On the concept and implementation of sequential analysis for linear random fields

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ABSTRACT

Formulas are derived for the optimum (least-mean-square) linear mixing of prognostic information and synoptic observations in objective analyses. Exploitation of these results makes possible a continuing synoptic analysis, updated as new measurements are reported, without the necessity of globally simultaneous observations. For the special case of an unbounded, spatially periodic observation lattice in a homogeneous field, solutions in the wave-number domain are obtained. A one-dimensional wave equation and a thermal diffusion problem are treated as examples.

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Introduction, motivation, and background

The concept of a "synoptic analysis" is central in dynamic meteorology and, indeed, in all disciplines in which forecasts of physical field variables are of interest. This central position is implied by the form of the mathematical models of field dynamics in which time derivatives are expressed as functions of spatial values and spatial derivatives and integrals. In the mathematical theory of stochastic processes this same idea is embodied in the concept of a (simple) Markov process.

As a natural consequence of this physical motivation, the design of observation and data collection systems for analysis and prediction of physical fields has been oriented toward periodic simultaneous measurements throughout the medium, and collection of the resulting data as rapidly as possible at a central processing facility. Analysis, either manual or "objective", then consists of a reconstruction of the continuous spatial field (or a dense grid representation of the same), to which the dynamic equations of motion are applied for extrapolation into future time.

Although the applicable equations imply that a continuous "initial condition" field description is sufficient to define the values of the field for all future time, the uncertainty of the synoptic reconstruction caused by finite sampling density and inaccurate measurements suggests that continuity of development from previous analyses may assist in defining the present distribution of values. The human analyst utilizes this principle intuitively by inspecting the current array of data for patterns previously identified. In objective analyses, previous data enter in the form of a "first-guess" field of forecast or extrapolated values, which are interpolated to the locations of current measurements, and the differences used to provide weighted corrections to the surrounding grid point values (Cressman, 1959). In neither case, according to present practice, is the relative weighting of current and previous data objectively and rationally related to the statistics of forecast and observation errors, to the density of observations, or to the time interval between analyses.

In some situations, synoptic observations are not feasible, as in the use of vertically rising sounding balloons and in the reports of casual or reconnaissance aircraft operating at finite speed within the medium. In other cases, efficient use of communication links dictates sequential collection of observation reports at uniform rates from one synoptic hour to the next. Of particular interest at the present time and of increasing future significance is the measurement of geophysical parameters by remote sensor devices in low-altitude orbiting satellites—observations which are inherently sequential rather than synoptic in character. At present, no definitive techniques are available for the inclusion of this type of information in objective (numerical) analyses.

The above considerations suggest the abandonment of the notion that synoptic analysis must be inextricably linked to simultaneous observations. What is required is an adequate theory for the mixture of forecast and interpolated fields so that observations at any times and any locations in the medium may be utilized optimally to update a *continuing* "synoptic analysis". The concept of sequential estimation was studied by the present author in 1965 (Petersen, 1965). Two approaches to the problem of the mixing of forecast and observed data have been published by Russian authors (Gandin, 1963; Mashkovich, 1965). The work of Jones (Jones, 1965) and Thompson (Thompson, 1961) should also be mentioned. The present approach is an extension and application of Kalman filtering and prediction theory (Kalman, 1960) to the problem of sampled multi-dimensional fields. It is closely related to the work of Petersen and Middleton (Petersen & Middleton, 1962, 1963, 1964, 1965) with the specific additional ingredients of a dynamic equation of motion for the medium, and constraints on excitative and measurement noise to ensure the Markovian character of the resulting random process.

A by-product of the development here, of interest in its own right, is the explicit specification of the relative weighting to be placed on data of varying accuracy, whether obtained at synoptic or non-synoptic times. At present, no objective methods exist by which the "value" of various modes of observation may determine their relative influence on a synoptic analysis; the credibility, for example, of pilot reports vs. upper-air balloon soundings may only be inserted subjectively into the analysis. This problem will be accentuated in the future when indirect soundings via satellites may be used for the bulk of upper air observations, with relatively sparse and infrequent "bench mark" measurements by *in situ* sensors. The fact that the results of such indirect sounding and measurements of electromagnetic radiation will inherently represent some sort of spatial average of the field rather than discrete point observations (thus enhancing their "representativeness" (Petersen & Middleton, 1963) can also be explicitly accounted for.

The methods described here can be interpreted as giving objective significance to the concept of "information perishability" often alluded to by meteorologists (Air Navigation Development Board, 1952). As such, they can introduce an important ingredient into the design of effective communication systems for real-time collection and dissemination of environmental data.

It is recognized, of course, that the more general models of meteorological and oceanographic dynamics are significantly nonlinear and thus their analysis by means of the methods described below is not strictly optimal. However, the difficulties involved are inherent in the models (and physical processes) themselves, regardless of the data processing technique used. For instance, if a mean-square evaluation criterion is used, the optimum prognostic field is *not* obtained by applying the dynamic model to the optimum synoptic analysis! Uncorrected "dynamical" methods are in general unsuitable under a statistical evaluation criterion.

It is intuitively clear, however, that if the errors in analysis can be considered as small perturbations superimposed (additively) upon a general flow, and if the prognostic errors remain small, at least from one observation time to the next, the methods derived for linear systems should yield reasonably useful results. It is probable, however, that forecasts far into the future, when the results of perturbations in the analysis can no longer be considered "small", should be statistically corrected for nonlinear effects.

Assumptions and problem formulation

Our basic assumption is the existence of a mathematical model for the dynamic behavior of the physical field, expressible as a linear (partial) integro-differential equation (Langevin equation (Middleton, 1960))

$$\frac{\partial \alpha(\mathbf{x}, t)}{\partial t} + \int_X \lambda(\mathbf{x}, \mathbf{y}, t)\, \alpha(\mathbf{y}, t)\, d\mathbf{y} = f(\mathbf{x}, t), \qquad (1)$$

or as its solution

$$\alpha(\mathbf{x}, t) = \int_X h(\mathbf{x}, t; \mathbf{y}, t_0)\, \alpha(\mathbf{y}, t_0)\, d\mathbf{y} + \int_X \int_{t_0}^{t} h(\mathbf{x}, t; \mathbf{y}, \tau)\, f(\mathbf{y}, \tau)\, d\mathbf{y}\, d\tau \qquad (2)$$

in terms of a characteristic "Green's Function" (Hildebrand, 1952) or impulse response, $h(\mathbf{x}, t; \mathbf{y}, s)$, $t > s$. Here α is a field defined over a multi-dimensional spatial domain X and time t, and $f(\mathbf{x}, t)$ is a driving force or excitation. For the purposes of this paper, we shall think of α as a scalar variable; however, the extension to vector fields is straightforward.

