Sources and sinks on a β-earth

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ABSTRACT

This paper is concerned with flow on a β-earth toward a sink. A perfect-fluid solution for a strong sink at the equator midway between two rigid latitude circles shows that the streamline that divides flow from the west and flow from the east is a straight, north-south line through the sink when the β-effect is neglected, but tilts toward the west when it is taken into account. The β-effect also causes the sink to draw more strongly from its own latitude in the region west of the sink and less strongly from its own latitude east of the sink. This agrees with qualitative reasoning based on the vorticity equations. Solutions are also found for a weak, viscous sink at an arbitrary latitude. Approximate analytical expressions, valid far from the origin, indicate greatly different flow patterns in the western and eastern sectors. In the former the motion is divided into two components. One is a jet at the latitude of the sink, and this jet carries the entire flux into the sink. The other corresponds to motion induced by the basic rotation itself and is a flattened portion of a vortex in the same region occupied by the westerly jet. The jet component dominates at sufficiently great distances. The flow from the east dies off exponentially with distance and has a wave-like character.

1. Introduction

The study of basic fluid systems in geophysical fluid mechanics has led to fundamental understanding of some of the important effects of rotation and stratification. This paper seeks to extend this understanding by an analysis of flow toward a sink on a β-earth.

The first investigation of sinks in rotating and stratified fluids was given by Long (1956) who solved for flow in a rotating cylinder of homogeneous fluid toward a strong point sink at the axis. A few cases are shown in Fig. 1. When the rotation is zero, the motion is irrotational. Rotation tends to cause a concentration of the streamlines near the axis. This is strongly emphasized for the lowest value of the Rossby number, $R_0 = .261$, for which a solution was found. The fluid then approaches in a jet at the axis, and there is reverse flow at the walls. Superimposed on this motion is a circulating velocity component which becomes quite strong when the jet forms. The flow at $R_0 = .261$ is caused by the propagation up the cylinder of a long wave which creates a concentrated velocity at the axis and reverse flow further out.

The same kind of solution may be found for the case of a stratified fluid moving in a semi-infinite channel toward a sink at one end. This was done by Yih (1958). The results were similar, with the role of the Rossby number now played by the internal Froude number F_1. When F_1 is infinite, the effect of stratification is zero, and the motion is irrotational. At lower values of F_1 the sink draws more strongly from its own level until, at a critical value of F_1, it approaches in a jet, similar to that shown in Fig. 1, with reverse flow at the top and bottom.

Experiments (Long, 1956; Debler, 1960) reveal motions similar to theoretical predictions for large F_1 or R_0. But the flows at the critical values of R_0 and F_1 are unstable, and one would hardly expect to see them in experiments. In fact, under such conditions (and for still smaller values of F_1 and R_0), the observed motion is in the form of a single jet which becomes more

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and more concentrated as the flux becomes weaker and weaker; the reverse flow of the theory is not observed. Obviously viscosity and diffusion dominate when the strong gradients characteristic of jets occur.

It is well known that steady flow on a β-earth[1] is strongly analogous to stratified flow, and one might suppose that it would be relatively easy to obtain solutions of the type discussed above, i.e., for flow toward a point sink at (0, 0) midway between two latitude circles. Two difficulties arise, however. One is related to the form at the vorticity equation. Thus, if Ψ is the streamfunction, the steady-state vorticity equation, assuming uniform approach flow at great distances east and west, is

$$\nabla^2\Psi + \beta y = -\frac{\beta}{U}\Psi, \tag{1}$$

where U is the uniform westerly flow far from the sink. However, since the constant β/U is positive in the region west of the sink and negative in the region east of the sink, the corresponding solutions of (1) differ fundamentally. For example, the flow is not symmetrical about the line $x = 0$, and we must not, therefore, assume that $x = 0$ is a streamline.[2] As a result, if there is a streamline that divides the two regions of flow governed by the two different forms of Eq. (1), it must be located from the non-linear kinematic and dynamic conditions to be imposed at this boundary. The second difficulty relates to the effect of the basic rotation or f_0. This does not appear in the vorticity equation, but it is obvious from the equations of motion that the pressure distribution in the vicinity of a sink, arising from the tendency for geostrophy, will cause a local vortex to form. The combined problem appears to be a very difficult one for strong sinks, and we will limit ourselves to the special case of a sink at the equator. Here, $f_0 = 0$, and the "β-effect" contributes all of the rotational aspects of the motion. It is likely that the results of this specialized investigation will reveal basic effects of the β-term in the vorticity equation for a sink at an arbitrary latitude. An approximate analysis along these lines is in Section 2. The solution for slow, viscous flow toward a sink at an arbitrary latitude is in Section 3.

