Total potential energy of finite columns

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ABSTRACT

The relation between enthalpy and total potential energy in a column of finite horizontal extent is investigated. The limitations of the quasistatic approximation are introduced *a priori* rather than *a posteriori*, i.e., cyclone-scale air masses are assumed to be in hydrostatic equilibrium, but any differential equation implying small-scale hydrostatic balance is deliberately avoided. It is found that the enthalpy and total potential energy are equal for columns over ocean, but in general, there is an orographic term which cannot be eliminated by redefining the zero-level of geopotential. There is also a term consequent of the departure from hydrostatic balance in the neighborhood of surface irregularities; this term is presumably negligible for columns of cyclone-scale lateral extent.

1. Introduction

Treatment of cyclone-scale meteorological problems is virtually always based on the quasistatic approximation, i.e., the differential equation governing vertical momentum is replaced by the hydrostatic equation. This involves two physical assumptions:

(I) Pressure and earth-gravity are the only non-negligible vertical forces;

(II) Vertical acceleration of the air is small compared with g.

Assumption (I) is beyond cavil. Only a few forces are available as contenders, and each of them is several orders of magnitude smaller than earth-gravity.

Strictly speaking, assumption (II) is erroneous. Vertical acceleration comparable with g is a rather ordinary atmospheric event. Nevertheless, the quasi-static approximation leads to correct results when applied to atmospheric motions of sufficiently broad lateral extent: local errors cancel out, so that motions of cyclone scale or larger are adequately described.

Reasonably persuasive apologia for the quasi-static approximation is given by Kibel (1963), who compares orders of magnitude for a typical case and by Eliassen and Kleinschmidt (1957), who employ a hydrodynamical argument based on the properties of nearly solenoidal flows. Both references stress the importance of horizontal scale: while the quasi-static approximation is appropriate for cyclone-scale motions, the closer one looks at the flow, the worse the error.

Using the quasi-static system of differential equations therefore introduces a conceptual difficulty: each time one writes a derivative he is, so to speak, looking at the flow infinitely closely. The difficulty evaporates if we take the view recommended in Hilbert's Göttingen lectures, viz., that physical principles be expressed as integral conservation laws. This permits us to incorporate the scale of observation directly into the mathematical formation and to replace (II) by a more palatable assumption:

(II′) Masses of air with cyclone-scale horizontal extent do not undergo mass-average vertical accelerations of order g.

The fluid-dynamical differential equations, sometimes regarded as fundamental, are in fact results derived from the conservation laws by appropriate use of Gauss' divergence theorem. The integral theorems which occasionally appear in the literature are often recognizable inversions of the procedure. So are finite-difference analogs of the differential equations, or at least the work of Arakawa (1963) indicates that they should be: formal derivation of difference equations from the differential equations, without regard to the underlying balance laws, can lead to computational instability. The finite-difference equations are thus essentially the integral balance laws, applied to regions of a particular size and shape. The author has suggested (1965) that there may be value in using this point explicitly, i.e., in deriving one's differencing scheme directly from the balance laws without introducing differential equations as an intermediate step. A recent paper by Kurihara and Holloway (1967), which comes quite close to taking this view, has cleared up certain computational difficulties of general circulation models. Reduction of meteorological problems to a system of differential equations offers certain advantages, but when one begins and ends with statements about broad air masses it is at best a mathematical tautology. Moreover, it requires that a useful method—the quasi-static approximation—be based upon an obviously erroneous assumption which somehow leads to valid results.

A well-known result of the quasi-static approximation based upon assumptions (I) and (II) is that the total potential energy and enthalpy of a vertical column are the same. In the present paper we re-examine this result starting from (I) and (II′). We find that it holds valid for a column over ocean, but in general the total potential energy differs from the enthalpy by an orographic term which cannot be removed by redefining the zero level of potential energy.

2. The quasi-static approximation

Let V denote a volume of atmosphere, fixed with respect to the earth, and let S denote the surface of V. Further, let M denote the vertical momentum of the mass of air which occupies V at time t. Two sorts of forces act to accelerate this air mass: body forces acting thruout V and surface forces acting on S. By assumption (I), earth-gravity is the only important vertical body force and only pressure contributes importantly to the vertical surface force. Thus

$$dM/dt = -\iiint \varrho g dV - \oint\!\!\oint p\mathbf{k}\cdot\mathbf{n}dS, \quad (2.1)$$

where ϱ is the density, p is the pressure, $\mathbf{k}$ is the upward unit vector, and $\mathbf{n}$ is the unit outward normal to S. Observe that dM/dt is not the same thing as the rate at which total vertical momentum within V is changing. The latter quantity is influenced by another contribution, viz., convection of vertical momentum inward across S.

