Solitary waves in a two-fluid system in a channel of arbitrary cross section

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ABSTRACT

The problem of solitary wave at the interface of two immiscible ideal fluids in a channel of arbitrary cross section and infinite length is investigated, and expressions for wave profile and speed are obtained. It is shown that when the two fluids are of nearly equal density, then the wave is of elevation (or depression) when the lower (or upper) layer is shallower of the two. The particular case of a rectangular channel is discussed in detail, and the results correspond to those of Long [1956] and Benjamin [1966].

1. Introduction

In recent years there has been a growing interest in the subject of solitary waves [Long and Morton (1966), Ter-Krikorov (1961), Benjamin (1962, 1966), Shen (1965, 1966) and Milne-Thomson (1964)]. The problem of solitary waves in two-fluid system and stratified medium with free surface has been studied by Peters and Stoker (1960). Earlier Long (1956) had investigated solitary waves in a two-layer medium between two plane fixed boundaries. Recently Peters (1966) tackled the problem of irrotational and rotational solitary waves in a channel of arbitrary cross section wherein he studied in detail the three-dimensional aspects of these waves.

In this paper we study the problem of solitary waves at the interface of two immiscible ideal fluids in a closed channel of arbitrary cross section and of infinite length.

2. Formulation of the problem

Two immiscible ideal fluids of constant densities ϱ_1 and ϱ_2 ($\varrho_2 \leqslant \varrho_1$) are enclosed in a closed channel of arbitrary cross section and of infinite length. In the undisturbed state, let the depths of the fluids be H_1 and H_2 respectively. We assume that a wave of permanent type exists and moves at constant speed C towards left over the interface of the fluids which are at rest at infinity. We take x_1, x_3-plane in the undisturbed interface with x_1-axis along the length of the channel. Take x_2-axis vertically upwards and passing through the crest of the wave. Thus our coordinate system is convected with the wave. Fig. 1 shows the model adopted for this investigation.

3. Basic equations

The relevant equations for the lower fluid with the assumptions made in the previous section are the equation of continuity

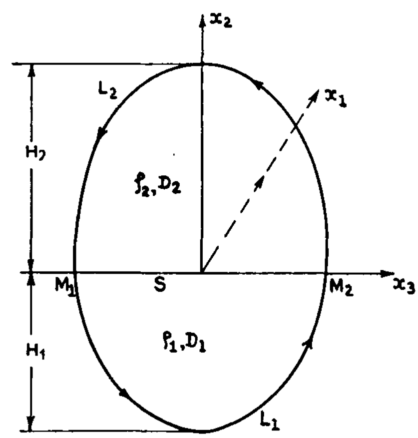

Fig. 1. The cross section of the channel at infinity.

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial x_2} + \frac{\partial \omega_1}{\partial x_3} = 0, \tag{1}$$

and the Bernoullis' equation

$$\left. \frac{p}{\varrho_1} + \frac{q_1^2}{2} + g x_2 \right|_{Q_0}^{Q_1} = 0, \tag{2}$$

where Q_1 and Q_0 are any two points, while g is acceleration due to gravity.

The kinematic boundary conditions are:

(i) at the channel $x_2 = A(x_3)$

$$v_1 = \omega_1 \frac{\partial A}{\partial x_3}, \tag{3}$$

(ii) at the interface $x_2 = f(x_1, x_3)$

$$v_1 = u_1 \frac{\partial f}{\partial x_1} + \omega_1 \frac{\partial f}{\partial x_3}, \tag{4}$$

where u_1, v_1 and ω_1 are velocity components of the lower fluid.

The dynamical boundary condition at the interface requires that pressure should be continuous across it. Further we have the conditions at infinity as

$$\left.\begin{aligned} &f(-\infty, x_3) = 0,\ u_1(-\infty, 0, 0) = C, \\ &v_1(-\infty, 0, 0) = \omega_1(-\infty, 0, 0) = 0, \\ &p(-\infty, 0, 0) = p^{(1)} = \text{Const.} \end{aligned}\right\} \tag{5}$$

If we choose Q_0 to be the point $(-\infty, 0, 0)$ and Q_1 a point on the interface $x_2 = f(x_1, x_3)$, equation (2) reduces to

$$p_1 + \varrho_1 \frac{q_1^2}{2} + \varrho_1 g f = \frac{C^2}{2} \varrho_1 \tag{6}$$

where p_1 is the excess of pressure at (x_1, x_2, x_3) over that at $(-\infty, 0, 0)$ i.e. $p_1 = p - p^{(1)}$.

