A composite Ekman boundary layer problem

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ABSTRACT

The steady, axially symmetric, viscous, incompressible flow induced by an infinite rotating disk in the presence of a stationary coaxial disk is studied numerically for the complete laminar range of a Taylor number, R. Emphasis is placed on qualitative interpretation of the numerical solutions. The asymptotic state for large R is shown to consist of two well-known special Ekman boundary layers on the disks which are linked by a Taylor column, with Ekman layer suction into one layer just balanced by axial outflow from the other layer.

1. Introduction

The concern here is with steady viscous axially symmetric flow between infinite coaxial disks separated by a distance d when one disk (at $z=0$, say) rotates at constant angular speed Ω, and the other (at $z=d$) is held at rest. The relevant parameter characterizing the flow may be taken as a rotational Reynolds number, a Taylor number, or an Ekman number. Following Greenspan & Howard (1963), we choose a Taylor number $R = d^2\Omega/\nu$ and note that it is simply the square of the ratio of total fluid depth to the depth of an Ekman layer. The interest is in the complete laminar range of R.

This generalization of the classical von Kármán (1921), single disk rotating boundary layer was first discussed qualitatively by Batchelor (1951). He suggested that for large R the flow ought to consist of two thin boundary layers on the disks separated by an essentially inviscid region in uniform rotation. The angular velocity of the core, Ω_c, was presumed to be that value which just linked the two boundary layers by matching the axial outflow from one with the inflow to the other. Thus, in Batchelor's view the rotating boundary layers are capable of exerting a strong influence on the interior flow far away from the boundaries and this distinctive feature of rotating boundary layers is what endows the problem with geophysical significance.

In 1953, Stewartson proposed an alternative description of the flow based on analytical-numerical methods applied to approximate boundary layer equations. He concluded that the interior could be completely free of rotation, in which case the two disks are effectively uncoupled and no torque is transmitted to the fixed disk. This last prediction was "confirmed" experimentally by measuring the torque developed in a finite disk apparatus. It is now clear (e.g. Rott & Lewellen, 1966), that either type of flow is possible for finite disks depending crucially upon whether or not the disks are encased in a shroud. Here Batchelor's similarity solution for infinite disks is studied.

More recently, accurate numerical solutions have been obtained by various workers for several Taylor numbers. For example, Lance & Rogers (1962), present plots of two of the three velocity components for $R = 25$, 81, 225 and 441 and Pearson's (1965), transient calculations extend to the steady state for $R = 100$, 1000. Nevertheless, a complete understanding of the flow has not yet been attained.

Since the problem represents such a pure example of the strong interaction aspect of rotating boundary layers, it deserves a careful and complete examination. In this paper, the Taylor number has been varied systematically from zero up to 2500 (which lies well within the asymptotic range) and the smooth dependence of the flow on R is exposed. Three distinct regimes in Taylor number are revealed and a simple but complete qualitative interpretation of the asymptotic regime is made.

2. Formulation

The steady, axially symmetric, incompressible continuity and Navier Stokes equations in non-rotating cylindrical coordinates r, θ, z with respective velocity components u, v, w are:

$$(ru)_r + rw_z = 0, \tag{1}$$

$$uu_r + wu_z - \frac{1}{r}v^2 = -\frac{1}{\varrho}p_r + \nu\left[u_{rr} + \left(\frac{1}{r}u\right)_r + u_{zz}\right], \tag{2}$$

$$uv_r + \frac{1}{r}uv + wv_z = \nu\left[v_{rr} + \left(\frac{1}{r}v\right)_r + v_{zz}\right], \tag{3}$$

$$uw_r + ww_z = -\frac{1}{\varrho}p_z + \nu\left[w_{rr} + \frac{1}{r}w_r + w_{zz}\right]. \tag{4}$$