If the volume V has cyclone-scale horizontal extent, we can invoke (II′). In the notation introduced above, this assumption reads

$$\left(\iiint \varrho dV\right)^{-1} dM/dt < < g. \quad (2.2)$$

Hence, the left side of (2.1) is negligible compared with the first term on the right. We thus obtain the *integral form of the quasi-static approximation:*

$$\oint\!\!\oint p\mathbf{k}\cdot\mathbf{n}dS = -\iiint_V \varrho g dV. \quad (2.3)$$

By use of Gauss' divergence theorem, this result can be written

$$\iiint \left(\partial p/\partial z - \varrho g\right) dV = 0, \quad (2.4)$$

where z is the upward vertical coordinate.

By taking (2.2), and hence (2.4), to hold for an arbitrary volume V, i.e., by assuming (II) rather than (II′), one concludes that the integrand in (2.4) vanishes identically. This yields the hydrostatic differential equation for the vertical variation of pressure. As indicated in the introduction, we deliberately refrain from taking this final step.

3. Vertical structure

A weakened form of the hydrostatic differential equation can be obtained within the framework of the present study. Assumption (II′) restricts us to a volume of broad horizontal extent, but permits us to partition it vertically in any way that we find convenient.

Let X denote a cyclone-scale region of the sea-level ($z = 0$) plane and let C denote the atmospheric column over X. Thus, if $\mathbf{x}$ denotes a generic point of X and $h(\mathbf{x})$ is the elevation of the earth's surface over $\mathbf{x}$, we have (Fig. 1)

$$C = \{(\mathbf{x}, z) \text{ such that } \mathbf{x} \in X \text{ and } z > h(\mathbf{x})\}.$$

Corresponding to each altitude z, we define a subset A_z of X, consisting of those points of X

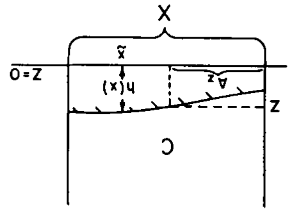

Fig. 1

whose projections onto the z-level are in the atmosphere rather than underground, i.e.,

$$A_z = \{\mathbf{x} \quad \text{such that} \quad (\mathbf{x}, z) \in C\}.$$

We take

$$\tilde{\varrho}(z, t) = \|X\|^{-1} \iint_{Az} \varrho(\mathbf{x}, z, t)\, dX, \qquad (3.1.)$$

where $\|X\|$ denotes the area of X. Thus, if z is high enough that A_z coincides with X, then $\tilde{\varrho}$ is the average density of the z-level air over X.

More generally, $\tilde{\varrho}\|X\|dz$ approximates the mass of air in a layer of thickness dz over X at altitude z. A more accurate value is $\tilde{\varrho}\|X\|(1 + z/a)^2 dz$, where a denotes the radius of the earth. Thruout the analysis we neglect z/a in comparison with unity.

The total hydrostatic force supporting that part of the air over X which lies above the z-level is $\tilde{p}\|X\|$, where

$$\tilde{p}(z, t) = \|X\|^{-1} \times \left[\iint_{A_z} p(\mathbf{x}, z, t)\, dX + \iint_{(X - A_z)} p_s(\mathbf{x}, t)\, dX\right], \qquad (3.2)$$

in which $p_s(\mathbf{x}, t)$ denotes the surface pressure, i.e., $\tilde{p}(\mathbf{x}, h(\mathbf{x}), t)$. When $A_z = X$, $\tilde{p}$ is the average z-level pressure over the region X.

We now choose the volume V of $\oint$ 2 to be a horizontal slice of the column C, i.e. (Fig. 2), the portion of C lying between two levels, say z_0 and $z (z_0 < z)$. Since $\mathbf{k} \cdot \mathbf{n}$ vanishes on the vertical sides of C, the quasi-static approximation (2.3) can be written

$$\int_{z_0}^{z} g\tilde{\varrho}(z, t)\, dz = \tilde{p}(z_0, t) - \tilde{p}(z, t), \qquad (3.3)$$

or in differential form

$$\partial\tilde{p}/\partial z = -g\tilde{\varrho}. \qquad (3.4)$$

This result holds provided z is high enough that the lateral extent of A_z is sufficient for (2.3) to be valid. Physically, the quasi-static approximation cannot be assumed to hold within fjords, canyons, etc., but it is presumably all right when applied to a broad lateral sheet of atmosphere overlying a region having such irregularities.