Further we will have similar equations for the upper fluid, the flow variables of which shall be denoted by capital letters while those of the lower fluid by small letters unless stated.

We can write the above equations in the dimensionless form by introducing a typical length h in the vertical direction and a stretching parameter $\sqrt{\varepsilon}$. We take the dimensionless variables

$$\left.\begin{aligned} &x = \sqrt{\varepsilon} h^{-1} x_1,\ y = h^{-1} x_2,\ z = h^{-1} x_3 \\ &u = (gh)^{-1} u_1,\ v = \sqrt{\varepsilon}(gh)^{-1} v_1,\ \omega = \sqrt{\varepsilon}(gh)^{-1} \omega_1 \\ &\pi = (\varrho_1 gh)^{-1} p_1,\ \eta = h^{-1} f,\ B = h^{-1} A. \end{aligned}\right\} \tag{7}$$

Since the fluids are inviscid and start from rest the flow is irrotational. In terms of the above dimensionless variables with

$$u = \varphi_x,\ v = \varphi_y,\ \omega = \varphi_z$$

the basic equations reduce to

$$\varepsilon \varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0, \tag{8}$$

$$\varphi_y = \varphi_z B_z \quad \text{at} \quad y = B(z), \tag{9}$$

$$\varepsilon \varphi_x \eta_x = \varphi_y - \varphi_z \eta_z \quad \text{at} \quad y = \eta(x, z). \tag{10}$$

The dynamical condition at the interface $y = \eta(x, z)$ reduces to

$$\varepsilon\{\varphi_x^2 + 2\eta(1-\varrho) - \varrho\, \Phi_x^2\} + (\varphi_y^2 + \varphi_z^2) - \varrho(\Phi_y^2 + \Phi_z^2) = (1-\varrho)\frac{C^2}{gh}\varepsilon, \tag{11}$$

where $\varrho = \varrho_2/\varrho_1$.

Now the conditions at infinity are

$$\eta(-\infty, z) = 0,\ \varphi_x(-\infty, 0, 0) = \Phi_x(-\infty, 0, 0) = \frac{C}{\sqrt{gh}}. \tag{12}$$

As stated above Φ denotes the velocity potential of the upper fluid.

In the same way the equations for the upper fluid are

$$\varepsilon \Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0, \tag{13}$$

$$\Phi_y = \Phi_z B_z \quad \text{at} \quad y = B(z), \tag{14}$$

and

$$\varepsilon \Phi_x \eta_x = \Phi_y - \Phi_z \eta_z \quad \text{at} \quad y = \eta(x, z). \tag{15}$$

4. Approximations

Following Peters (1966) we assume here that the dependent variables can be developed in integral powers of ε for our three-dimensional problem. That is we can write

$$\varphi(x, y, z) = \varphi_0 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \dots,$$

$$\Phi(x, y, z) = \Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2 + \dots,$$

and

$$\eta(x, y, z) = \varepsilon\eta_1 + \varepsilon^2\eta_2 + \dots. \tag{16}$$

Since $C = \sqrt{gh \cdot \varphi_x(-\infty, 0, 0)}$, we have

$$C^2 = C_0^2 + \varepsilon 2 C_0 C_1 + \varepsilon^2 (C_1^2 + 2 C_0 C_2) + \dots. \tag{17}$$

We shall substitute these expansions in the basic equations (8) to (15) and equate the coefficients of like powers of ε.

(i) Zeroth order approximation:

Equations (8) to (10) and (13) to (15) give for $\varphi_0(x, y, z)$

$$\left.\begin{aligned} &\varphi_{0yy} + \varphi_{0zz} = 0 \quad \text{in} \quad D_1, \\ &\varphi_{0y}(x, 0, z) = 0 \quad \text{on} \quad S, \\ &\varphi_{0y} - \varphi_{0z} B_z = 0 \quad \text{on} \quad L_1, \end{aligned}\right\} \tag{18}$$

for $\Phi_0(x, y, z)$

$$\left.\begin{aligned} &\Phi_{0yy} + \Phi_{0zz} = 0 \quad \text{in} \quad D_2 \\ &\Phi_{0y}(x, 0, z) = 0 \quad \text{on} \quad S, \\ &\Phi_{0y} - \Phi_{0z} B_z = 0 \quad \text{on} \quad L_2, \end{aligned}\right\} \tag{19}$$