The underlying purpose of the manipulations of the mathematical model are to obtain extrapolations in time (predictions or forecasts) beyond the latest available observations or measurements. The criterion by which these estimates are to be evaluated is their mean-square error or deviation from the true value of the field variable at the given location and time. It is well known (Papoulis, 1965) that the optimal solution to this problem is that the estimate should coincide with the mean value or expectation of the variable, conditional upon the available data. Applying this criterion to eq. (2), we obtain

$$\hat{\alpha}(\mathbf{x}, t) = \int_X h(\mathbf{x}, t; \mathbf{y}, t_0)\, E\{\alpha(\mathbf{y}, t_0)\}\, d\mathbf{y} + \int_X \int_{t_0}^{t} h(\mathbf{x}, t; \mathbf{y}, \tau)\, E\{f(\mathbf{y}, \tau)\}\, d\mathbf{y}\, d\tau. \qquad (3)$$

We next place a constraint on the excitation function such as to make the second term on the right of eq. (3) vanish identically. We demand that the mean value of the excitation at all spatial locations, conditional upon *past* observations of the (excited) field, be zero. This ensures that the excitation has no "memory"—in effect represents "white noise".[1] The dynamics of the system which involve "memory" must all be incorporated into the Green's function $h(\mathbf{x}, t; \mathbf{y}, s)$ and thus, by implication, into the "state representation" $\alpha(\mathbf{x}, t)$. Under these circumstances the Green's function may be used, step-by-step or directly, to "forecast" the field indefinitely into the future in an optimal manner, starting from an "initial condition" or "synoptic analysis" $E\{\alpha(\mathbf{y}, t_0)\}$ which is itself an optimal estimate based on all available data up to and including time t_0.

[1] This implies that $E\{f(\mathbf{x}, t) f(\mathbf{y}, \tau)\} = F(\mathbf{x}, \mathbf{y}, t)\, \delta(t - \tau)$, where δ is the Dirac delta function.

What is the nature of the "available data"? It consists, in general, of a set of observations or measurements of the physical field, made at various (past) times and places by means of instruments having certain physical and statistical characteristics. We require, quite reasonably, that the totality of this information, whether counted in units of real numbers or of bits, be finite. We also demand, somewhat less realistically, that measurements be instantaneous in time (i.e., that instrumentation incorporate no "memory" of past conditions). Again, "memory" in instrumentation must be included in the system state description.

We thus express the observation process as a linear operation on the instantaneous spatial distribution of the field variable:

$$\beta_k(t) = \int_X \alpha(\mathbf{y}, t)\, g_k(\mathbf{y}, t)\, d\mathbf{y} + n_k(t). \qquad (4)$$

Here the index k denotes the kth observation at time t, and $n_k(t)$ is the "noise" or error associated with that measurement operation The weighting function $g_k(\mathbf{y}, t)$ incorporates both a *location* and a *resolution* characteristic for each observation; for discrete point measurements it becomes a Dirac delta function $\delta(\mathbf{x}_k - \mathbf{y})$, while for remote (e.g., electromagnetic) observations it represents a form of aperture function. The noise is also assumed to be "white", so that an optimal estimate of the *next* observation may be made by applying the measurement operator to the estimated (predicted) field:

$$\hat{\beta}_k(t) = \int_X \hat{\alpha}(\mathbf{y}, t)\, g_k(\mathbf{y}, t)\, d\mathbf{y}. \qquad (5)$$

It will be shown in the next section that the optimum "synoptic analysis" of the field at time t_0 consists of the field *predicted* for time t_0 on the basis of all previous data (i.e., extrapolated from the last "synoptic analysis" through the Green's Function), plus an additive correction at each point consisting of a linear weighting on the *difference* between each new observation at time t_0 and its corresponding predicted value. Furthermore, the only statistical information needed to derive the appropriate error weighting functions are the prediction error covariance function for the field,

$$P_t(\mathbf{x},\mathbf{y}) = E\{[\alpha(\mathbf{x},t)-\hat{\alpha}(\mathbf{x},t)]\,[\alpha(\mathbf{y},t)-\hat{\alpha}(\mathbf{y},t)]\}, \quad (6)$$

where the estimated values of the field variable are those based on the data available prior to time t, and the statistics of observation or measurement errors.

Derivation of the formulas for optimum synoptic analysis

Let us consider time instants $\ldots, t_{-2}, t_{-1}, t_0, t_1, t_2, \ldots$, not necessarily equally spaced, at which observations of the field are made and an updated "synoptic analysis" prepared. At each instant the available measurements are well defined in the sense of eq. (4) (even if ex-post-facto), and their predicted values have been determined according to eq. (5). Let us denote the totality of all measurements obtained up to and including time t_{-1} as Y_{-1}; and up to and including time t_0 as Y_0. Let $\hat{\alpha}_{Y-1}$ be the estimate of the field based on the data set Y_{-1}, etc. Then, by the principle of orthogonal projection (Papoulis, 1965), the errors in predicting the observations at time t_0,

$$\tilde{\beta}_k(t_0) = \beta_k(t_0) - \hat{\beta}_{k\,Y_{-1}}(t_0), \quad k = 1, 2, \ldots, N_0, \quad (7)$$

are orthogonal to Y_{-1} and may be considered as the "new information" contained in the measurements $\{\beta_k(t_0)\}$. But we may then write the optimum linear estimate of the field itself at time t_0, based on data up to and including time t_0, as the sum of the estimate based on data up to and including time t_{-1} and a linear weighting applied to the observation prediction errors:

$$\hat{\alpha}_{Y_0}(\mathbf{x},t_0) = \hat{\alpha}_{Y-1}(\mathbf{x},t_0) + \sum_{k=1}^{N_0} \Delta_k(\mathbf{x},t_0)\,[\beta_k(t_0) - \hat{\beta}_{k\,Y_{-1}}(t_0)]. \quad (8)$$

In eq. (8), N_0 component functions $\{\Delta_k(\mathbf{x}, t_0)\}$ appear, each of which applies a weighting to the error of the corresponding observation at time t_0 to update the current estimate of the field at all points $\mathbf{x}$.

To determine the form of the functions $\{\Delta_k(\mathbf{x}, t_0)\}$, we observe that the error field $[\alpha(\mathbf{x}, t_0) - \hat{\alpha}_{Y_0}(\mathbf{x}, t_0)]$ is orthogonal to the estimate $\hat{\alpha}_{Y_0}(\mathbf{x}, t_0)$; i.e., that

$$E\{[\alpha(\mathbf{x},t_0) - \hat{\alpha}_{Y_0}(\mathbf{x},t_0)]\,[\hat{\alpha}_{Y_0}(\mathbf{y},t_0)]\} = 0 \quad (9)$$

for all $\mathbf{x}$ and $\mathbf{y}$. This also implies that the error covariance function

$$\begin{aligned} P_0(\mathbf{x},\mathbf{y}) &= \\ &= E\{[\alpha(\mathbf{x},t_0) - \hat{\alpha}_{Y-1}(\mathbf{x},t_0)]\,[\alpha(\mathbf{y},t_0) - \hat{\alpha}_{Y-1}(\mathbf{y},t_0)]\} \\ &= E\{[\alpha(\mathbf{x},t_0) - \hat{\alpha}_{Y-1}(\mathbf{x},t_0)]\,[\alpha(\mathbf{y},t_0)]\} \\ &= P_0(\mathbf{y},\mathbf{x}) \end{aligned} \quad (10)$$

for all $\mathbf{x}$ and $\mathbf{y}$. Substituting eqs. (4) and (5) into (8) and (9), we obtain

$$\begin{aligned} E\Bigg\{\Bigg[&\alpha(\mathbf{x},t_0) - \hat{\alpha}_{Y-1}(\mathbf{x},t_0) - \sum_{k=1}^{N_0}\Delta_k(\mathbf{x},t_0) \\ &\times\left[\int_X g_k(\mathbf{z},t_0)\,[\alpha(\mathbf{z},t_0) - \hat{\alpha}_{Y-1}(\mathbf{z},t_0)]\,d\mathbf{z} + n_k(t_0)\right]\Bigg] \\ &\times\Bigg[\hat{\alpha}_{Y-1}(\mathbf{y},t_0) + \sum_{k=1}^{N_0}\Delta_k(\mathbf{y},t_0) \\ &\times\Bigg[\int_X g_k(\mathbf{z},t_0)\,[\alpha(\mathbf{z},t_0) - \hat{\alpha}_{Y-1}(\mathbf{z},t_0)]\,d\mathbf{z} \\ &+ n_k(t_0)\Bigg]\Bigg]\Bigg\} = 0. \end{aligned} \quad (11)$$