If orography is ignored, (3.4) becomes the usual hydrostatic differential equation, but with the pressure and density replaced by their average values.

4. Total potential energy

We neglect the variation of g and take the air to be an ideal gas with constant specific heats The hydrostatic differential equation then leads to the well-known result that the potential energy and thermal energy of a column are in constant ratio. This result is usually derived for a column of infinitesimal cross section. To apply it to a column of finite cross-sectional area involves an additional assumption, viz., that the ground under the column is all at the same elevation, which is then taken as the zero-level of potential energy. Since the present study is restricted to atmospheric regions of very broad lateral extent, we can make this additional assumption only over ocean; the general result will reflect the influence of orography. We take sea-level as the zero level of geopotential. The potential energy P in the column C is then given by

$$P = \int_{h_0}^{\infty} \iint_{A_z} \varrho g z\, dX\, dz, \qquad (4.1)$$

where h_0 is the lowest point of the column, i. e.,

$$h_0 = \min_{\mathbf{x} \in X} h(\mathbf{x}). \qquad (4.2)$$

We do not exclude the possibility that h_0 is

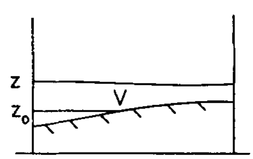

Fig. 2

negative: part of the column may extend below sea level. With the definition (3.1) of $\tilde{\varrho}$ we have

$$P = g \| X \| \int_{h_0}^{\infty} z\tilde{\varrho}(z, t)\, dz, \tag{4.3}$$

which we rewrite

$$P = -\| X \| \int_{h_0}^{\infty} z \frac{\partial \tilde{p}}{\partial z} dz + \varepsilon \tag{4.4}$$

where

$$\varepsilon = \| X \| \int_{h_0}^{\infty} z \left[\frac{\partial \tilde{p}}{\partial z} + g\tilde{\varrho}(z, t) \right] dz. \tag{4.5}$$

Since the quasi-static approximation (3.4) holds except possibly at low altitudes, ε is small compared with the leading term on the right side of (4.4). Integrating (4.4) by parts, we obtain

$$P = \| X \| \, h_0 \tilde{p}(h_0, t) + \| X \| \int_{h_0}^{\infty} \tilde{p}\, dz + \varepsilon; \tag{4.6}$$

this result assumes that $\tilde{p}$ falls off faster than z^{-1} as $z \to \infty$.

We now use the definition (3.2) of $\tilde{p}$. Since $h(\mathbf{x}) = h_0$ identically thruout A_{h_0}, we have

$$\| X \| \, h_0 \tilde{p}(h_0, t) = \iint_X h_0 p_s(\mathbf{x}, t)\, dX. \tag{4.7}$$

The integral term in (4.6) becomes

$$\int_{h_0}^{\infty} \iint_{A_z} p(\mathbf{x}, z, t)\, dX\, dz + \int_{h_0}^{\infty} \iint_{(X - A_z)} p_s(\mathbf{x}, t)\, dX\, dz.$$

The first of these terms is simply the volume integral of pressure over the column C. If the order of integration is reversed, the second term becomes

$$\iint_X \int_{h_0}^{h(\mathbf{x})} p_s(\mathbf{x}, t)\, dz\, dX,$$

which can be simplified by noting that the integrand is independent of z. Thus we obtain

$$\| X \| \int_{h_0}^{\infty} \tilde{p} dX = \iiint_C p dV + \iint_X (h(\mathbf{x}) - h_0)\, p_s(\mathbf{x}, t)\, dX. \tag{4.8}$$

With (4.7) and (4.8), the equation (4.6) for the potential energy within C becomes

$$P = \iiint_C p dV + \iint_X h(\mathbf{x})\, p_s(\mathbf{x}, t)\, dX + \varepsilon. \tag{4.9}$$

The thermal energy I in the column C is given by

$$I = \iiint_C \varrho e dV, \tag{4.10}$$

where e denotes the specific internal energy, i. e., $e = c_v T$. Since $\varrho T = p/R$,

$$I = (c_v/R) \iiint_C p dV, \tag{4.11}$$

If X is ocean, so that ε vanishes and $h(\mathbf{x}) \equiv h_0 = 0$, comparison of (4.9) and (4.11) reveals that P and I are indeed proportional:

$$(P/I)_{\text{ocean}} = R/c_N. \tag{4.12}$$

Using the relation $(c_p - c_v) = R$, we find that the "total potential energy" $(P + I)$ is given by

$$(P + I)_{\text{ocean}} = \iiint_C \varrho i d V, \tag{4.13}$$

where i denotes the specific enthalpy of the dry phase, i.e., $i = c_p T$.

