf{u}} \cdot \overline{\mathbf{u}''\mathbf{u}''}) = (\bar{\mathfrak{J}} \cdot \nabla) \cdot \hat{\mathbf{u}} + \hat{\mathbf{u}} \cdot (\nabla \cdot \bar{\mathfrak{J}}). \tag{6.4}$$

Longuet-Higgins, Stewart and Bretherton identify the first term on the right as the rate at which the mean flow does work on the waves and treat it as a local source term for wave energy. Similarly, a divergence of the stress tensor, $\bar{\mathfrak{J}}$, signifies an exchange of momentum with the mean flow; and the second term can be regarded as a local source of kinetic energy of the mean flow.

The important point to note is that a rate of increase of wave energy at some point is not necessarily associated with an equal rate of decrease of mean flow kinetic energy *at the same point* (Lumley and Panofsky, 1964, p. 69). Consider a bounded fluid which may have sources and sinks of wave energy, but which has no energy exchange with external sources. Then

$$\int \nabla \cdot (\bar{\mathfrak{J}} \cdot \hat{\mathbf{u}})\, dx = 0. \tag{6.5}$$

Thus, a local source of wave energy must be balanced by equivalent sinks of either wave energy or mean flow energy, which may be located elsewhere in the fluid. $\bar{\mathfrak{J}} \cdot \hat{\mathbf{u}}$ is then an energy flux connecting these sources and sinks.

7. The friction stress term

The term $\overline{\mathfrak{F} \cdot \mathbf{u}''}$ may be thought of in much the same sense considered in discussing the kinetic energy flux. The term may be construed as a molecular advection of kinetic energy. It may also be considered as a flux connecting regions where viscous dissipation adds to internal energy, $(\overline{(\mathfrak{F} \cdot \nabla) \cdot \mathbf{u}''} < 0)$, and subtracts from kinetic energy, $(\mathbf{u}'' \cdot \nabla \cdot \mathfrak{F} > 0)$.

If there is no mean fluid stress, this term reduces to $\overline{\mathfrak{J}'' \cdot \mathbf{u}''}$, which represents, for example, the entire energy transport of viscous shear waves in a homogeneous incompressible fluid with no mean velocity.

If there is a mean flow shear and stress, some care must be taken. The wave mean mass transport, $\overline{\varrho' \mathbf{u}''}$, interacts with the mean stress; and fluctuations in $\mathfrak{J}$ may arise not only from fluctuations in shear, but also from fluctuations in the viscosity of the fluid.

8. Summary

We have defined a second form of wave energy flux, the correlation wave energy flux, which differs from the more normally used external wave energy flux. A rather indirect approach through turbulence theory techniques was chosen primarily to make clearer the difference between this flux and the second-order reaction of the fluid to the effects of the wave.

The differences between the Hines and Reddy concept of wave energy flux and that of the other authors cited is understandable in the light of these two different definitions. Hines and Reddy have been dealing with the correlation wave energy flux; and this is the relevant quantity for the questions they have tried to answer, questions as to the role waves play in the energy balance of the upper atmosphere.

On the other hand, Longuet-Higgins, Stewart, Whitham, and Bretherton have been more concerned with the structure and behavior of the waves *per se*. For this type of problem, the wave energy density is the pertinent quantity.

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ПЕРЕНОС ЭНЕРГИИ ВНУТРЕННИМИ ВОЛНАМИ

Развиваются и сравниваются две различные концепции переноса энергии волнами. «Волновой поток энергии» связан с переносом количества, известного как волновая энергия, и часто используется при анализе характеристик самих волн. «Корреляционный волновой поток энергии», связанный с переносом энергии феноменом, называемым волной, аналогичен турбулентному переносу энергии и используется в общем энергетическом бюджете системы. В диссипативных жидкостях и в жидкостях с пространственным изменением средних потоков эти понятия не совпадают. В ходе анализа проводится сравнение между разложением Стокса, обычно используемым в линейной волновой теории, и турбулентным подходом, основанном на рассмотрении возмущений.