where D_1 and D_2 are respectively the domains of the cross section of the channel (Fig. 1) occupied by lower and upper fluids at infinity. S is the segment of the z-axis determined by the points where the closed curve $L = L_1 + L_2$ bounding the region $D = D_1 + D_2$ intersects $y = 0$. We assume L to be a piecewise smooth curve which crosses $y = 0$ just in two points M_1 and M_2 such that the length of the segment $M_2 - M_1 = b > 0$. L_1 and L_2 are the lower and upper parts of the curve L devided by S. In other words D_1 is bounded by the curves L_1 and S, and D_2 by the curves L_2 and S. Equation (11) gives

$$\begin{aligned} &\varphi_{0y}^2(x, 0, z) + \varphi_{0z}^2(x, 0, z) - \\ &\quad - \varrho\{\Phi_{0y}^2(x, 0, z) + \Phi_{0z}^2(x, 0, z)\} = 0. \end{aligned} \tag{20}$$

The above equations (18) to (20) yield

$$\left.\begin{aligned} \varphi_0 &= F_0(x), \\ \Phi_0 &= G_0(x). \end{aligned}\right\} \tag{21}$$

From equation (12) we get

$$\varphi_{0x}(-\infty, 0, 0) = \Phi_{0x}(-\infty, 0, 0) = \frac{C_0}{\sqrt{gh}}. \tag{22}$$

(ii) First approximation:

The first order approximation (i. e. coefficient of ε) gives

$$\left.\begin{aligned} &\varphi_{1yy} + \varphi_{1zz} = -F_0''(x) \quad \text{in} \quad D_1, \\ &\varphi_{1y}(x, 0, z) = 0 \quad \text{on} \quad S, \\ &\varphi_{1y} - \varphi_{1z} B_z = 0 \quad \text{on} \quad L_1, \end{aligned}\right\} \tag{23}$$

$$\left.\begin{aligned} &\Phi_{1yy} + \Phi_{1zz} = -G_0''(x) \quad \text{in} \quad D_2, \\ &\Phi_{1y}(x, 0, z) = 0 \quad \text{on} \quad S, \\ &\Phi_{1y} - \Phi_{1z} B_z = 0 \quad \text{on} \quad L_2, \end{aligned}\right\} \tag{24}$$

and

$$\{F_0'^2(x) - \varrho G_0'^2(x)\} = (1 - \varrho)\frac{C_0^2}{gh}, \tag{25}$$

where prime indicates differentiation with respect to x.

From equations (22) to (25) we get

$$\varphi_0(x) = \frac{C_0}{\sqrt{gh}} x, \quad \Phi_0(x) = \frac{C_0}{\sqrt{gh}} x. \tag{26}$$

Equation (11) gives

$$\varphi_{1x}(-\infty, 0, 0) = \Phi_{1x}(-\infty, 0, 0) = \frac{C_1}{\sqrt{gh}}. \tag{27}$$

The above equations enable us to write

$$\left.\begin{aligned} \varphi_1 &= F_1(x) + \frac{C_1}{\sqrt{gh}} x, \\ \Phi_1 &= G_1(x) + \frac{C_1}{\sqrt{gh}} x, \end{aligned}\right\} \tag{28}$$

where $\quad F_1'(-\infty) = G_1'(-\infty) = 0.$

(iii) Second approximation:

The set of equations to be satisfied by φ_2 and Φ_2 are

$$\left.\begin{aligned} &\varphi_{2yy} + \varphi_{2zz} = -F_1''(x) \quad \text{in} \quad D_1, \\ &\varphi_{2y}(x, 0, z) = \frac{C_0}{\sqrt{gh}} \eta_1'(x) \quad \text{on} \quad S, \\ &\varphi_{2y} - \varphi_{2z} B_z = 0 \quad \text{on} \quad L_1, \end{aligned}\right\} \tag{29}$$

$$\frac{C_0}{\sqrt{gh}}\{F_1'(x) - \varrho G_1'(x)\} + (1 - \varrho)\eta_1(x) = 0 \quad \text{on} \quad S, \tag{30}$$

and

$$\left.\begin{aligned} &\Phi_{2yy} + \Phi_{2zz} = -G_1''(x) \quad \text{in} \quad D_2, \\ &\Phi_{2y}(x, 0, z) = \frac{C_0}{\sqrt{gh}} \eta_1'(x) \quad \text{on} \quad S, \\ &\Phi_{2y} - \Phi_{2z} B_z = 0 \quad \text{on} \quad L_2. \end{aligned}\right\} \tag{31}$$