Using eq. (10) and the requirement that measurement noise be uncorrelated with present and past values of the true field variable, we may express eq. (11) as

$$\begin{aligned} &\sum_{k=1}^{N_0}\Delta_k(\mathbf{y},t_0)\left[\int_X g_k(\mathbf{z},t_0)\,P_0(\mathbf{x},\mathbf{z})\,d\mathbf{z}\right] \\ &\quad - \sum_{k=1}^{N_0}\sum_{j=1}^{N_0}\Delta_k(\mathbf{x},t_0)\,\Delta_j(\mathbf{y},t_0) \\ &\quad\times\left[\int_X\int_X g_k(\mathbf{z},t_0)\,g_j(\mathbf{w},t_0)\,P_0(\mathbf{z},\mathbf{w})\,d\mathbf{z}\,d\mathbf{w} \right. \\ &\quad\left. + R_{kj}(t_0)\right] = 0, \end{aligned} \quad (12)$$

where $R_{kj}(t_0) = E\{n_k(t_0)n_j(t_0)\}$ is the covariance of the observation errors of the kth and jth observations.

The bracket in the second term of eq. (12) has an important physical significance. It is (one term of) the "verification error" covariance matrix of the observations at time t_0: the second-order statistics of the *difference* between the predicted values of the measurements and the actual raw (error-contaminated) observations at the synoptic time. These statistics are directly obtainable from the data processing system, provided that the dynamic model of the system is correct and the optimum anlysis/forecast procedure is used. Since the statistics are themselves needed to design the optimum processing formulas, one can envisage an iterative (adaptive) program by which this optimum may be approached *without specific knowledge of the statistics of the driving forces.*

The solution of eq. (12) for the error weighting functions $\{\Delta_k(\mathbf{x}, t_0)\}$ is straightforward provided the above-mentioned error covariance matrix is nonsingular. Designating by $\mathbf{\Delta}(\mathbf{y}, t_0)$ the column vector of components $\{\Delta_k(\mathbf{y}, t_0)\}$, by $\mathbf{\Gamma}(\mathbf{x}, t_0)$ the integrals $\{\int_X g_k(\mathbf{z}, t_0) P_0(\mathbf{x}, \mathbf{z}) d\mathbf{z}\}$, and by $\mathbf{K}(t_0)$ the matrix whose entries are

$$K_{kj}(t_0) = \int_X \int_X g_k(\mathbf{z}, t_0)\, g_j(\mathbf{w}, t_0)\, P_0(\mathbf{z}, \mathbf{w})\, d\mathbf{z}\, d\mathbf{w} + R_{kj},\ (t_0) \tag{13}$$

we may write eq. (12) in the form

$$\mathbf{\Gamma}^\dagger(\mathbf{x}, t_0)\, \mathbf{\Delta}(\mathbf{y}, t_0) = \mathbf{\Delta}^\dagger(\mathbf{x}, t_0)\, \mathbf{K}(t_0)\, \mathbf{\Delta}(\mathbf{y}, t_0), \tag{14}$$

where the superscript † denotes the transposed matrix. Then, the desired weighting functions are

$$\mathbf{\Delta}(\mathbf{x}, t_0) = [\mathbf{K}^{-1}(t_0)]^\dagger \mathbf{\Gamma}(\mathbf{x}, t_0); \tag{15}$$

or, specifically for the kth function,

$$\Delta_k(\mathbf{x}, t_0) = \sum_{j=1}^{N_0} [\mathbf{K}^{-1}(t_0)]_{jk} \int_X g_j(\mathbf{z}, t_0)\, P_0(\mathbf{x}, \mathbf{z})\, d\mathbf{z}, \quad k = 1, 2, \ldots, N_0. \tag{16}$$

Equation (16) implies that the influence of a given observation on the updating of the predicted synoptic field is primarily dependent on the covariance (correlation) between the errors of prediction from the last analysis at the point to be updated and the observation point. If the observation is diffuse (e.g., as made by a remote sensor with finite aperture), the covariance function is similarly averaged in the vicinity of the observation focus, thus tending to smooth out short wavelength fluctuations (Petersen & Middleton, 1963). Finally, the inverted observation error matrix accounts for varying density and/or accuracy of observations to achieve a balanced optimum reconstruction of the field at every point.

It is possible to write a recursion relationship connecting the prediction error covariance function $P(\mathbf{x}, \mathbf{y})$ and the reconstruction weighting functions $\{\Delta_k(\mathbf{x})\}$, using the Green's function description of field dynamics and the statistics of field excitation. To do this, we express the error matrix at time t_1 as

$$P_1(\mathbf{x}, \mathbf{y}) = E\{[\alpha(\mathbf{x}, t_1) - \hat{a}_{Y_0}(\mathbf{x}, t_1)]\, [\alpha(\mathbf{y}, t_1) - \hat{a}_{Y_0}(\mathbf{y}, t_1)]\}. \tag{17}$$

But, as stated following eq. (3), the forecasted field is simply

$$\hat{a}_{Y_0}(\mathbf{x}, t_1) = \int_X h(\mathbf{x}, t_1; \mathbf{y}, t_0)\, \hat{a}_{Y_0}(\mathbf{y}, t_0)\, d\mathbf{y}. \tag{18}$$

Substituting from eqs. (2), (4), (5), and (8), we have an expression for the error at time t_1,

$$\begin{aligned} \alpha(\mathbf{x}, t_1) - \hat{a}_{Y_0}(\mathbf{x}, t_1) &= \int_X h(\mathbf{x}, t_1; \mathbf{y}, t_0)\, [\alpha(\mathbf{y}, t_0) - \hat{a}_{Y_0}(\mathbf{y}, t_0)]\, d\mathbf{y} \\ &\quad + \int_X \int_{t_0}^{t} h(\mathbf{x}, t_1; \mathbf{y}, \tau)\, f(\mathbf{y}, \tau)\, d\mathbf{y}\, d\tau \\ &= \int_X h(\mathbf{x}, t_1; \mathbf{y}, t_0) \Bigg[\alpha(\mathbf{y}, t_0) - \hat{a}_{Y-1}(\mathbf{y}, t_0) \\ &\quad - \sum_{k=1}^{N_0} \Delta_k(\mathbf{y}, t_0) \int_X g_k(\mathbf{z}, t_0) \\ &\quad \times [\alpha(\mathbf{z}, t_0) - \hat{a}_{Y-1}(\mathbf{z}, t_0)]\, d\mathbf{z} \\ &\quad - \sum_{k=1}^{N_0} \Delta_k(\mathbf{y}, t_0)\, n_k(t_0)\Bigg]\, d\mathbf{y} \\ &\quad + \int_X \int_{t_0}^{t_1} h(\mathbf{x}, t_1; \mathbf{y}, \tau)\, f(\mathbf{y}, \tau)\, d\mathbf{y}\, d\tau. \end{aligned} \tag{19}$$

