On the energy integral for satellites[1]

By ARNE BJERHAMMAR, *Institute of Geodesy, Royal College of Technology, Stockholm*

(Manuscript received March 12, 1968)

ABSTRACT

The energy integral can be used for studing the gravity field of the earth from satellite orbits. With known satellite velocities in the orbit we can compute the potential in an extremely simple way. Polar satellites give full coverage all over the earth and permit a high accuracy. Non-polar satellites require a small correction for the rotation of the earth.

The orbital theory lends itself in a natural way to the study of the gravity field of the earth using artificial satellites. Various authors have successfully applied an expansion in *spherical harmonics* combined with an *integration* procedure for a determination of the low order harmonics of the gravity field of the earth (Kozai, King-Hele, Kaula, Anderle). The integration of the orbit is made either by analytical or numerical methods. Present solutions have given promising results up to order fifteen of an expansion in spherical harmonics and higher order solutions can be expected. The advantage of the present technique is rather obvious for all low order solutions. Only a rather limited number of observations are needed for a determination of the complete low order geoids and the internal consistency is high. However, the differences between various methods can be considerable. This means that our present technique gives a misleading presentation with respect to the accuracy of the solution and there is a possibility that the very complex mathematical models so far used include critical biases of different types. The truncation of the spherical harmonic expansion of the gravity field has a tendency to give a bias in the determination of the individual coefficients and various authors find considerable differences between most tesseral harmonics. It is obvious that all higher harmonics are excluded in the present technique and this paper presents a new method for analyzing the gravity field of the earth using satellite data.

An orbiting satellite is exposed to the gravity field of the earth and all other celestial bodies. If we disregard the influence of solar radiation, the air drag, and distant celestial bodies we have the following relation between the gravitational and the kinetic potential of an orbiting satellite

$$V_j + \frac{GM_1}{r_1} + \frac{GM_2}{r_2} - \frac{v_j^2}{2} = c \text{ [see footnote 2]}, \tag{1}$$

where

V_j = potential of the earth in the orbit (at point P_j),
G = gravitational constant,
r_1 = distance to the moon from the satellite,
M_1 = mass of moon,
r_2 = distance to the sun from the satellite,
M_2 = mass of the sun,
v_j = velocity of the satellite,
c = constant.

This relation is based on the laws of conservation of energy and needs no direct explanation. The gravitational energy and the kinetic energy must balance in such a way that there always is a constant difference.

[1] This study has mainly been made at the Research Institute for Geodetic Sciences and presented in a paper 'A new approach to satellite geodesy', (Alexandria, 1967).

[2] The energy integral of a stationary gravity field. Cf. also equation (11).

We are here not directly interested in the energy and express instead our relation as a potential difference.

If we also consider the influence of air drag and solar radiation we have

$$V_j = \frac{v_j^2}{2} - \frac{GM_1}{r_1} - \frac{GM_2}{r_2} + c + \Delta c_1 + \Delta c_2, \qquad (2)$$

where

Δc_1 = correction for drag,
Δc_2 = correction for solar radiation.

Equation (2) is transcribed

$$V_j = \frac{v_j^2}{2} + k_j + c, \qquad (3)$$

where

$$k_j = -\frac{GM_1}{r_1} - \frac{GM_2}{r_2} + \Delta c_1 + \Delta c_2.$$

Equation (3) can be used for a determination of potential differences in the gravitational field of the earth. The constant c is not directly available but the quantity k can be determined with high degree of accuracy for any position of the satellite. In most applications we want the potential at the surface of the earth and in consequence of the harmonic properties we can include the necessary reduction in the following integral equation

$$V_j = 0.5v_j^2 + k_j + c = \frac{r_j^2 - r_0^2}{4\pi r_0} \iint_S \frac{V^*}{r_{ij}^3} dS, \qquad (4)$$

where

r_{ij} = distance to the satelliie from the moving point,
r_j = geocentric distance to the satellite,
r_0 = radius of the reference sphere ("radius of the earth"),
V^* = potential at the reference sphere,
S = reference surface.