Appropriate boundary conditions are:

$$u = w = 0, \quad v = r\Omega \quad \text{at} \quad z = 0, \tag{5}$$

$$u = v = w = 0 \quad \text{at} \quad z = d. \tag{6}$$

Non-dimensionalization and reduction of the system to ordinary differential equations can be accomplished in several ways. The most compact and revealing one for the present purposes is (following Batchelor) to retain von Kármán's notation slightly generalized and write:

$$u(r,z) = -\tfrac{1}{2}r\Omega H'(\zeta), \quad v(r,z) = r\Omega G(\zeta) \tag{7}$$

$$w(z) = (\nu\Omega)^{\frac{1}{2}}H(\zeta), \quad p(r,z) = K\varrho r^2\Omega^2 - \mu\Omega P(\zeta),$$

$$\zeta = (\Omega/\nu)^{\frac{1}{2}}z, \quad R = d^2\Omega/\nu.$$

Here K is a constant to be determined and the continuity equation is automatically satisfied. With this scaling, since the fluid occupies the region $0 \leqslant z \leqslant d$, the range of z is from 0 to $R^{\frac{1}{2}}$. This explains why Lance & Rogers in practice found $R^{\frac{1}{2}}$ rather than R to be the relevant parameter. Also, increase in Taylor number (more precisely $R^{\frac{1}{2}}$) is associated, not with increase in angular speed Ω, as in Pearson's work, but rather with increase in disk separation, d. With these definitions equation (4) determines the pressure function $P(z)$ after u and w have been found. The unknown constant K, which will occur in equation (2) is conveniently eliminated by differentiating (2) with respect to z. Thus, the problem reduces to solving the sixth order coupled system of non-linear ordinary differential equations:

$$G'' - HG' + H'G = 0, \tag{8}$$

$$H'''' - HH''' = 4GG'. \tag{9}$$

The dependence on Taylor number enters through the boundary conditions which are now

$$G(0) = 1, \quad G(R^{\frac{1}{2}}) = H(0) = H'(0) = H(R^{\frac{1}{2}}) = H'(R^{\frac{1}{2}}) = 0. \tag{10}$$

3. Method of numerical solution

The two point boundary value problem posed by equations (8)–(10) was accurately solved for $0 \leqslant R^{\frac{1}{2}} \leqslant 50$ by essentially the same technique used by Lance and Rogers, the only difference being that it was not found necessary to treat each boundary layer separately for large R. The numerical method rests on a simple yet accurate analytic solution valid for small R. Such a "narrow gap" solution is used as an iterative basis for estimating extra boundary conditions so that the problem becomes a one-point boundary value problem. Convergence is assured by taking sufficiently small steps in parameter space.

Numerical accuracy (not discussed by Lance and Rogers) was verified in several ways in addition to the customary check of refining the mesh size until it produced no appreciable change in results (the final mesh used was $\Delta\zeta = 0.01$). For one thing, the velocity profiles coincide with those found by other workers at appropriate values of R. An independent check is obtained by comparing the numerical solutions with the first integral of equation (9) which is:

$$H''' - HH'' + \tfrac{1}{2}H'H' - 2G^2 = \text{constant}. \tag{11}$$

Numerical evaluation of the left hand member typically remained independent of ζ to one part in 10^6. A final check was made by substituting the boundary conditions (10) into (11) which shows that

$$H'''(0) - H'''(R^{\frac{1}{2}}) = 2 \tag{12}$$

and this equation was also satisfied to one part in a million.

4. Results and discussion

The velocity profiles as functions of ζ are shown in Figs. 1–3. The value of Taylor number

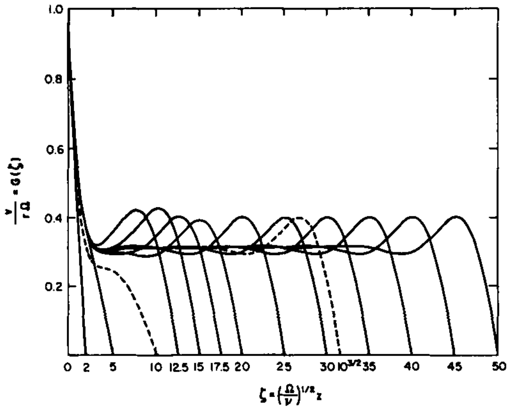

Fig. 1. Circumferential flow profiles for Taylor numbers up to 2500.