In substituting (19) into (17), we assume that the excitation $f(\mathbf{x}, t)$, $t > t_0$, and the observation errors $n_k(t_0)$ are uncorrelated with the true field, with the observations available at time t_0, and with each other. We also use the definitions of $P_0(\mathbf{x}, \mathbf{y})$, R_{ij}, and $Q_1(\mathbf{x}, \mathbf{y})$:

$$Q_1(\mathbf{x},\mathbf{y}) = \int_X \int_X \int_{t_0}^{t_1} \int_{t_0}^{t_1} h(\mathbf{x}, t_1; \mathbf{z}, \tau)\, h(\mathbf{y}, t_1; \mathbf{w}, \sigma) \times E\{f(\mathbf{z},\tau) f(\mathbf{w},\sigma)\}\, d\mathbf{z}\, d\mathbf{w}\, d\tau\, d\sigma$$

$$= \int_X \int_X \int_{t_0}^{t_1} h(\mathbf{x}, t_1; \mathbf{z}, \tau)\, h(\mathbf{y}, t_1; \mathbf{w}, \tau) \times F(\mathbf{z}, \mathbf{w}, \tau)\, d\mathbf{z}\, d\mathbf{w}\, d\tau, \quad (20)$$

the cumulative effect of excitations occurring in the time interval (t_0, t_1). Thus,

$$\begin{aligned} P_1(\mathbf{x},\mathbf{y}) = & \int_X \int_X h(\mathbf{x}, t_1; \mathbf{z}, t_0)\, P_0(\mathbf{z}, \mathbf{w})\, h(\mathbf{y}, t_1; \mathbf{w}, t_0) \\ & \times d\mathbf{z}\, d\mathbf{w} - \int_X \int_X \int_X h(\mathbf{x}, t_1; \mathbf{z}, t_0)\, h(\mathbf{y}, t_1; \mathbf{w}, t_0) \\ & \times \sum_{k=1}^{N_0} \Delta_k(\mathbf{w}, t_0)\, g_k(\mathbf{v}, t_0)\, P_0(\mathbf{z}, \mathbf{v})\, d\mathbf{z}\, d\mathbf{w}\, d\mathbf{v} \\ & - \int_X \int_X \int_X h(\mathbf{x}, t_1; \mathbf{z}, t_0)\, h(\mathbf{y}, t_1; \mathbf{w}, t_0) \\ & \times \sum_{k=1}^{N_0} \Delta_k(\mathbf{z}, t_0)\, g_k(\mathbf{v}, t_0)\, P_0(\mathbf{v}, \mathbf{w})\, d\mathbf{z}\, d\mathbf{w}\, d\mathbf{v} \\ & + \int_X \int_X \int_X \int_X h(\mathbf{x}, t_1; \mathbf{z}, t_0)\, h(\mathbf{y}, t_1; \mathbf{w}, t_0) \\ & \times \sum_{k=1}^{N_0} \sum_{j=1}^{N_0} \Delta_k(\mathbf{z}, t_0)\, \Delta_j(\mathbf{w}, t_0) \\ & \times g_k(\mathbf{u}, t_0)\, g_j(\mathbf{v}, t_0)\, P_0(\mathbf{u}, \mathbf{v})\, d\mathbf{z}\, d\mathbf{w}\, d\mathbf{u}\, d\mathbf{v} \\ & + \int_X \int_X h(\mathbf{x}, t_1; \mathbf{z}, t_0)\, h(\mathbf{y}, t_1; \mathbf{w}, t_0) \\ & \times \sum_{k=1}^{N_0} \sum_{j=1}^{N_0} \Delta_k(\mathbf{z}, t_0)\, \Delta_j(\mathbf{w}, t_0)\, R_{kj}(t_0)\, d\mathbf{z}\, d\mathbf{w} \\ & + Q_1(\mathbf{x}, \mathbf{y}). \end{aligned} \quad (21)$$

In eq. (21), the fourth and fifth terms may be combined by use of the definition (13); this in turn cancels the third term because of relation (14). Thus,

$$P_1(\mathbf{x},\mathbf{y}) = \int_X \int_X h(\mathbf{x}, t_1: \mathbf{z}, t_0)\, h(\mathbf{y}, t_1; \mathbf{w}, t_0) \times \left[P_0(\mathbf{z}, \mathbf{w}) - \sum_{k=1}^{N_0} \Delta_k(\mathbf{w}, t_0 \times \int_X g_k(\mathbf{v}, t_0)\, P_0(\mathbf{z}, \mathbf{v})\, d\mathbf{v}\right] d\mathbf{z}\, d\mathbf{w} + Q_1(\mathbf{x}, \mathbf{y}) \quad (22)$$

expresses P_1 in terms of $P_0(\mathbf{x}, \mathbf{y})$ and $\mathbf{\Delta}(\mathbf{x}, t_0)$. Equation (22) in conjunction with (16) allows iterative solution for the error statistics from one time step to the next. In addition, if steady-state conditions may be postulated, the subscript on P may be dropped and eqs. (22) and (16) solved for P and the analysis updating functions $\{\Delta_k(\mathbf{x})\}$.

Equation (22) may also be used to assess the error in forecasting over any time interval. It is probably of interest to the operational user to have only the means-quare error $P(\mathbf{x}, \mathbf{x})$ or possibly some areal average of $P(\mathbf{x}, \mathbf{x})$. The analysis error correlation is expressed by the bracket in the integral of eq. (22).

Application to real-time geophysical analyses

Let us conceive of the following mode of operation for a real-time geophysical data processing center, organized to exploit the technique of continuing analysis described above.

Stored in the computer memory in any suitable form (e.g., grid-point samples) is a synoptic analysis of the geophysical field, valid for current time and extrapolated step by step in real time according to the applicable equations of field dynamics. Also stored is a covariance function embodying the statistics of the errors in the current analysis, updated in step with the analysis itself.[1]

When one or more data are received as a result of planned or casual observations, they are categorized as to locality, instrument characteristics, and expected accuracy. The stored analysis is then operated on in a manner corresponding to the reporting instrument characteristic (e.g., sampling or local spatial

[1] It may be desirable to have the stored analysis lag behind "real time" by an amount equivalent to a reasonable communication delay.

averaging), and these results are compared with the received data. Weighting functions are calculated, based on the current analysis error statistics and the characteristics of the new observations, and used to update both the synoptic analysis and its associated error function.

At desired intervals, the current analysis may be extrapolated into the future faster than real time for the issuance of forecasts. The accompanying error field is also readily available; and since errors are not expected to be homogeneous, this should prove a useful adjunct for the purpose of assessing the deterioration of forecast validity with time.

The author is not, of course, proposing that observations of geophysical phenomena become completely random in space and time. Emphasis is rather placed on the fact that suitable methods of analysis can be developed for any arrangement of observations in space and time. Promising instrumentation techniques (e.g., scanning from low-orbit satellites) need not be discarded or used only peripherally because they are inherently non-synoptic; nor is great expense justified to obtain large numbers of spatially-distributed sensors and wide-bandwidth communications. Certainly, some techniques will prove more efficacious than others in terms of output (forecast) performance, others more economical; the point is that many more possibilities may now be explored.

Measurements on periodic spatial lattices

A special situation, but one of considerable interest as a limiting case, is one in which similar measurements are made simultaneously on an unbounded periodic spatial lattice in a homogeneous physical field. The matrix $\mathbf{K}(t_0)$ is then of infinite order; however, the periodicity of the observation lattice and the resulting homogeneity of the error weighting function allow a solution to be obtained in a wave-number domain.