Here we have for a constant V_0^* the simple solution

$$\frac{r_j^2 - r_0^2}{4\pi r_0} \iint_S \frac{V_0^*}{r_{ij}^3} dS = \frac{r_0}{r_j} V_0^* = V_j. \qquad (5)$$

Thus we find from equations (4) and (5) the approximative solution in the general case

$$V_0^* = \frac{r_j}{r_0}\left[(0.5v_j^2 + k_j + c) + \frac{r_j^2 - r_0^2}{4\pi r_0} \iint_S \frac{0.5(v_j^2 - v^2) + (k_j - k)}{r_{ij}^3} dS\right], \qquad (6)$$

where

V_0^* = reouced potential of V_j (on reference sphere),
v_j = velocity of the satellite at the fixed point,
v = velocity of the satellite,
k_j = parameter value for the fixed point,
k = parameter value for the moving point.

The final solution has to satisfy the equation

$$V_0^* = \frac{r_j}{r_0}\left[V_j + \frac{r_j^2 - r_0^2}{4\pi r_0} \iint_S \frac{V_0^* - V^*}{r_{ij}^3} dS\right] \qquad (7)$$

and we can insert any approximate solution (for example, from equation (6)) below the integral sign of equation (7) for an iteration approach. This operation can then be repeated with corrected values of the potentials to any accuracy obtainable from the observations.

Also, a solution with spherical harmonics can be contemplated

$$V_j = \frac{GM}{r_j} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{r_0}{r_j}\right)^n \times (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_{nm}(\sin \varphi), \qquad (8)$$

where

M = mass of the earth,
$P_{nm}(\sin \varphi)$ = associated Legendre polynomial,
C_{nm}, S_{nm} = spherical harmonic coefficients.

This approach will in all practical application exclude higher harmonics.

Gravity in space is obtained after derivation of (4)

$$g = -\frac{\partial V_j}{\partial r_j} = -\frac{1}{8\pi r_0 r_j}\iint_S \times \frac{[4r_j^2 r_{ij}^2 - 3(r_j^2 - r_0^2 + r_{ij}^2)(r_j^2 - r_0^2)]\,V^*}{r_{ij}^5}\,dS. \quad (9)$$

For the reference sphere we obtain the simplified expression

$$g = -\frac{1}{2\pi}\iint_S \frac{V^* - V_0^*}{r_{ij}^3}\,dS + \frac{V_0^*}{r_0}. \quad (9\,\mathrm{a})$$

The multi-body problem for point masses

The exact relation between the gravitational and kinetic energy can be given for all bodies considered to be point masses or spherical bodies with concentric mass distribution.

$$\frac{GM_0M_1}{r_{01}} + \frac{GM_0M_2}{r_{02}} + \frac{GM_0M_3}{r_{03}} + \frac{GM_1M_2}{r_{12}} + \frac{GM_1M_3}{r_{13}} + \frac{GM_2M_3}{r_{23}} = \tfrac{1}{2}(M_0v_0^2 + M_1v_1^2 + M_2v_2^2 + M_3v_3^2) + c + \Delta c_1 + \Delta c_2 \quad (10)$$

or

$$G\sum_{i=0}^{n-1}\sum_{j=1}^{n}\frac{M_iM_j}{r_{ij}} - \frac{1}{2}\sum_{i=0}^{n} M_iv_i^2 = c + \Delta c_1 + \Delta c_2 \quad (i<j), \quad (10\,\mathrm{a})$$

where

M_0 = mass of the satellite,
M_1 = mass of the earth,
M_2 = mass of the moon,
M_3 = mass of the sun,
v_0 = inertial velocity of the satellite,
v_1 = inertial velocity of the earth,
v_2 = inertial velocity of the moon,
v_3 = inertial velocity of the sun,
r_{01} = distance satellite to earth,
r_{02} = distance satellite to moon,
r_{03} = distance satellite to sun,
r_{12} = distance earth to moon,
r_{13} = distance earth to sun,
r_{23} = distance sun to moon.

Thus we have for the potential of the earth

$$V_j = \frac{1}{2M_0}[(M_0v_0^2 + M_1v_1^2 + M_2v_2^2 + M_3v_3^2)] - \frac{1}{M_0}\left[\frac{GM_0M_2}{r_{02}} + \frac{GM_0M_3}{r_{03}} + \frac{GM_1M_2}{r_{12}} + \frac{GM_1M_3}{r_{13}} + \frac{GM_2M_3}{r_{23}} - c - \Delta c_1 - \Delta c_2\right]. \quad (11)$$

This expression is also a useful approximation for an aspherical earth when the terrestrial potential at the satellite is given by the expression

$$V_j = G\iiint_M \frac{1}{r}\,dM, \quad (12)$$

where

dM = mass element of the earth,
r = distance between the actual mass element and the satellite.