The last equations in (29) and (31) with equation (30) give

$$\left.\begin{aligned}(1-\varrho)\varphi_{2y}(x,0,z)+\frac{C_0^2}{gh}\{F_1''(x)-\varrho G_1''(x)\}=0,\\(1-\varrho)\Phi_{2y}(x,0,z)+\frac{C_0^2}{gh}\{F_1''(x)-\varrho G_1''(x)\}=0.\end{aligned}\right\}\quad(32)$$

We know that if χ_n is the outward normal derivative of χ at the boundary of D then

$$\iint_D(\chi_{yy}+\chi_{zz})\,dy\,dz=\int_L\chi_n\,ds,\qquad(33)$$

where ds denotes the arc length along the boundary L and L is oriented counterclockwise. This identity imposes a condition for the existence of a function χ which has a prescribed normal derivative on the boundary of D and which satisfies

$$\chi_{yy}+\chi_{zz}=T(y,z)\qquad(34)$$

in D.

Such a function does not exist if the assigned normal derivative violates

$$\iint_D T(y,z)\,dy\,dz=\int_L\chi_n\,ds.\qquad(35)$$

Replacing χ in equation (33) by φ_2 and Φ_2 in turn we have

$$\left.\begin{aligned}d_1F_1''(x)&=\frac{C_0^2}{g(1-\varrho)}\{F_1''(x)-\varrho G_1''(x)\},\\d_2G_1''(x)&=-\frac{C_0^2}{g(1-\varrho)}\{F_1''(x)-\varrho G_1''(x)\},\end{aligned}\right\}\quad(36)$$

where $d_1=A_1h^2/bh$, $d_2=A_2h^2/bh$; A_1h^2 and A_2h^2 are the areas of D_1 and D_2 respectively, while bh is the breadth of the interface at infinity.

Equation (36) shows that

$$G_1''(x)=-dF_1''(x),\qquad(37)$$

and
$$C_0^2=gd_1\cdot\frac{(1-\varrho)}{(1+\varrho d)},$$

where $d=d_1/d_2=A_1/A_2$, while d_1 and d_2 are the mean depths of the lower and upper fluids respectively.

Now the equations for φ_2 reduce to

$$\begin{aligned}&\varphi_{2yy}+\varphi_{2zz}=-F_1''(x)\quad\text{in}\quad D_1,\\&\varphi_{2y}(x,0,z)=-k_1F_1''(x)\quad\text{on}\quad S,\\&\varphi_{2y}-\varphi_{2z}B_z=0\quad\text{on}\quad L_1,\qquad(38)\\&\varphi_{2x}(-\infty,0,0)=\frac{C_2}{\sqrt{gh}},\end{aligned}$$

with $\qquad k_1=d_1/h.$

This equation shows that

$$\varphi_2=-F_1''(x)\,\psi_2(y,z)+\frac{C_2}{\sqrt{gh}}x+F_2(x),\qquad(39)$$

where
$$\begin{aligned}&\psi_{2yy}+\psi_{2zz}=1\quad\text{in}\quad D_1,\\&\psi_{2y}(0,z)=K_1\quad\text{on}\quad S,\qquad(40)\\&\psi_{2n}=0\quad\text{on}\quad L_1.\end{aligned}$$

To make $\psi_2(y,z)$ unique we can assume without loss of generality, $\psi_2(0,0)=0$. Thus from the above system ψ_2 can be calculated.

In a similar manner we have

$$\Phi_2=dF_1''(x)\,\Psi_2(y,z)+\frac{C_2}{\sqrt{gh}}x+G_2(x),\qquad(41)$$

where
$$\begin{aligned}&\Psi_{2yy}+\Psi_{2zz}=1\quad\text{in}\quad D_2,\\&\Psi_{2y}(0,z)=k_2\quad\text{on}\quad S,\\&\Psi_{2n}=0\quad\text{on}\quad L_2,\qquad(42)\\&\Psi_2(0,0)=0,\end{aligned}$$

with $\qquad k_2=-d_2/h.$

Equation (42) determines $\Psi_2(y,z)$.

The last equation in (29) gives

$$F_1'(x)=-k\eta_1'(x),\quad k=\frac{C_0}{\sqrt{gh}}\frac{h}{d_1}.\qquad(43)$$

To determine $F_1(x)$ we proceed to the third order system (coefficient of ε^3) of equations.