Let the observation operation be expressed as

$$g_k(\mathbf{x}, t) = g(\mathbf{x} - \mathbf{v}_{[k]}, t), \tag{23}$$

where

$$\mathbf{v}_{[k]} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_N\mathbf{v}_N;\ k_1, k_2, \ldots, k_N = 0, \pm 1, \pm 2, \ldots, \tag{24}$$

and the vectors $\{\mathbf{v}_j\}$ form a basis for the N-dimensional spatial domain (usually, $N = 1$, 2 or 3). It is clear that the error covariance functions $P(\mathbf{x}, \mathbf{y})$ are then also periodic in the sense that

$$P(\mathbf{x} - \mathbf{v}_{[k]}, \mathbf{y} - \mathbf{v}_{[k]}) = P(\mathbf{x}, \mathbf{y}), \tag{25}$$

so that the integrals $\int_x g_k(\mathbf{z}, t_0) P_0(\mathbf{x}, \mathbf{z}) d\mathbf{z} = \Gamma_k(\mathbf{x}, t_0)$ are also functions of $(\mathbf{x} - \mathbf{v}_{[k]})$. Furthermore, the elements K_{jk} of the matrix $\mathbf{K}$ become functions of $(\mathbf{v}_{[j]} - \mathbf{v}_{[k]})$. Thus, the elements of the error weighting functions $\{\Delta_k(\mathbf{x}, t_0)\}$ are also functions of the displacement of the point $\mathbf{x}$ from the observation point $\mathbf{v}_{[k]}$:

$$\Delta_k(\mathbf{x}, t_0) = \Delta(\mathbf{x} - \mathbf{v}_{[k]}, t_0). \tag{26}$$

We thus have, from eq. (14),

$$\Gamma_0(\mathbf{x} - \mathbf{v}_{[k]}) = \sum_{[j]} \Delta_0(\mathbf{x} - \mathbf{v}_{[j]}) K_0(\mathbf{v}_{[j]} - \mathbf{v}_{[k]}), \tag{27}$$

where the subscript now indicates the corresponding time index, and the summation is of order N over a doubly-infinite range in all indices (see eq. (24)).

Equation (27) may readily be solved in the wave-number domain. By a change of variables we write

$$\Gamma_0(\mathbf{x}) = \sum_{[k]} K_0(\mathbf{v}_{[k]}) \Delta_0(\mathbf{x} - \mathbf{v}_{[k]}), \tag{28}$$

and express the right-hand side as a delta-modulated field,

$$\Gamma_0(\mathbf{x}) = \sum_{[k]} \int_X K_0(\mathbf{z}) \Delta_0(\mathbf{x} - \mathbf{z}) \delta(\mathbf{z} - \mathbf{v}_{[k]}) d\mathbf{z}. \tag{29}$$

Expanding the delta summation in Fourier series, we have

$$\Gamma_0(\mathbf{x}) = \frac{1}{Q} \sum_{[k]} \int_X K_0(\mathbf{z}) \Delta_0(\mathbf{x} - \mathbf{z}) \exp i\mathbf{z} \cdot \mathbf{u}_{[k]} d\mathbf{z}, \tag{30}$$

in which Q is the hypervolume of the cell defined by the basis vector set $\{\mathbf{v}_j\}$ and the vectors $\{\mathbf{u}_k\}$ are orthogonal to the set $\{\mathbf{v}_j\}$, i.e.,

$$\mathbf{u}_k \cdot \mathbf{v}_j = 2\pi\delta_{kj} = 2\pi \begin{Bmatrix} 1, & k = j \\ 0, & k \neq j \end{Bmatrix}. \tag{31}$$

Equation (30) is now transformed to the wave-number domain for the final result,

$$\Gamma_0(\mathbf{v}) = \frac{\Delta_0(\mathbf{v})}{Q} \sum_{[k]} K_0(\mathbf{v} - \mathbf{u}_{[k]}), \qquad (32)$$

which is immediately solvable for the weighting transfer function $\Delta_0(\mathbf{v})$.

The striking similarity of eq. (32) to results previously published (Petersen & Middleton, 1962, 1963) is, of course, not unexpected. Whereas, in the basic theory of multidimensional sampling, the field itself is reconstructed on the basis of its own statistics, in sequential analysis it is the *new information* afforded by the current set of observations which is interpolated and added to the field extrapolated from previous data. The statistics involved are those of the *forecast error* rather than those of the field itself.

The recursion formula (22) for the error covariance function can also be written in the wave-number domain. However, since the error covariance is in general nonhomogeneous, a doubly-dimensional transformation is required. Defining

$$P(\boldsymbol{\mu}, \mathbf{v}) = \int_X \int_X P(\mathbf{x}, \mathbf{y}) \exp -i\mathbf{x} \cdot \boldsymbol{\mu} \exp -i\mathbf{y} \cdot \mathbf{v} \, d\mathbf{x} \, d\mathbf{y}, \qquad (33)$$

we write

$$P_1(\boldsymbol{\mu}, \mathbf{v}) = \int_X \int_X \int_X \int_X h(\mathbf{x} - \mathbf{z}, t_1 - t_0) \, h(\mathbf{y} - \mathbf{w}, t_1 - t_0) \times \left[P_0(\mathbf{z}, \mathbf{w}) - \sum_{[k]} \Delta_0(\mathbf{w} - \mathbf{v}_{[k]}) \, \Gamma_0(\mathbf{z} - \mathbf{v}_{[k]}) \right] \times \exp -i\mathbf{x} \cdot \boldsymbol{\mu} \exp -i\mathbf{y} \cdot \mathbf{v} \, d\mathbf{x} \, d\mathbf{y} \, d\mathbf{z} \, d\mathbf{w} + Q_1(\boldsymbol{\mu}, \mathbf{v}). \qquad (34)$$

Using Parseval's theorem, we can express eq. (34) as

$$P_1(\boldsymbol{\mu}, \mathbf{v}) = H(\boldsymbol{\mu}, t_1 - t_0) \, H(\mathbf{v}, t_1 - t_0) \times \left[P_0(\boldsymbol{\mu}, \mathbf{v}) - \sum_{[k]} \Delta_0(\mathbf{v}) \, \Gamma_0(\boldsymbol{\mu}) \exp -i\mathbf{v}_{[k]} \cdot (\boldsymbol{\mu} + \mathbf{v}) \right] + Q_1(\boldsymbol{\mu}, \mathbf{v}). \qquad (35)$$

Substituting from eq. (32), and observing that the sum of exponentials may be expressed as a delta-function series, we obtain

$$P_1(\boldsymbol{\mu}, \mathbf{v}) = H(\boldsymbol{\mu}, t_1 - t_0) \, H(\mathbf{v}, t_1 - t_0) \times \left[P_0(\boldsymbol{\mu}, \mathbf{v}) - \frac{(2\pi)^n \Gamma_0(\mathbf{v}) \, \Gamma_0(\boldsymbol{\mu}) \sum_{[k]} \delta(\boldsymbol{\mu} + \mathbf{v} - \mathbf{u}_{[k]})}{\sum_{[k]} K_0(\mathbf{v} - \mathbf{u}_{[k]})} \right] + Q_1(\boldsymbol{\mu}, \mathbf{v}). \qquad (36)$$