If we consider the velocity of the earth constant the only additional correction needed will be a change of c. In all other combinations of equations (10) we can still consider the bodies concerned to be point masses if the distances are measured to the gravity center.

In the two body problem we have the following relation for a determination of c

$$\frac{G(M_0+M_1)}{r_{01}} - \frac{v^2}{2} = c = \frac{G(M_0+M_1)}{2a} = \frac{G(M_0+M_1)}{r_{01_{\max}} + r_{01_{\min}}}, \quad (13)$$

where

v = relative velocity,
a = semi major axis of the orbit,
r_{01} = distance between the two bodies.

For the multi-body problem we can compute a value of c using the known value of gravity at the surface of the earth (equation 9a) or any known value of the potential at the surface of the earth (equation 6).

Orbital approach

All present studies of the gravity field of the earth seem to use an orbital approach. We are here making a condensed presentation based on the classical procedure.

The terrestrial gravity field can be computed anywhere in space directly in the local coordinate system. However, a computation using a previous expansion in spherical harmonics simplifies the presentation in most applications with satellite orbits. Then the principal problem will be a transformation from the equatorial rectangular coordinate system of the earth to a satellite centered system. Irrespective of the method chosen, we can apply the following technique.

A series expansion, according to Taylor, of the potential of a discrete point in space gives (only the potential disturbance considered)

$$T = T_0 + (\Delta x, \Delta y, \Delta z) \begin{bmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \\ \dfrac{\partial T}{\partial z} \end{bmatrix} + \tfrac{1}{2}(\Delta x, \Delta y, \Delta z) \begin{bmatrix} T_{xx}, & T_{xy}, & T_{xz} \\ T_{yx}, & T_{yy}, & T_{yz} \\ T_{zx}, & T_{zy}, & T_{zz} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} + \cdots \text{ terms of higher order,} \quad (20)$$

where T = disturbance potential

$$(T = V_j - GM/r_j).$$

$$\begin{bmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \\ \dfrac{\partial T}{\partial z} \end{bmatrix} = -\begin{bmatrix} \Delta g_x \\ \Delta g_y \\ \Delta g_z \end{bmatrix} = \nabla T \quad \text{(potential gradient using rectangular coordinates).} \quad (21)$$

For T we introduce the position vector $\bar{r}_j$ defined by polar coordinates.

$$\bar{r}_j = \begin{bmatrix} r_j \cos\phi \cos\lambda \\ r_j \cos\phi \sin\lambda \\ r_j \sin\phi \end{bmatrix}, \quad (22)$$

where

ϕ = latitude (geocentric),

λ = longitude,

r_j = geocentric distance to the actual point.

Thus we have the following unit vectors

$$\hat{r} = \begin{bmatrix} \cos\phi \cos\lambda \\ \cos\phi \sin\lambda \\ \sin\phi \end{bmatrix}, \quad \hat{r}_\phi = \begin{bmatrix} -\sin\phi \cos\lambda \\ -\sin\phi \sin\lambda \\ \cos\phi \end{bmatrix}, \quad \hat{r}_\lambda = \begin{bmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{bmatrix}, \quad (23)$$

where

$$\hat{r} \cdot \hat{r}_\phi = 0, \quad \hat{r}_\phi \cdot \hat{r}_\lambda = 0, \quad \hat{r} \cdot \hat{r}_\lambda = 0. \quad (24)$$

Furthermore we have for the gradient in a polar coordinate system

$$\begin{bmatrix} \Delta g_x \\ \Delta g_y \\ \Delta g_z \end{bmatrix} = -\left[\hat{r}, \frac{\hat{r}_\lambda}{r_j \cos\phi}, \frac{\hat{r}_\phi}{r_j}\right] \begin{bmatrix} T_r \\ T_\lambda \\ T_\phi \end{bmatrix} = -\nabla T, \quad (25)$$

where

$$\left[\hat{r}, \frac{\hat{r}_\lambda}{r_j \cos\phi}, \frac{\hat{r}_\phi}{r_j}\right] = \begin{bmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial \lambda}{\partial x} & \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial r}{\partial y} & \dfrac{\partial \lambda}{\partial y} & \dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial r}{\partial z} & \dfrac{\partial \lambda}{\partial z} & \dfrac{\partial \phi}{\partial z} \end{bmatrix}$$