(iv) Third approximation:

For φ the third approximation gives

$$\left.\begin{aligned}&\varphi_{3yy}+\varphi_{3zz}=F_1''''(x)\psi_2(y,z)-F_2''(x)\quad\text{in}\quad D_1,\\&\varphi_{3y}(x,0,z)=-\eta_1(x)\eta_1'(x)k\psi_{2yy}(0,z)+\\&+\eta_2'(x,z)\frac{C_0}{\sqrt{gh}}+\eta_1'(x)\left\{\frac{C_1}{\sqrt{gh}}-k\eta_1(x)\right\}\quad\text{on}\quad S,\\&\varphi_{3y}-\varphi_{3z}B_z=0\quad\text{on}\quad L_1.\end{aligned}\right\}\quad(44)$$

Equation (11) gives

$$(1-\varrho)\eta_2(x,z)+\eta_1''(x)\frac{kC_0}{\sqrt{gh}}\{\psi_2(0,z)+\varrho d\Psi_2(0,z)\}$$

$$+\frac{\eta_1^2(x)}{2}k^2(1-\varrho d^2)-c_1\eta_1(x)(1+\varrho d)\frac{k}{\sqrt{gh}}$$

$$+\frac{C_0}{\sqrt{gh}}\{F_2'(x)-\varrho G_2'(x)\}=0 \quad \text{on } S, \quad (45)$$

while equation (45) and the second equation in (44) give

$$\varphi_{3y}(x,0,z)=-\eta_1'''(x)\frac{C_0^2 k}{gh(1-\varrho)}$$

$$\times\{\psi_2(0,z)+\varrho d\Psi_2(0,z)\}$$

$$-\eta_1\eta_1'(x)k\left\{1+\psi_{2yy}(0,z)+\frac{C_0 k}{\sqrt{gh}}\frac{(1-\varrho d^2)}{(1-\varrho)}\right\}$$

$$+\frac{C_1\eta_1'(x)}{\sqrt{gh}}\left\{1+\frac{C_0 k}{\sqrt{gh}}\frac{(1+\varrho d)}{1-\varrho}\right\}$$

$$-\frac{C_0^2}{gh(1-\varrho)}\{F_2''(x)-\varrho G_2''(x)\}. \quad (46)$$

Similarly we have for Φ_3 the equations

$$\left.\begin{aligned}&\Phi_{3yy}+\Phi_{3zz}=-d\Psi_2(y,z)F_1^{\text{iv}}(x)-G_2''(x) \quad \text{in } D_2,\\ &\Phi_{3y}-\Phi_{3z}B_z=0 \quad \text{on } L_2,\end{aligned}\right\} \quad (47)$$

$$\Phi_{3y}(x,0,z)=-\eta_1'''(x)\frac{kC_0^2}{gh(1-\varrho)}$$

$$\times\{\psi_2(0,z)+\varrho d\Psi_2(0,z)\}$$

$$-\eta_1\eta_1'(x)k\left\{\frac{C_0 k}{\sqrt{gh}}\frac{1-\varrho d^2}{1-\varrho}-d-d\Psi_{2yy}(0,z)\right\}$$

$$+\frac{C_1\eta_1'(x)}{\sqrt{gh}}\left\{1+\frac{kC_0}{\sqrt{gh}}\cdot\frac{1+\varrho d}{1-\varrho}\right\}$$

$$-\frac{C_0^2}{gh(1-\varrho)}\{F_2''(x)-\varrho G_2''(x)\} \quad (48)$$

Application of equation (33) to φ_3 and Φ_3 gives

$$\frac{k_1}{1+\varrho d}\frac{b}{k}(F_2''-\varrho G_2'')-\frac{A_1}{k}F_2''=$$

$$-\eta_1\eta_1'(x)\left\{\int_{M_1}^{M_2}\left(1+\frac{1-\varrho d^2}{1+\varrho d}+\psi_{2yy}(0,z)\right)dz\right\}$$

$$+\eta_1'''(x)\left[\iint_{D_1}\psi_2(y,z)\,dz\,dy\right.$$

$$\left.-\frac{k_1}{1+\varrho d}\cdot\int_{M_1}^{M_2}\{\psi_2(0,z)+\varrho d\Psi_2(0,z)\}dz\right]$$

$$+\frac{2bC_1}{k\sqrt{gh}}\eta_1'(x), \quad (49)$$

and

$$\frac{k_1 b}{k(1+\varrho d)}(F_2''-\varrho G_2'')+\frac{A_2}{k}G_2''$$

$$=\eta_1\eta_1'(x)\cdot\int_{M_1}^{M_2}\left(d+d\Psi_{2yy}(0,z)-\frac{1-\varrho d^2}{1+\varrho d}\right)dz$$