The recursion is completed by writing

$$\Gamma_m(\mathbf{v}) = \frac{1}{(2\pi)^n} \int_X P_m(\mathbf{v}, \boldsymbol{\zeta}) \, g(-\boldsymbol{\zeta}) \, d\boldsymbol{\zeta} \qquad (37)$$

and [from eq. (13)]

$$K(\mathbf{v}) = G(-\mathbf{v}) \, \Gamma(\mathbf{v}) + R(\mathbf{v}). \qquad (38)$$

If desired, we may reconstruct the synoptic field in the wave-number domain directly from the observation errors. We transform eq. (8) [in view of eq. (26)] to

$$\hat{a}_{Y_0}(\mathbf{v}, t_0) = \hat{a}_{Y-1}(\mathbf{v}, t_0) + \Delta_0(\mathbf{v}) \sum_{[k]} \exp -i\mathbf{v} \cdot \mathbf{v}_{[k]} \times [\beta_{[k]}(t_0) - \hat{\beta}_{[k]Y-1}(t_0)]. \qquad (39)$$

Then, from eq. (18), we have

$$\hat{a}_{Y_0}(\mathbf{v}, t_1) = H(\mathbf{v}, t_1 - t_0) \, \hat{a}_{Y_0}(\mathbf{v}, t_0) \qquad (40)$$

as the description of the prognostic field in the wave-number domain. The mean-square prognostic error at time t_m is

$$P_m(\mathbf{x}, \mathbf{x}) = \frac{1}{(2\pi)^{2N}} \int_X \int_X P_m(\boldsymbol{\mu}, \mathbf{v}) \exp i\mathbf{x} \cdot (\boldsymbol{\mu} + \mathbf{v}) \, d\boldsymbol{\mu} \, d\mathbf{v}, \qquad (41)$$

while the error of the synoptic analysis is

$$P_m^{(a)}(\mathbf{x}, \mathbf{x}) = P_m(\mathbf{x}, \mathbf{x}) - \sum_{[k]} \frac{(2\pi)^N}{Q} \exp i\mathbf{u}_{[k]} \cdot \mathbf{x} \times \int_X \Gamma_m(\boldsymbol{\mu}) \, \Delta_m(\mathbf{u}_{[k]} - \boldsymbol{\mu}) \, d\boldsymbol{\mu}. \qquad (42)$$

A simple wave equation

Let us consider, as a first example, a field $\alpha(\mathbf{x}, t)$ governed by the simple unforced wave equation

$$\frac{\partial \alpha}{\partial t} + U \frac{\partial \alpha}{\partial x} = 0, \qquad (43)$$

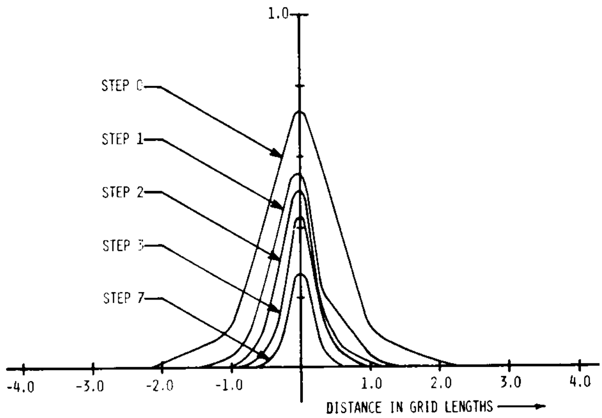

Fig. 1. Interpolation weighting functions—simple wave.

whose wave-number/time transfer function is

$$H(\nu, t) = \exp -i\nu Ut. \tag{44}$$

We assume that, at $t = 0$, the spatial autocorrelation function of the wave is $\exp -\lambda|x|$; the corresponding wave-number spectrum is

$$P_0(\mu, \nu) = \frac{4\pi\lambda}{\mu^2 + \lambda^2}\,\delta(\mu + \nu). \tag{45}$$

We postulate discrete sampling measurements to be made, starting at $t = 0$, at intervals of $1/4\lambda$ over the entire spatial line,[1] and repeated at time intervals τ_0. Noise of variance 0.1, uncorrelated from point to point, contaminates the observations. Note that, since the initial wave merely translates in time, if $U\tau_0$ is incommensurate with $1/4\lambda$ we will eventually "fill in" the entire function with vanishing error; if the original sampling points recur after N time steps, we will eventually have the equivalent of noise-free observations with a density N times that actually existing.

For this situation, the measurement function $g_k(x)$ becomes a Dirac impulerse $\delta(x - k/4\lambda)$, and its transfer function is unity. The aliasing wave-number interval is here $u = 8\pi\lambda$. We make take the measurement "noise" in the wave-number domain to be flat, of amplitude $0.1/4\lambda$, for $|\nu| < 4\pi\lambda$ and zero elsewhere. Thus, from eqs. (32), (35), (38), and (42) we have

$$\Gamma_m(\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_m(\mu, \zeta)\,d\zeta; \tag{46}$$

[1] This is $1/4\pi$ times the nominal "Nyquist" interval.

$$\Delta_m(\nu) = \frac{1}{4\lambda}\,\frac{\Gamma_m(\nu)}{\sum_{-\infty}^{\infty} \Gamma_m(\nu - 8\pi k\lambda) + 0.1/4\lambda}; \tag{47}$$

$$P_{m+1}(\mu, \nu) = \exp -i(\mu + \nu)\,U\tau_0 \times [P_m(\mu, \nu) - 8\pi\lambda\Gamma_m(\mu)\,\Delta_m(\nu) \sum_{-\infty}^{\infty} \delta(\mu + \nu - 8\pi k\lambda)]. \tag{48}$$

The above equations were programmed on a digital computer. The iterative evolution of the interpolation weighting function $\Delta_m(x)$ and the mean-square analysis error $P_m(x, x)$, for $U\tau_0 = \frac{1}{4}$ of the sampling interval, are shown in Figs. 1 and 2. We note that, as the wave extrapolated from previous analyses becomes ever more precisely known, the weighting placed on new data decreases at sampling points and drops off more and more sharply away from the sampling points. At the same time, the influence of the prognostic information approaches unity at all points. In the limit (for this example), the wave is effectively sampled, with vanishing noise, at 16π times the Nyquist density. The steady-state error is shown in Fig. 2; the weighting on new observations vanishes as all observed deviations are attributed to instrument noise.

A diffusion problem

We next postulate a one-dimensional spatial field which may, for example, represent the distribution of temperature along an infinite bar.

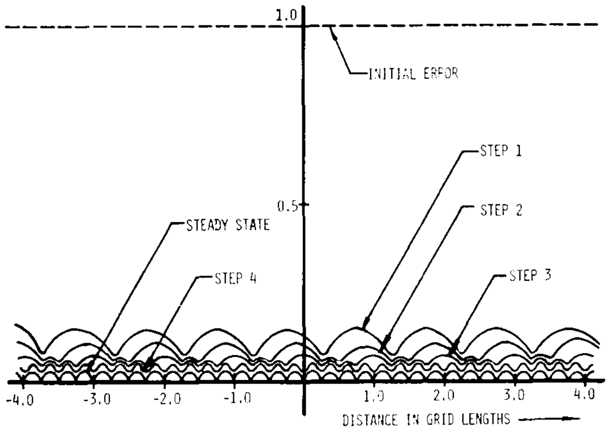

Fig. 2. Mean-square analysis/prognosis error—simple wave.