and furthermore

$$-\frac{\partial T}{\partial r_j} = \frac{GM}{r_j} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left[\frac{C_{nm} \cos m\lambda + S_{nm} \sin m\lambda}{r_j^{n+1} r_0^{-n}}\right] \times (n+1) P_n^m (\sin\phi), \quad (25\,\text{a})$$

$$\frac{\partial T}{\partial \lambda_j} = \frac{GM}{r_j} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \frac{-C_{nm} \sin m\lambda + S_{nm} \cos m\lambda}{r_j^n r_0^{-n}} \times m P_n^m (\sin\phi), \quad (25\,\text{b})$$

$$\frac{\partial T}{\partial \phi_j} = \frac{GM}{r_j} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \frac{C_{nm} \cos m\lambda + S_{nm} \sin m\lambda}{r_j^n r_0^{-n}} [P_n^{m+1}(\sin\phi) - m \tan\phi P_n^m (\sin\phi)]. \quad (25\,\text{c})$$

Finally we obtain the potential gradient at any point in the orbit using equation (25).

In most applications it is simpler to use a gradient presented in the orbital system and we have to make corresponding mathematical transformations.

$$\begin{bmatrix} \Delta g_{x*} \\ \Delta g_{y*} \\ \Delta g_{z*} \end{bmatrix} = \begin{bmatrix} A_{xx*} & A_{xy*} & A_{xz*} \\ A_{yx*} & A_{yy*} & A_{yz*} \\ A_{zx*} & A_{zy*} & A_{zz*} \end{bmatrix} \begin{bmatrix} \Delta g_x \\ \Delta g_y \\ \Delta g_z \end{bmatrix}, \qquad (26)$$

where

x^* = radial direction,

y^* = direction normal to radius vector in the plane of the orbit (direction of the motion of the satellite),

z^* = direction normal to the plane of the orbit positive north

and

$A_{xx*} = \cos\bar{\omega}\cos\Omega - \sin\bar{\omega}\sin\Omega\cos i$,

$A_{xy*} = \cos\bar{\omega}\sin\Omega + \sin\bar{\omega}\cos\Omega\cos i$,

$A_{xz*} = \sin\bar{\omega}\sin i$,

$A_{yx*} = -\sin\bar{\omega}\cos\Omega - \cos\bar{\omega}\sin\Omega\cos i$,

$A_{yy*} = -\sin\bar{\omega}\sin\Omega + \cos\bar{\omega}\cos\Omega\cos i$,

$A_{yz*} = \cos\bar{\omega}\sin i$,

$A_{zx*} = \sin\Omega\sin i$,

$A_{zy*} = -\cos\Omega\sin i$,

$A_{zz*} = \cos i$,

where

i = inclination of the orbital plane to the equatorial plane,

Ω = right ascension of the ascending node of the orbit,

$\bar{\omega}$ = argument of perigee + true anomaly.

Corrections to the Keplerian elements are obtained according to Gauss in the following way

$$\frac{\partial M^*}{\partial t} = \left(\frac{GM}{a^3}\right)^{\frac{1}{2}} - \frac{1}{na}\left(\frac{2r}{a} - \frac{1-e^2}{e}\cos v\right)\Delta g_{x*} - \frac{1-e^2}{nae}\left(1+\frac{r}{p}\right)\sin v\,\Delta g_{y*}, \qquad (27\text{a})$$

$$\frac{\partial a}{\partial t} = \frac{2}{n\sqrt{1-e^2}}\left(e\sin v\cdot\Delta g_{x*} + \frac{p}{r}\Delta g_{y*}\right), \qquad (27\text{b})$$

$$\frac{\partial e}{\partial t} = \frac{\sqrt{1-e^2}}{na}(\sin v\,\Delta g_{x*} + (\cos E + \cos v)\Delta g_{y*}), \qquad (27\text{c})$$

$$\frac{\partial \omega}{\partial t} = \frac{\sqrt{1-e^2}}{nae}\left(-\cos v\,\Delta g_{x*} + \left(\frac{r}{p}+1\right)\sin v\,\Delta g_{y*}\right) - \cos i\frac{\partial\Omega}{\partial t}, \qquad (27\text{d})$$