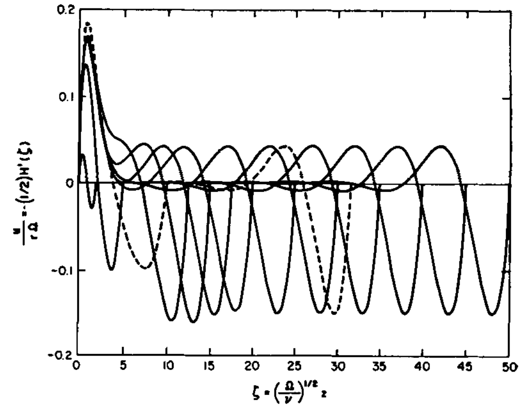

Fig. 3. Radial flow profiles for Taylor numbers up to 2500.

appropriate to each curve is simply the square of the right hand endpoint value of ζ; for example the dashed curves terminating at $\zeta = 10$, $10^{3/2}$ are the cases treated by Pearson ($R = 100$, 1000). The initially unknown extra boundary conditions at $\zeta = 0$ required to make the problem a one-point boundary value problem, and found in the numerical solution, have been plotted by Lance & Rogers for R up to 441, and will not be repeated here. However, the asymptotic values found for $R = 2500$, listed for reference, are $G'(0) = -0.5243$, $H''(0) = -0.9454$, $H'''(0) = 1.804$.

It is of interest to know how much of the viscous torque developed on the rotating disk is transmitted to the fixed disk. Ordinarily, one might expect equal and opposite torques, but the fluid here is not fully confined and some of it leaks out to radial infinity where there may well be an effective sink for angular momentum. Since each disk is of unbounded radius, the total torque on each is infinite. However, if T_0, T_1 are the torque magnitudes developed on the rotating and stationary disks, respectively, out to some finite radius $r = a$, then the torque transmission ratio is

$$\frac{T_1}{T_0} = \frac{G'(R^{1/2})}{G'(0)}. \tag{13}$$

This ratio is plotted as a function of $R^{1/2}$ in Fig. 4. When the disks are sufficiently close together, the torque transmission is complete, but following some interesting variations it

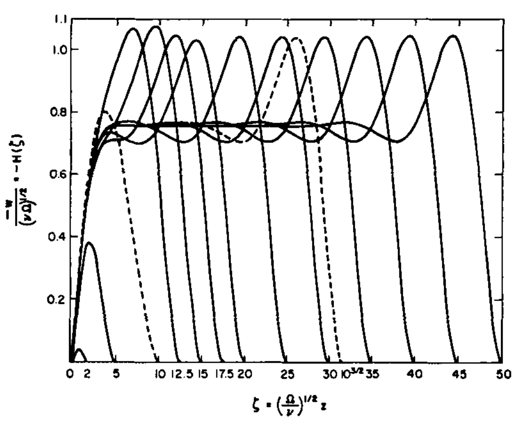

Fig. 2. Axial flow profiles for Taylor numbers up to 2500.

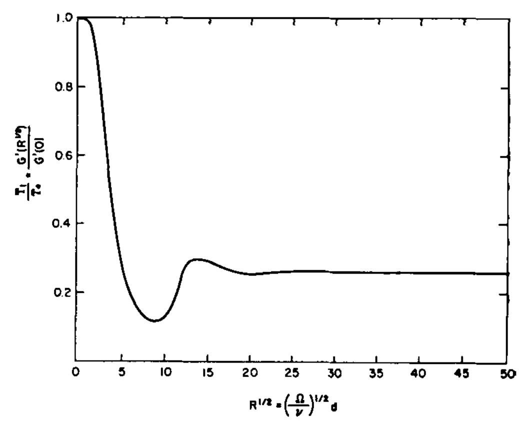

Fig. 4. Torque transmission ratio as a function of disk spacing.

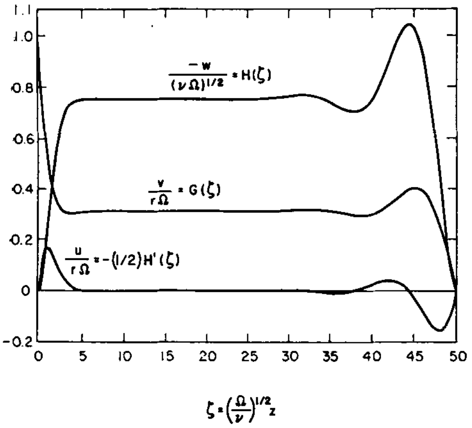

Fig. 5. Velocity profiles for Taylor number = 2500.

reaches an asymptotic value of only 0.258 when the disks are widely separated.