$$+\eta_1'''(x)\left[d\iint_{D_2}\Psi_2(y,z)\,dz\,dy\right.$$

$$\left.-\frac{k_1}{1+\varrho d}\cdot\int_{M_1}^{M_2}\{\psi_2(0,z)+\varrho d\Psi_2(0,z)\}dz\right]$$

$$+\frac{2bC_1}{k\sqrt{gh}}\eta_1'(x). \quad (50)$$

Equations (49) and (50) give

$$G_2''(x)=-dF_2''(x)+\eta_1'''(x)\frac{k}{A_2}$$

$$\times\left\{d\iint_{D_2}\Psi_2(y,z)\,dz\,dy-\iint_{D_1}\psi_2(y,z)\,dz\,dy\right\}$$

$$+\eta_1\eta_1'(x)\frac{k}{A_2}\left[b(1+d)\right.$$

$$\left.+\int_{M_1}^{M_2}\{\psi_{2yy}(0,z)+d\Psi_{2yy}(0,z)\}dz\right] \quad (51)$$

Substituting equation (51) in (49) we get

$$m_0\eta_1'''(x)=m_1\eta_1\eta_1'(x)+m_2\eta_1'(x) \quad (52)$$

where

$$m_0=-\frac{1}{1+\varrho d}\left\{\varrho d^2\iint_{D_2}\Psi_2(y,z)\,dz\,dy\right.$$

$$+ \iint_{D_1} \psi_2(y,z)\,dz\,dy$$

$$\left. - \frac{k_1}{1+\varrho d} \int_{M_1}^{M_2} [\psi_2(0,z) + \varrho d \Psi_2(0,z)]\,dz \right\},$$

$$m_1 = \frac{1}{\varrho d + 1} \int_{M_1}^{M_2} \psi_{2zz}(0,z) - \varrho d^2 \Psi_{2zz}(0,z)\,dz$$

$$+ 3b \frac{(\varrho d^2 - 1)}{1+\varrho d}, \quad (53)$$

and $$m_2 = \frac{2b}{k} \frac{C_1}{\sqrt{gh}}.$$

Equation (53) can further be simplified with the help of (40) and (42) to give

$$m_0 = \frac{1}{1+\varrho d}\left[\iint_{D_1} (\psi_{2y}^2 + \psi_{2z}^2)\,dz\,dy\right.$$

$$\left. + \varrho d^2 \iint_{D_2} (\Psi_{2y}^2 + \Psi_{2z}^2)\,dz\,dy\right],$$

$$m_1 = -3bT \cdot \frac{(1-\varrho d^2)}{1+\varrho d}, \quad (54)$$

and $$m_2 = \frac{2b}{k_1} \frac{C_1}{C_0},$$

where

$$T = 1 + \frac{1}{3b(\varrho d^2 - 1)} \times \int_{M_1}^{M_2} (\psi_{2zz}(0,z) - \varrho d^2 \Psi_{2zz}(0,z))\,dz.$$

We note here that when the curve L intersects the z-axis orthogonally then $T = 1$ and T can always be made positive by taking the breadth of the channel large.

Equation (52) is a well known basic equation in the theory of solitary and conoidal waves. As we are interested in the investigation of solitary waves we impose the conditions

$$\varphi_{5y}(-\infty, 0, z) = \Phi_{sy}(-\infty, 0, z) = 0, \ \eta_1(x) = \eta_1(-x) \quad \text{for} \quad s = 1, 2, 3 \ldots. \quad (55')$$

These conditions along with equation (11) reduce to

$$\eta_1(-\infty) = \eta_1'(-\infty) = \eta_1''(-\infty) = 0 \quad (55)$$

The solution of (52) with the above boundary conditions, can be written as

$$\eta_1(x) = -\frac{3m_2}{m_1} \operatorname{Sech}^2 \sqrt{\frac{m_2}{m_0}} \cdot \frac{x}{2}. \quad (56)$$

Let us define

$$q = -\frac{3m_2}{m_1} h\varepsilon = \frac{2\varepsilon h}{k_1 T} \cdot \frac{C_1}{C_0} \cdot \frac{(1+\varrho d)}{1-\varrho d^2},$$

$$l = \sqrt{\frac{m_2 \varepsilon}{m_0}} = \sqrt{\frac{qTb}{m_0 h} \frac{1-\varrho d^2}{1+\varrho d}}. \quad (57)$$

The equation of the solitary wave, at the interface of two immiscible ideal fluids, to the first-order of approximation can be written as

$$f = q \operatorname{Sech}^2 \frac{x_1}{2} \frac{l}{h} + o(q), \quad (58)$$

where $o(q)$ means $\lim_{q \to o} o(q)/q = 0$.