$$\frac{\partial i}{\partial t} = \frac{1}{na\sqrt{1-e^2}}\frac{r}{a}\cos\bar{\omega}\,\Delta g_{z*}, \qquad (27\text{e})$$

$$\frac{\partial \Omega}{\partial t} = \frac{1}{na\sqrt{1-e^2}}\frac{r}{a}\frac{\sin\bar{\omega}}{\sin i}\cdot\Delta g_{z*}, \qquad (27\text{f})$$

where

a = semi major axis of the satellite orbit,

e = excentricity of the satellite orbit,

n = mean angular velocity of the satellite,

$p = a(1-e^2)$,

r = distance of the satellite from the gravity center of the earth,

v = true anomaly $(\bar{\omega}-\omega)$,

E = eccentric anomaly $(\cos E = (1-r/a)/e)$,

M^* = mean anomaly $(M^* = E - e\sin E)$.

Now the disturbing accelerations are known in the direction of radius vector, normal to radius vector in the plane of the orbit and perpendicular to the plane of the orbit. After integration with respect to time we obtain the disturbances in relation to a pure Keplerian orbit.

Orbits from gravity anomalies

The satellites have been widely used for a determination of the gravity field of the earth. However, if the gravity anomaly field is known from terrestrial date we can compute the orbit using the expressions for the potential in space.

$$T_j = \frac{1}{4\pi r_j}\iint_S \Delta g^* \sum_{n=2}^{\infty}\left(\frac{r_0}{r_j}\right)^n \frac{2n+1}{n-1}P_n(\cos\omega)\,dS + \frac{GM - GM_0}{r_j}$$

$(\omega$ = geocentric angular distance), (30)

$$\frac{\partial T}{\partial r_j} = \frac{r_0^2 - r_j^2}{4\pi r_j} \iint_S \frac{\Delta g^*}{r_{ij}^3} dS - \frac{2T_j}{r_j} - \frac{GM - GM_0}{r_j^2}$$

[see footnote 1], (31)

$$\frac{\partial T}{\partial \lambda_j} = \frac{\cos\phi\, r_0}{4\pi r_j^2} \iint_S \Delta g^* \sin\omega \sin\alpha\, k' dS, \quad (32)$$

$$\frac{\partial T}{\partial \phi_j} = \frac{r_0}{4\pi r_j^2} \iint_S \Delta g^* \sin\omega \cos\alpha\, k' dS, \quad (33)$$

$$k' = \frac{2}{r^3} - 8 + \frac{3(r+1)}{2r\varkappa} - 3 \ln \varkappa, \quad (34)$$

$$\varkappa = \frac{(r+1)^2 - \left(\dfrac{r_0}{r_j}\right)^2}{4}, \quad (35)$$

$$r = r_{ij}/r_j \quad (36)$$

(M = mass of the actual earth,)

(M_0 = mass of the reference ellipsoid),

$$\Delta g^0 = \frac{r_j^2 - r_0^2}{4\pi r_j} \iint_S \frac{\Delta g^*}{r_{ij}^3} dS = g - \gamma \quad (37)$$

g = observed gravity,

γ = theoretical gravity (from the international gravity formula)[2], cf. also Arnold (1965).

It should be noted that the international gravity formula is related to the Potsdam system which gives approximately 13 milligal too high values. The international ellipsoid has a flattening of 1 : 297 and an equator radius of 6,378, 388 m. Modern satellite data give a flattening 1 : 298.3 for an equator of the earth 6,378, 165 m.

Cf. Arnold, Kaula and Gaposchkin.

Equations (30)–(33) define the orbit after proper transformation to the orbital elements by equation (25)–(27f).

[1] Zero and first order harmonics are included in the integral.

[2] Harmonics of order two and four of the International Gravity formula have to be removed, Cf. p. 43, 'On a coalescent world geodetic system' (Bjerhammar, 1967). Here the "sperical gravity anomaly" (Δg^0) has been used.

Coordinate systems

Inertial system

A coordinate system with inertial orientation and with inertial origin.

Equation:

$$V_j M_0 + \frac{GM_0 M_2}{r_{02}} + \frac{GM_0 M_3}{r_{03}} + \frac{GM_1 M_2}{r_{12}} + \frac{GM_1 M_3}{r_{13}} + \frac{GM_2 M_3}{r_{23}} - \frac{M_0 v_0^2}{2} - \frac{M_1 v_1^2}{2} - \frac{M_2 v_2^2}{2} - \frac{M_3 v_3^2}{2} = c^*, \quad (38)$$

where

$$c^* = c + \Delta c_1 + \Delta c_2.$$

The equation has been evaluated in the following way:

1. Exact in the Newtonian way for point masses and if all external forces are excluded. (Correction for a symmetry of Eq. (46).)
2. "Exact" in geodetic applications for polar satellite orbits.
3. A useful approximation for non polar orbits.