An examination of Figs. 1–3 reveals that the flow dependence on R can be subdivided into three regimes. For $0 \leqslant R^{\frac{1}{2}} \leqslant 2$, or for Taylor numbers up to about 4, the circumferential flow profile is nearly linear, so we have rotating Couette flow. In this regime of small R, the analytic small gap approximation for what is essentially a Stokes flow is accurate (see Lance & Rogers, 1962). The disks are so close together that viscous forces dominate throughout. Centrifugal forces arising from the Couette-type primary flow drive a secondary flow with negligible convective acceleration. For the intermediate range of $R^{\frac{1}{2}}$ from 2 to about 25 the nonlinear convective acceleration grows in importance. The Stokes-type flow undergoes a complicated transition to a state in which a jet-like region of high circumferential flow exists near the fixed disk. In this range boundary layers begin to form on each disk but they overlap and therefore interact strongly. Finally, for $R > 625$, the asymptotic state is approached in which the flow near each disk becomes an asymptotic boundary layer with a well developed core between them. Further increase in R merely thickens the core, in which there is nearly uniform rotation (Fig. 1) with uniform axial flow (Fig. 2) and essentially no radial motion (Fig. 3). It is evident that the large qualitative changes with R that appear in Pearson's work are due to the fact that $R = 100$ falls in the intermediate range whereas for $R = 1000$ the flow has approached closely to its asymptotic state.

The most interesting regime in Taylor number, both physically and mathematically, is the asymptotic boundary layer regime and it deserves further attention (refer to Fig. 5 which displays the velocity profiles for the case $R = 2500$). It is to be expected that for large R the main body of fluid in the core, well away from the boundaries, is executing a cyclostrophic motion in which a radial pressure gradient balances centrifugal forces arising from the fluid rotation (it being obvious that there must be some angular velocity in the core since viscous effects, even though small, persist throughout the flow). This is verified by solving the inviscid versions of equations (8) and (9):

$$HG' - H'G = 0, \tag{14}$$

$$HH''' + 4GG' = 0. \tag{15}$$

Equation (14) implies that G and H are proportional, say $G = \alpha H$. Substitution into (15) then leads to inviscid solutions of the form

$$G(\zeta) = \alpha A + \alpha B \sin(2\alpha\zeta + \phi), \tag{16}$$

$$H(\zeta) = A + B \sin(2\alpha\zeta + \phi), \tag{17}$$

$$H'(\zeta) = 2\alpha B \cos(2\alpha\zeta + \phi), \tag{18}$$

where A, B, α, ϕ, are constants. It should be noticed that purely inviscid undamped spatial oscillations are possible. Equation (18) predicts that there can be cells of radial inflow and outflow in an inviscid interior. However, for consistency, the amplitude of these oscillations must be small (or the wavelength long) for otherwise the neglected viscous terms would not be small. As a result, the basic inviscid solution, apart from these small inertial oscillations consists of a uniform rigid body rotation at angular speed $\Omega_c = \alpha A \Omega$, a uniform axial flow of non-dimensional speed A, and no radial motion. (That is, a Taylor column motion in which $\partial v / \partial z = 0$). Fig. 5 exhibits such motion in the region from about $\zeta = 5$ to $\zeta = 28$. The pronounced spatial oscillation near the fixed boundary in Fig. 5 is evidently not of the inviscid type described by equations (16)–(18), since it is too large to be inviscid, is spatially damped, and has different phase relations.

Once the core has been identified as a region in cyclostrophic motion, then it follows that the boundary layers on the two disks are Ekman layers in a generalized sense (e.g. Faller

& Kaylor, 1966). It is easily confirmed that the boundary layer on the fixed disk is precisely equivalent to that analyzed by Bödewadt (1940), provided that the angular velocity of the outer flow is taken to be $\Omega_c = 0.313\ \Omega$ as required by the present numerical results.

The boundary layer on the rotating disk is clearly a modified von Kármán (1921), boundary layer because the outer flow is also rotating. This extension of von Kármán's problem has been studied by Fettis (1955) and Rogers & Lance (1960). With $\Omega_c/\Omega = 0.313$ the present results compare very favorably with an interpolation of those of Rogers and Lance (for example, the latter would predict $G'(0) = -0.521, H''(0) = -0.935, H(\infty) = -0.751$ whereas the corresponding present values are -0.524, -0.945, -0.754).