It can easily be verified that for the lower fluid

$$u_1 = C_0 \left[1 + \frac{Tq}{2d_1} \frac{1-\varrho d^2}{1+\varrho d} - \frac{q}{d_1} \operatorname{Sech}^2 \frac{x_1}{2} \frac{l}{h}\right] + o(q),$$

$$v_1 = -\frac{ql}{d_1} C_0 \psi_{2y}(y,z) \operatorname{Sech}^2 \frac{x_1 l}{2h} \tanh \frac{x_1 l}{2h} + o(q^{\frac{3}{2}}),$$

$$\omega_1 = -\frac{ql}{d_1} C_0 \psi_{2z}(y,z) \operatorname{Sech}^2 \frac{x_1 l}{2h} \tanh \frac{x_1 l}{2h} + o(q^{\frac{3}{2}}), \quad (59)$$

and

$$p_1 = \varrho_1 g \left[\frac{1-\varrho}{1+\varrho d} q \operatorname{Sech}^2 \frac{x_1 l}{2h} - x_2\right] + o(q).$$

Similarly for the upper fluid we have

$$u_1 = C_0 \left[1 + \frac{Tq}{2d_1} \frac{1-\varrho d^2}{1+\varrho d} + \frac{q}{d_2} \operatorname{Sech}^2 \frac{x_1 l}{2h}\right] + o(q),$$

$$v_1 = \frac{ql}{d_2} C_0 \Psi_{2y}(y,z) \operatorname{Sech}^2 \frac{x_1 l}{2h} \cdot \tanh \frac{x_1 l}{2h} + o(q^{\frac{3}{2}}),$$

$$w_1 = \frac{ql}{d_2} C_0 \Psi_{2z}(y,z) \operatorname{Sech}^2 \frac{x_1 l}{2h} \cdot \tanh \frac{x_1 l}{2h} + o(q^{\frac{3}{2}}), \quad (60)$$

and

$$P_1 = -\varrho_2 g\left[qd\frac{1-\varrho}{1+\varrho d}\,\text{Sech}^2\frac{x_1 l}{2h}+x_2\right]+o(q).$$

It is to be noted that p_1 and P_1 occurring in (59) and (60) represent the excess of pressures over the pressure at $(-\infty, 0, 0)$.

From (59), we obtain the speed of the solitary wave at the interface, as

$$C^2 = C_0^2\left[1+\frac{1-\varrho d^2}{1+\varrho d}\frac{qT}{d_1}\right]+o(q). \tag{61}$$

For $\eta_2(x,z)$, the second-order approximation of the wave profile, we have the equation

$$\begin{aligned}\eta_2(x,z) = {} & \frac{\eta_1''(x)}{1+\varrho d}\left[\frac{\varrho}{A_2}\left\{d\iint_{D_2}\Psi_2\,dz\,dy\right.\right.\\ & \left.\left. -\iint_{D_1}\psi_2\,dz\,dy\right\}-\psi_2(0,z)-\varrho d\Psi_2(0,z)\right]\\ & -\frac{\eta_1(x)}{2(1+\varrho d)}\left[-\frac{\varrho}{A_2}\left\{(1+d)b\right.\right.\\ & \left. +\int_{M_1}^{M_2}(\psi_{2yy}(0,z)+d\Psi_{2yy}(0,z))\,dz\right\}\\ & \left. +\frac{1-\varrho d^2 h}{d_1}\right]+\frac{C_1}{C_0}\eta_1(x)-F_2'(x)\frac{1+\varrho d}{1-\varrho}\frac{C_0}{\sqrt{gh}}.\end{aligned} \tag{62}$$