It should be noted that if non-polar satellites are used then they should be matched in such a way that equal distribution on both sides of the pole is obtained. Circular orbits of shortest possible radius are to be preferred.

Heliocentric system

A coordinate system with inertial orientation and with the origin in the sun or more correctly in the gravity center of all bodies concerned.

Equation:

$$V_j M_0 + \frac{GM_0 M_2}{r_{02}} + \frac{GM_0 M_3}{r_{03}} + \frac{GM_1 M_2}{r_{12}} + \frac{GM_1 M_3}{r_{13}} + \frac{GM_2 M_3}{r_{23}} - \frac{M_0 v_{03}^2}{2} - \frac{M_1 v_{13}^2}{2} - \frac{M_2 v_{23}^2}{2} = c^*, \quad (39)$$

where

v_{03} = the heliocentric velocity of the satellite,
v_{13} = the heliocentric velocity of the earth,
v_{23} = the heliocentric velocity of the moon.

This system is from geodetic point of view equivalent with the inertial system. However, the velocities do not include the velocity of the coordinate system.

Geocentric system

An inertial orientation of the system with the origin of the coordinate system in the center of the earth.

Equation:

$$V_j M_0 + \frac{GM_0 M_2}{r_{02}} + \frac{GM_0 M_3}{r_{03}} + \frac{GM_1 M_2}{r_{12}} + \frac{GM_1 M_3}{r_{13}} + \frac{GM_2 M_3}{r_{23}} - \frac{M_0 v_{01}^2}{2} - \frac{M_2 v_{21}^2}{2} - \frac{M_3 v_{31}^2}{2} = c^*, \quad (40)$$

where

v_{01} = geocentric velocity of the satellite,
v_{21} = geocentric velocity of the moon,
v_{31} = geocentric velocity of the sun.

1. Exact only with the joint gravity center in M_1.
2. The accelerations of M_1 are transferred to all extra-terrestial bodies. This means that the sun will receive accelerations that are 333,000 times larger than the original ones of the earth!—The system is justified only in the combination of the earth, the satellite and the moon.

The numerical example is computed in the geocentric system with exclusion of the sun and the moon. Cf. p. 8. Explorer 22 is a "polar satellite" with the inclination 79.69°.

Rotating geocentric system

An earth fixed orientation of the coordinate system with the origin of the coordinate system in the center of the earth which rotates with the "constant" angular velocity ω. (The classical geodetic system for geopotentials.)

For the two body problem (the earth and the satellite) we have the so called modified energy integral (cf. textbooks).

$$V_j + \frac{\omega^2 r_{01}^2 \cos^2\phi}{2} - \frac{v_{01}^2}{2} = c^*, \quad (41)$$

where

v_{01} = the velocity of the satellite in relation to the rotating earth (relative velocity)
r_{01} = the geocentric distance to the satellite,
ϕ = the latitude of the satellite (geocentric),
ω = the angular velocity of the earth,
$V_j + (\omega^2 r_{01}^2 \cos^2\phi/2)$ = the geopotential of the satellite.

There is no variation of the geopotentials with respect to time and therefore equation (41) it valid also for a rotating asymmetrical gravity field.

This equation is transcribed using geocentric velocity vectors

$$V_j + \frac{\dot{r}_\omega \dot{r}_\omega}{2} - \frac{(\dot{r}_{01} - \dot{r}_\omega)^2}{2} = c^*, \quad (42)$$

where

$\dot{r}_{01}$ = the geocentric velocity vector of the satellite $|\dot{r}_{01}|(\doteq v_{01})$,
$\dot{r}_\omega$ = the geocentric velocity vector of a point of the rotating coordinate system at the position of the satellite.

Here we have

$$\dot{r}_\omega = \bar{\omega} \times \bar{r}_{01}, \quad (43)$$

where

$\bar{r}_{01}$ = the geocentric position vector of the satellite,
$\bar{\omega}$ = the angular velocity vector of the earth.