We write the differential equation for this system as

$$D\frac{\partial^2\theta(x,t)}{\partial x^2}-\frac{\partial\theta(x,t)}{\partial t}-\frac{\theta(x,t)}{\tau}=f(x,t), \quad (49)$$

in which D is a diffusion parameter, τ is a leakage constant, and $f(x, t)$ is a thermal excitation which we take to be "white" in both distance and time:

$$E\{f(x,t)f(y,u)\}=K_s\delta(x-y)\delta(t-u). \quad (50)$$

The "Green's function" corresponding to eq. (49) is readily found to be

$$h(x,t)=\begin{Bmatrix}0, t<0 \\ \frac{1}{\sqrt{4\pi Dt}}\exp -x^2/4Dt-t/\tau, \quad t>0\end{Bmatrix}. \quad (51)$$

The space/time statistics of the field defined by eq. (49), excited by the input (50), have been calculated (Petersen, 1963). We require here only the steady-state spatial autocovariance function

$$K(x)=\frac{K_s}{4}\sqrt{\frac{\tau}{D}}\exp -|x|/\sqrt{\tau D}, \quad (52)$$

which serves as the initial prediction error covariance at the time the first measurements are made.[1] In the wave-number domain, the expressions corresponding to (51) and (52) are

$$H(\nu,t)=\begin{Bmatrix}0, t<0 \\ \exp -(D\nu^2+1/\tau)t, \quad t>0\end{Bmatrix} \quad (53)$$

and
$$K(\nu)=\frac{K_s}{2D}\cdot\frac{1}{\nu^2+1/\tau D}. \quad (54)$$

We again need to postulate a measurement scheme. To correspond with the previous example for comparison, we will investigate two situations: (1) Three observations, spaced at $\sqrt{\tau D}/4$,[2] are made every τ_0 seconds. Measurement variance is 0.1 and is uncorrelated from point to point; the constant $(K_s/4)\sqrt{\tau/D}$ is normalized to unity. (2) The field is sampled over the entire spatial line at $\sqrt{\tau D}/4$ intervals, with measurement variance 0.1 uncorrelated from point to point; the synoptic observation is repeated at τ_0-second intervals.

Case 1

The measurement operation $g_k(x)$ is taken as a delta modulation $\delta(x-k\sqrt{\tau D}/4)$. Thus, eqs. (13) and (16) become

$$K_{ij}(m\tau_0)=P_m\left(i\frac{\sqrt{\tau D}}{4}, j\frac{\sqrt{\tau D}}{4}\right)+0.1\delta_{ij};$$
$$i,j=1,2,3, \quad (55)$$

and

$$\Delta_k(x,m\tau_0)=\sum_{j=1}^{3}[\mathbf{K}^{-1}(m\tau_0)]_{jk}P_m\left(x,j\frac{\sqrt{\tau D}}{4}\right);$$
$$k=1,2,3. \quad (56)$$

[1] We assume that the field itself has reached steady state. For $t=t_0$ our "forecast" is the steady-state mean which is here zero.

[2] Note that the spatial sampling interval is again $1/4\pi$ times the "Nyquist" interval.

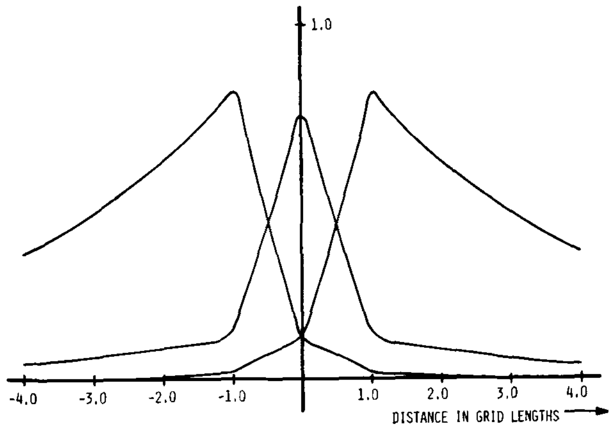

Fig. 3. Interpolation weighting functions—thermal field. (Case 1: time step 0.5 τ, steady state.)

The accumulated excitation $Q(x,y)$ may be found using eqs. (20), (50), and (51):

$$Q(x,y) = \int_{-\infty}^{\infty}\int_{\varrho=0}^{\tau_0} \frac{K_s}{4\pi D(\tau_0-\varrho)} \times \exp -\left(\frac{(x-z)^2}{4D(\tau_0-\varrho)} + \frac{(\tau_0-\varrho)}{\tau} - \frac{(y-z)^2}{4D(\tau_0-\varrho)} + \frac{(\tau_0-\varrho)}{\tau}\right) dz\,dp$$

$$= \int_0^{\tau_0} \frac{K_s}{2\sqrt{2\pi D\sigma}} \exp -\left(\frac{2\sigma}{\tau} + \frac{(x-y)^2}{8D\sigma}\right) d\sigma. \tag{57}$$

The iteration formula (22) for $P(x,y)$ becomes

$$P_{m+1}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi D\tau} \times \exp -\left(\frac{(x-z)^2}{2D\tau} + \frac{1}{2} + \frac{(y-w)^2}{2D\tau} + \frac{1}{2}\right) \times \left[P_m(z,w) - \sum_{k=1}^{3} \Delta_k(w, m\tau_0)\, P_m(z, k\sqrt{\tau D}/4)\right] \times dz\,dw + Q(x,y), \tag{58}$$

with $P_0(z,w) = \exp -(|z-w|/\sqrt{\tau D})$ from eq. (52).

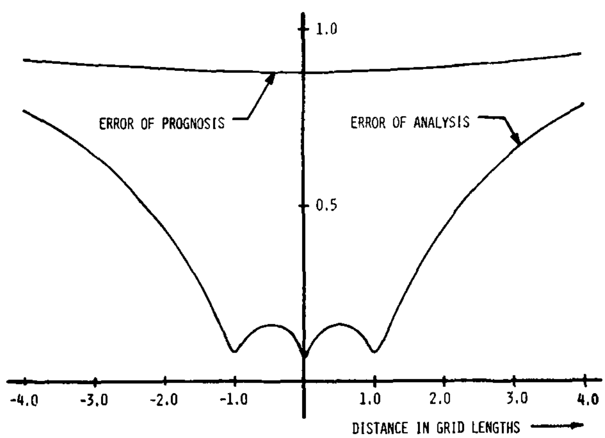

Fig. 4. Mean-square analysis/prognosis errors—thermal field. (Case 1: time step 0.5 τ, steady state.)

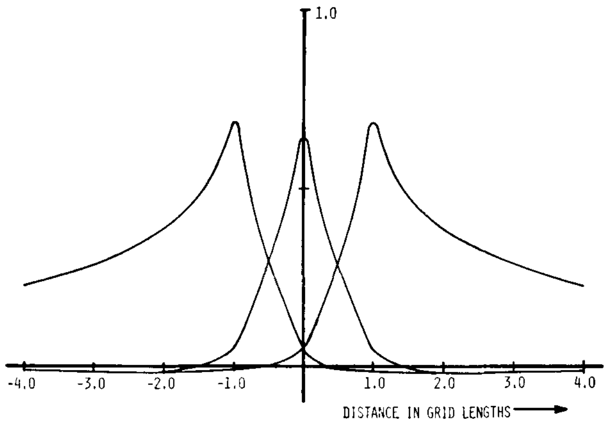

Fig. 5. Interpolation weighting functions—thermal field. (Case 1: time step 0.01 τ, steady state.)