More generally, the effects of orography and hydrostatic defect must beincluded. Adding (4.11) to (4.9) yields

$$(P + I) = \iiint_C \varrho i d V + \iint_X h(\mathbf{x})\, p_s(\mathbf{x}, t)\, dX + \varepsilon. \tag{4.14}$$

We thus see that the total potential energy and the total enthalpy of the column C are equal if and only if it is over ocean. Moreover, the orographic term cannot be eliminated by redefining the potential energy. If we take $z = H$ as the zero level of geopotential, the analysis leading to (4.14), *mutatis mutandis*, leads instead to

$$(P + I) = \iiint \varrho i d V + \iint [h(\mathbf{x}) - H] p_s(\mathbf{x}, t)\, dX + \varepsilon. \tag{4.15}$$

For any given instant, the orographic term vanishes if we choose

$$H = \frac{\iint_X h(\mathbf{x})\, p_s(\mathbf{x}, t)\, dX}{\iint_X p_s(\mathbf{x}, t)\, dX}, \tag{4.16}$$

but it cannot be made to vanish identically in time unless $h(\mathbf{x})$ is constant over X. It becomes fairly small if H is defined by (4.16), with $p_s(\mathbf{x}, t)$ replaced by a "standard atmosphere" expression for pressure as a function of height. Even if this procedure leads to an orographic term which is small compared with the enthalpy term, it entails the drawback, that the zero level of potential energy must be separately defined for each atmospheric column.

RÉSUMÉ

L'auteur étudie la relation entre l'enthalpie et l'énergie potentielle totale dans une collonne verticale d'étendue horizontale finie. Il introduit a priori plutôt qu'à posteriori les restrictions de l'hypothèse quasi statique, c'est-à-dire qu'à l'échelle d'un cyclone, il admet que les masses d'air sont en équilibre hydrostatique, mais il renonce à toute relation différentielle exprimant cet équilibre à petite échelle. L'auteur démontre que l'enthalpie et l'énergie potentielle totale d'une colonne au-dessus de l'océan sont égales, mais qu'en général il existe un terme d'origine orographique qu'on ne peut éliminer en choisissant un autre niveau zéro pour le géopotentiel. De plus, il existe aussi un terme résultant, dans le voisinage du relief du sol, de l'écart par rapport à l'équilibre hydrostatique; ce terme estvraisemblablement négligeable pour une colonne dont l'étendue horizontale est de l'ordre de grandeur de celle d'un cyclone.

REFERENCES

Arakawa, A. Computational Design for Long Term Numerical Integrations of the Equations of Atmospheric Motion. Paper presented at 44th Annual Meeting. American Geophysical Union, Washington D.C., April 1963.

Eliassen A. & Kleinschmidt, E., Jr. Dynamic Meteorology. *Handbuch der Physik*, XLVIII, 1957.

Kibel, I. A. *Hydrodynamic Methods of Short Period Weather Forecasting*. Pergamon Press, 1963.

Kurihara Y. & Holloway, J. L., Jr. Numerical Integration of a Nine-Level Global Primitive Equations Model Formulated by the Box Method. *Monthly Weather Review, 95,* Aug. 1967.

Langlois, W. E. Integral Formulation of the Global Circulation Problem. IBM Research Report RJ 359, November 1965.

ПОЛНАЯ ПОТЕНЦИАЛЬНАЯ ЭНЕРГИЯ КОНЕЧНОГО СТОЛБА

Исследуется соотношение между энтальпией и полной потенциальной энергией в столбе конечных горизонтальных размеров. Ограничения квазистатической аппроксимации вводятся априори, скорее, чем апостериори, т. е. предполагается, что воздушные массы циклонического масштаба находятся в гидростатическом равновесии, но при этом удается избежать использования уравнения, подразумевающего мелкомасштабный гидростатический баланс. Найдено, что энтальпия и полная потенциальная энергия равны для столба над океаном, но в общем случае имеется орографический член, который не может быть исключен путем нового определения нулевого уровня геопотенциала. Имеется также член, являющийся следствием отклонения от гидростатического баланса в окрестности нерегулярности поверхности; этот член, по-видимому, пренебрежим для столба с горизонтальными размерами циклонического масштаба.