The equation (42) is rewritten

$$V_j - \frac{\dot{r}_{01}\dot{r}_{01}}{2} = c^* - \dot{r}_\omega \dot{r}_{01}. \quad (44)$$

This equation is valid for a rotating *asymmetrical* gravity field ($M_0 \doteq 0$).

If there is no rotation we have the classical energy integral

$$V_j - \frac{\dot{r}_{01}\dot{r}_{01}}{2} = c^*. \tag{45}$$

The asymmetry correction (Δc) of equation (44) is now separated

$$\Delta c = -\dot{r}_\omega \dot{r}_{01} = -|\dot{r}_\omega||\dot{r}_{01}| \cos\beta \tag{46}$$

or

$$\Delta c = -v_{01} r_{01} \cos\phi \cos\beta, \tag{46a}$$

where

β = the angle between the vectors $\dot{r}_{01}$ and $\dot{r}_\omega$, and

$\beta = 90°$ for polar satellites.

The rotating geocentric system gives some practical problems when applied to the sun, the moon, the earth and the satellite. The sun will be rotating around the earth in 24 hours and considerable relativistic corrections have to be applied. The potential of the sun is approximately 14 times larger than the potential of the earth in the satellite orbit and should not be neglected.

The main advantage with this system is that tesseral harmonics are correctly reproduced for all inclinations. The practical importance of this improvement is rather limited because only polar orbits cover all the earth.

The correction of the constant c^* is difficult to compute in the inertial system and the angle between to two velocity vectors is needed free from errors for the total orbit. Any errors of this angle will give a very large contribution to the final error because they will also falsify the zonal harmonics which are up to 1000 times larger than the tesseral harmonics.

The asymmetry correction (Δc of Eq. 46) is zero for polar orbits and it seems justified to consider equation (38) as a "useful approximation" also for the rotating asymmetrical gravity field of the earth. Only polar satellites give full coverage of the gravity field of the earth. (For non polar satellites the asymmetry correction can also be considered for equation (38).) This can be verified using Lagrange equations.

Numerical results

Computations by G. Prine

Orbital data from Explorer 22 have been used for a numerical study where all spherical harmonic coefficients up to order 9.9 were uniquely defined. The potentials were computed for selected points in the orbit with the following results.

Geocentric coordinates and velocities

No.	ϕ	λ	r_j, km	$0.5v^2$ km/sec^2
1	−37°289056	209°68479	7427.2965	26.609251
2	−27.406516	211.48025	7439.5848	26.530560
3	−19.162304	212.63917	7448.0330	26.476387
4	− 9.269896	213.80737	7455.7257	26.426310
5	2.263073	215.04022	7461.0198	26.389836
6	12.138060	216.11588	7462.2415	26.378322
7	25.283632	217.79653	7459.0727	26.392189
8	40.018772	220.52081	7449.2582	26.447910
9	53.125694	224.75212	7435.4411	26.532860
10	61.597380	229.70330	7424.1617	26.605651
11	67.112441	235.11761	7415.7786	26.661442
12	72.155374	243.55883	7407.1401	27.720333
13	76.559800	258.27239	7398.0141	26.784031
14	57.471264	9.24347	7335.3089	27.259483
15	29.121549	17.87715	7297.9676	27.569213

Potentials

No.	W_1 = Gravitational potential (km^2/sec^2)	W_2 = Kinetic potential (km^2/sec^2)	$W_2 - W_1$
1	53.665152	53.665185	.000033
2	53.586466	53.586494	.000028
3	53.532305	53.532321	.000016
4	53.482229	53.482244	.000015
5	53.445763	53.445770	.000007
6	53.434249	53.434256	.000007
7	53.448129	53.448123	−.000006
8	53.503850	53.503844	−.000006
9	53.588806	53.588794	−.000012
10	53.661598	53.661585	−.000013
11	53.717389	53.717376	− 000013
12	53.776284	53.776267	−.000017
13	53.839981	53.839965	−.000016
14	54.315427	54.315417	−.000010
15	54.625168	54.625147	−.000021

$$W_1 = \frac{GM}{r_j}\left(\sum_{n=0}^{9}\sum_{m=0}^{9}\left(\frac{r_0}{r_j}\right)^n (C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\right.$$

P_{nm} sin ϕ) = gravitational potential,

$W_2 = 0.5\ v^2 + c$ = kinetic potential (air drag and solar radiation removed).