Case 2

This may be treated in the wave-number domain as in the simple wave example. Equations (46) and (47) continue to hold, while eq. (48) becomes

$$P_{m+1}(\mu,\nu)=\exp-(\tau_0/\tau)\,[2+\tau D(\mu^2+\nu^2)]$$
$$\times\left[P_m(\mu,\nu)-\frac{8\pi}{\sqrt{\tau D}}\Gamma_m(\mu)\,\Delta_m(\nu)\sum_{-\infty}^{\infty}\delta\left(\mu+\nu-\frac{8\pi k}{\sqrt{\tau D}}\right)\right]$$
$$+Q(\mu,\nu), \quad (59)$$

with the initial condition (45). The accumulated excitation $Q(\mu,\nu)$ is the two-dimensional Fourier transform of eq. (57):

$$Q(\mu,\nu)=\frac{4\pi}{\sqrt{\tau D}}\frac{1}{\mu^2+1/\tau D}$$
$$\times[1-\exp-(2\tau_0/\tau(1+\tau D\mu^2))]\,\delta(\mu+\nu). \quad (60)$$

Because of the rapid "wave-number extinction" characteristic of eq. (53), we need retain only the $\delta(\mu+\nu)$ term in eq. (59).[1] Also, since

$$H(\mu,t_1-t_0)\,H(\nu,t_1-t_0)\,P_0(\mu,\nu)+Q(\mu,\nu)=P_0(\mu,\nu), \quad (61)$$

we may write here

$$\Gamma_m(\nu)=\Gamma_0(\nu)-\frac{4}{\sqrt{\tau D}}\sum_{l=1}^{m}\exp-(2l(\tau_0/\tau))\,(1+\tau D\nu^2)$$
$$\times\Gamma_{m-l}(\nu)\,\Delta_{m-l}(-\nu). \quad (62)$$

Finally, the error of prognosis is obtained here as

[1] This implies that the prediction error is essentially stationary in space; i.e., is the same at sampling and non-sampling points. The error for time steps of 10^{-3} is on the order of 10^{-4}.

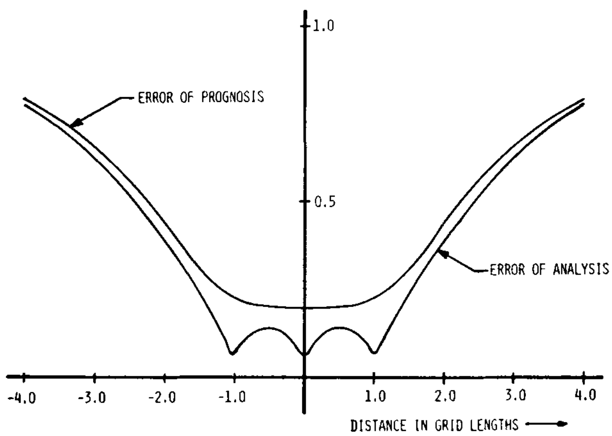

Fig. 6. Mean-square analysis/prognosis errors—thermal field. (Case 1: time step 0.01 τ, steady state.)

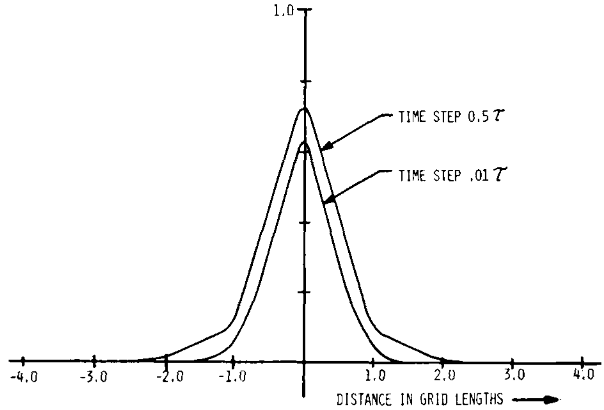

Fig. 7. Interpolation weighting functions—thermal field. (Case 2: steady state.)

$$P_m(x,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma_m(\mu)\,d\mu, \tag{63}$$

while the analysis error is found from eq. (42):

$$P_m^{(a)}(x,x) = P_m(x,x) - \sum_{-\infty}^{\infty} \frac{8\pi}{\sqrt{\tau D}} \exp i\,(8\pi k/\sqrt{\tau D})\,x \times \int_{-\infty}^{\infty} \Gamma_m(\mu)\,\Delta_m\left(\frac{8\pi k}{\sqrt{\tau D}} - \mu\right) d\mu. \tag{64}$$

The indicated iterations of eqs. (55), (56) and (58) for Case 1 and eqs. (47) and (62) for Case 2 were programmed for the IBM 360 computer for two values of τ_0. Fig. 3 shows the interpolation weighting functions attained in the steady state (after an extremely short transient) for Case 1, $\tau_0 = 0.5\ \tau$. Fig. 4 shows the mean-square errors $P(x, x)$ of the analysis and the prognosis for the same situation. In Figs. 5 and 6, the time step is 0.01 τ; here the prognosis is much more accurate and is more heavily weighted as compared with the new (noisy) observations. Figs. 7 and 8 refer to Case 2, in which an infinity of samples is available at each time step. Here the analysis error is little affected by the improvement in prognostic error under the shorter time step, although the new data are somewhat less strongly weighted.

Study is under way of the application of these techniques to a field whose dynamic properties are expressed by the Rossby wave equation.

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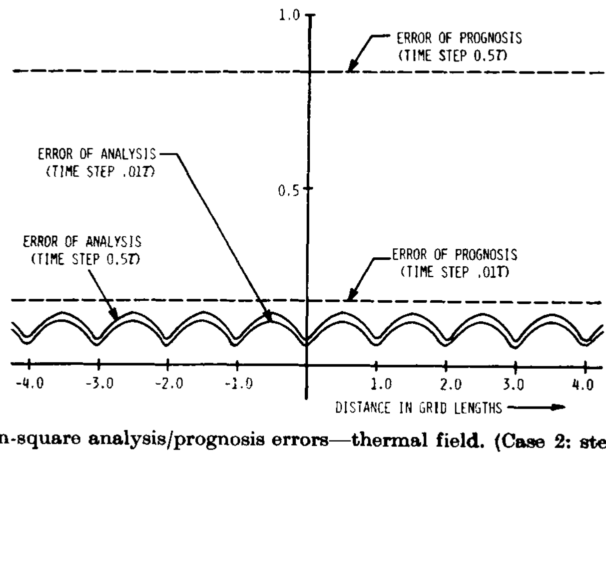

Fig. 8. Mean-square analysis/prognosis errors—thermal field. (Case 2: steady state.)

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О ПОНЯТИИ И ИСПОЛЬЗОВАНИИ ПОСЛЕДОВАТЕЛЬНОГО АНАЛИЗА ДЛЯ ЛИНЕЙНЫХ СЛУЧАЙНЫХ ПОЛЕЙ

Выведены формулы для оптимального (в смысле среднего квадратического) линейного смешения прогностической информации и синоптических наблюдений в объективном анализе. Использование этих результатов делает возможным непрерывный синоптический анализ, модернизирующейся по мере поступления новых данных, без необходимости одновременных глобальных наблюдений. Для случая неограниченной равномерной сетки в однородном поле получены решения в области волновых чисел. В качестве примеров рассмотрены одномерное волновое уравнение и задача о распространении тепла.