Once the complex flow profiles of Figs. 1–3 have been understood in terms of two Ekman layers linked by a Taylor column, it is straightforward to give a qualitative description of the flow in the limit of large Taylor number. For two reasons, the crucial property appears to be the essential absence of radial motion in the cyclostrophic interior. The first consequence of this is that the centrifugal forces associated with the uniformly rotating fluid, namely $\varrho r\Omega_c^2$, must be exactly counterbalanced by a pressure gradient force directed radially inward (i.e. $\partial p/\partial r = \varrho r\Omega_c^2$). From the geometry of the problem $\partial^2 p/\partial r \partial z \equiv 0$ (refer to equation (7)) so this radial pressure gradient extends undiminished through each boundary layer (without approximation). The viscously-induced more rapid rotation near $z = 0$ generates excess centrifugal forces which overcome the impressed radial pressure gradient and lead to a radial outflow in the Kármán layer (of smaller magnitude than in the pure Kármán case where there is no impressed radial pressure gradient). Through continuity, this radial outflow is accompanied by the well-known Ekman layer suction axially into the boundary layer (e.g. Greenspan & Howard, 1963). The fluid near the non-rotating boundary is viscously constrained to rotate more slowly than the fluid in the interior so its associated centrifugal forces are too weak to counterbalance the impressed radial pressure gradient. As a result, there is a slow radial inflow towards the axis and, by continuity, an axial outflow from the Bödewadt layer.

The second important effect of the absence of radial flow in the cyclostrophic interior is that it forces the two boundary layers to be linked in the following way. The fluid centrifugally pumped radially outward in the Kármán layer can only return radially inward in the Bödewadt layer. Through the continuity equation this forces the two boundary layers to have matching axial velocities, the axial outflow from the Bödewadt layer being the same as the axial flow into the Kármán layer. Fluid particles expelled from the Bödewadt layer follow nearly helical paths as they traverse the cyclostrophic region to enter the Kármán layer, where they are centrifugally pumped radially outward.

The only point on which the present description differs from Batchelor's is in the shape of the streamlines. Since the Bödewadt layer has large overshoot effects (due to the induced flow from rotated vortex lines, or see Rott & Lewellen, 1966), the streamline pattern is not as simple as that sketched by Batchelor. In fact, the substantial radial outflow near the top of the Bödewadt layer implies that the streamlines of the secondary flow fold back on themselves somewhat.

Detailed examination of the present numerical solution for $R = 2500$ reveals the presence of very small spatial oscillations in the Kármán layer and the cyclostrophic region, in addition to the pronounced oscillations in the Bödewadt layer. The analytic form of these oscillations has been deduced by Rogers and Lance by linearizing the full viscous equations about a uniform inviscid Taylor column motion. For $\Omega_c = 0.313\Omega$, they obtain harmonic oscillations of wavelength 6.4 (in units of ζ) which are exponentially damped away from the boundaries. This value is confirmed by the present numerical results. It is concluded that all spatial oscillations still present at $R = 2500$ are of the viscous type and that with further increase in R those in the interior will continue to decay until ultimately the Taylor column becomes strictly uniform in ζ.

It is perhaps worth mentioning that the problem treated here is highly idealized (because the disks extend to infinity) and furthermore these rotating flows are unstable beyond a critical radius (see Faller & Kaylor, 1966). Consequently, one should not expect such a flow to be producible in the laboratory. Nonetheless, the overall dynamics of such a rotating flow are believed to be of theoretical importance.

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СОСТАВНАЯ ЗАДАЧА ПОГРАНИЧНОГО СЛОЯ ЭКМАНА

Для полного интервала чисел Тэйлора R численно исследуется стационарное ламинарное аксиально симметричное течение вязкой несжимаемой жидкости, возбуждаемое бесконечным вращающимся диском в присутствие другого неподвижного коаксиального диска. Основное внимание уделяется качественной интерпретации численных решений. Показано, что асимптотическое состояние для больших чисел R состоит из двух хорошо известных специальных экмановских пограничных слоев на дисках, связанных тэйлоровской колонной, с подсосом от одного слоя в другой, сбалансированным аксиальным оттоком от другого слоя.