Discussion

The present study indicates that the kinetic potentials agree with the gravitational potentials within the computational accuracy. The simplifications obtained using the new approach are considerable in all cases where the velocity of the satellite can be determined with adequate consistency. Recent studies by Berbert *et al.* indicate that Goddard Range and Range Rate (RARR) System, SECOR AND TRANET Doppler data give initial velocities for short arcs with standard errors from ± 1.6 to ± 6.6 cm and positions from ± 6 to 16 m. These accuracies are sufficient for a determination of the total potentials to $\pm 1.10^{-6}$. There is little doubt that the low order harmonics will be obtained with the highest accuracy using the conventional integration procedure. If the total potential and the total gravity is considered the new procedure can be of interest.

Acknowledgements

The author is grateful for comments by Prof. A. Eriksson and trigadier M. Hotine.

REFERENCES

Anderle, R. J. 1965. Observations of resonance effects on satellite orbits arising from the thirteenth- and fourteenth-order tesseral gravitational coefficients. *J. Geophys. Res. 70*, 10, 2453–2485.

Arnold, K. 1965. Determination of the gravity field of the earth by successive passages of satellites. 2nd Intern. Symp.: The use of Artificial Satellites in Geodesy. Athens, 1965.

Bjerhammar, A. 1964. A new theory of geodetic gravity. Trans. Roy. Inst. Techn., No. 242. *Pure and Appl. Math. and Physics 27*, Stockholm.

Cook, A. H. 1961. Resonant orbits of artificial satellites and longitude terms in the earth's external gravitational potential (mit einem Anhang von H. J. Norton). *Geophys. J. 4*, 53–72.

Gaposchkin, E. M. 1964. Differential orbit improvement DOI-3. Smithsonian Astrophysical Observatory: Special Report 161.

Kaula, W. M. 1963. Determination of the earth's gravitational field. *Rev. Geophysics, 1*, No. 4, 507–551.

—— 1965. Comparison and combination of satellite with other results for geodetic parameters. 2nd Intern. Symp.: The use of Artificial Satellites in Geodesy. Athens, 1965.

—— 1964. Determination of variations in the gravitational field for calculation of orbit perturbations. Norton Air Force Base, California.

King-Hele, D. G., Cook, G. E., & Scott, D. W. 1965. The odd zonal harmonics in the earth's gravitational potential. 2nd Intern. Symp.: Use of Artificial Satellites for Geodesy. Athens, 1965.

Kozai, Y. 1961. Potential field of the earth derived from motions of artificial satellites. Symposium: *Geodesy in Space Age*, pp. 147–176. The Ohio State University, Columbus.

Kozai, Y. 1965. Long range analysis of satellite observations. 2nd Intern. Symp.: Use of Artificial Satellites for Geodesy. Athens, 1965.

Mancini, A. 1965. Space orientation and geodetic azimuths of long lines from observations of Anna satellite. *Bull. Géod. 76*, 97–114.

Moritz, H. 1964. The Computation of the external gravity field and the geodetic boundary value problem. Symposium: Extension of the Gravity Anomalies to Unsurveyed Areas. Columbus, Ohio.

O'Keefe, J. 1957 An application of Jacobi's integral to the motion of an earth satellite, *Astr. J. 62*, No. 1252.

Schmid, H. 1964. Accuracy aspects of a world-wide passive Satellite Triangulation System. US Dept of Commerce: Coast and Geodetic Survey. Washington, DC.

Schmid, H. 1964. Tracking of balloon-satellites for geodetic purposes. Cospar meeting, Florence, 1964.

Sigl, R. 1961. *Die Bedeutung künstlicher Erdsatelliten für die Geodäsie.* Stuttgart, 1961.

Veis, G. 1963. Optical tracking of artificial satellites. *Space Sci. Rev. 2*, 250–296.

Veis, G. & Whipple, F. L. 1962. Experience in precision optical tracking of satellites for geodesy. *Space Res. II*, 17–33.

ОБ ИНТЕГРАЛЕ ЭНЕРГИИ

Интеграл энергии может быть использован при изучении с орбит спутников гравитационного поля Земли. Зная скорости спутников, можно вычислить предельно простым образом потенциал. Спутники с полярной орбитой покрывают всю землю и дают высокую точность. Спутники с неполярными орбитами требуют внесения небольшой поправки на вращение Земли.