5. Rectangular channel

When the channel is a rectangular one of unit breadth, equations (40) and (42) reduce to

$$\psi_{2yy}+\psi_{2zz}=1 \quad \text{for} \quad \begin{cases}-r_1<y<0,\\ -\tfrac{1}{2}<z<\tfrac{1}{2}.\end{cases}$$

$$\psi_{2y}(0,z)=r_1 \quad \text{on} \quad y=0,$$

$$\psi_2(0,0)=0 \quad \text{with} \quad \psi_{2n}=0 \quad \text{on} \quad L_1, \tag{63}$$

and

$$\Psi_{2yy}+\Psi_{2zz}=1 \quad \text{for} \quad \begin{cases}0<y<r_2,\\ -\tfrac{1}{2}<z<\tfrac{1}{2},\end{cases}$$

$$\Psi_{2y}(0,z)=-r_2 \quad \text{on} \quad y=0,$$

$$\Psi_2(0,0)=0 \quad \text{with} \quad \Psi_{2n}=0 \quad \text{on} \quad L_2, \tag{64}$$

where r_1 and r_2 are non-dimensional depths of lower and upper fluids respectively.

The solutions (63) and (64) can easily be written as

$$\psi_2 = yr_1\left(\frac{y}{2r_1}+1\right),$$

$$\Psi_2 = yr_2\left(\frac{y}{2r_2}-1\right) \tag{65}$$

From (65) m_0, m_1 and m_2 reduce to

$$m_0=\frac{(1+\varrho d^{-1})}{1+\varrho d}\frac{r_1^3}{3}, \quad m_1=-3\frac{1-\varrho d^2}{1+\varrho d}. \tag{66}$$

and
$$m_2=\frac{2C_1}{C_0}\frac{h}{r_1}.$$

When we take the depth of the lower fluid as the vertical scale factor, then $r_1=1$, $r_2=d$ the ratio of the depths of the fluids.

Now the profile of the solitary wave, to the first-order of approximation, is

$$f=q\,\text{Sech}^2\frac{x_1 l}{2h}+o(q), \tag{67}$$

where
$$l^2=3q\frac{(1-\varrho d^2)}{1+\varrho d^{-1}}\frac{1}{h}.$$

The speed of the solitary wave becomes

$$\frac{C^2}{C_0^2}=1+\frac{q}{h}\frac{1-\varrho d^2}{1+\varrho d}+o(q), \tag{68}$$

and the critical speed is given by

$$\frac{C_0^2}{gh}=\frac{1-\varrho}{1+\varrho d}. \tag{69}$$

6. Conclusions

Equations (58) and (61) give the wave profile and speed for closed channel of arbitrary cross section. From the first-equation of (57), we have $C_1>0$ which shows that the solitary wave at the interface of two immiscible fluids always travels with super-critical speed $(C>C_0)$. Also from the last equation of (57), we observe that $q\gtrless 0$ if $1-d^2\gtrless 0$ when $\varrho\approx 1$; m_0 and T being positive. In other words, the wave is of elevation or depression depending on the lower or upper layer is shallower of the two. This result does not depend on the cross-section of the channel.

From (58) and (62), we find that the amplitude of the solitary wave does not suffer lateral change to the first-order of approximation while it does suffer lateral change in the higher approximations. In other words, the approximation giving the wave profile,

$$x_2 = \varepsilon h\eta_1\left(\sqrt{\varepsilon}\frac{x_1}{h}\right) + \varepsilon^2 h\eta_2\left(\sqrt{\varepsilon}\frac{x_1}{h}, \frac{x_3}{h}\right) + o(\varepsilon^2),$$

represents the three-dimensional solitary wave. For $\varrho = 0$ and $d_1 = h$, our results reduce to those of Peters [1966].

In case of a rectangular channel (67) with $\varrho \approx 1$ yields $q \gtrsim 0$ if $1 - d^2 \gtrsim 0$ for l to be real. This condition and the wave profile given by (67) correspond to equations (24) and (27) of Long [1956]. Equations (68) and (69) giving the speed of the solitary wave and the critical speed are in agreement with (4.9) and (4.7) of Benjamin [1966].

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ОДИНОЧНЫЕ ВОЛНЫ ДВУХСЛОЙНОЙ ЖИДКОСТИ В КАНАЛЕ С ПРОИЗВОЛЬНЫМ ПОПЕРЕЧНЫМ СЕЧЕНИЕМ

Исследуется задача об одиночной волне на поверхности раздела двух несмешивающихся идеальных жидкостей в бесконечно длинном канале с произвольным сечением. Получены выражения для профиля волны и ее скорости. Показано, что если эти две жидкости обладают почти одинаковой плотностью, то волна является волной возвышения (или депрессии), если нижний (или верхний) слой тоньше другого. Детально обсуждается частный случай канала прямоугольного сечения, причем полученные результаты совпадают с результатами Лонга (1956) и Бенджамина (1966).