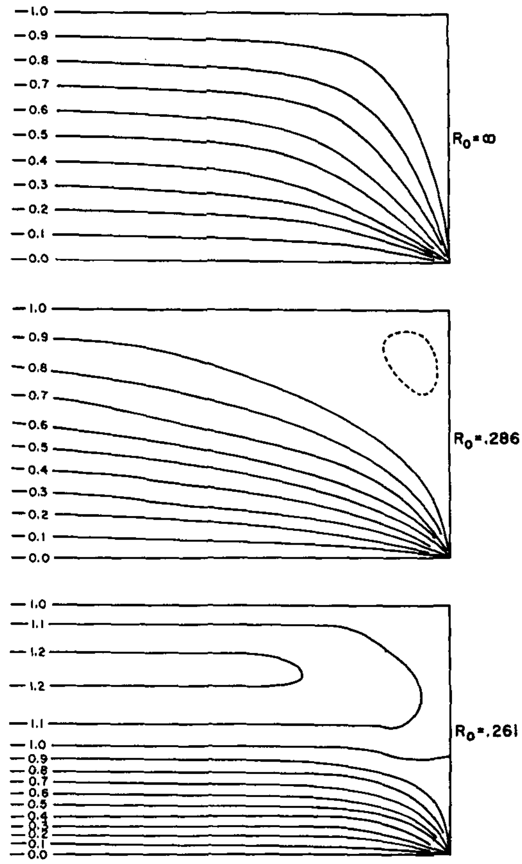

Fig. 1. Flow toward a sink at the axis of a rotating fluid. The streamlines are for flow in a plane containing the axis of the cylinder, but only the upper half is shown. The Rossby number is $R_0 = U/2\Omega a$ where a is the radius of the cylinder, Ω is the basic angular velocity and the flux into the sink is $U\pi a^2$.

2. Perfect-fluid sink at the equator of a β-earth

We suppose that there is a sink at the equator midway between two rigid latitude lines. We choose a unit length $h/2$, where h is the total

[1] The term β-earth is now used to describe a model of barotropic, planetary flow first introduced by Rossby (1939). In the model, the earth is considered to be a plane with x-axis east-west and y-axis north-south. The Coriolis parameter f is approximated by a linear function of y, i.e., $f = f_0 + \beta y$. A more detailed discussion for problems similar to the one of this paper has been given by Long (1960).

[2] Of course we could demand that $x = 0$ is a streamline. The resulting velocities, however, would be discontinuous across this line.

width of the channel, and a unit speed $Q/2h$, where Q is the total flux into the sink. Some question arises as to whether the speeds of the zonal currents at great distances from the sink in the two directions are the same. We can imagine that a sink is started impulsively, and that the β-effect modifies the motion as time goes on. The initial flow will be irrotational, and the nondimensional, uniform speeds of the approaching easterly and westerly currents will be the same (unity). Since the only possible wave in the system, the Rossby wave, can only move from east to west, it follows that no disturbance can ever propagate toward the east, so that the fluid speed at $x = +\infty$ remains the same, or unity. Since the total flux is held fixed the *mean* flow at $x = -\infty$ will also remain unity, but it is possible that the uniformity of the flow may be altered by the propagation upstream of long Rossby waves. In this section, however, we confine attention to strong sinks; consequently, no wave can progress toward the west against the current, and we may assume a uniform speed of unity at $x = -\infty$ as well.

The basic equations of the problem are

$$\nabla^2\Psi + \varepsilon y = -\varepsilon\Psi, \tag{2a}$$

$$\nabla^2\varphi + \varepsilon y = \varepsilon\varphi, \tag{2b}$$

where $$\varepsilon = \frac{\beta h^3}{2Q},$$

and we have chosen the streamfunctions to be Ψ and φ in the western and eastern sections, respectively. We take ε to be small (strong sink), and expand in powers of ε. For example, the streamline dividing the two regions is given by the equation

$$x = \varepsilon\eta_1(y) + \varepsilon^2\eta_2(y) + \dots . \tag{3}$$

We separate into a uniform flow and a disturbance:

$$\Psi = -y + \Psi', \tag{4a}$$

$$\varphi = y + \varphi'. \tag{4b}$$

Then $$\nabla^2\Psi' + \varepsilon\Psi' = 0, \tag{5a}$$

$$\nabla^2\varphi' - \varepsilon\varphi' = 0. \tag{5b}$$

The dividing streamline is given by $\Psi = -1$ or $\varphi = 1$ in the region $0 < y \leqslant 1$, and $\Psi = 1$ or $\varphi = -1$ in $-1 \leqslant y < 0$. Assuming antisymmetry of the streamfunctions about $y = 0$, we need only impose the conditions

$$\left.\begin{aligned}\Psi' &= (y-1) \quad (6\text{a})\\ \varphi' &= -(y-1) \quad (6\text{b})\end{aligned}\right\} 0 < y \leqslant 1, \quad x + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \dots .$$

Continuity of the velocity across the dividing streamline[1] requires that

$$\frac{\partial\Psi}{\partial x} - \frac{\partial\varphi}{\partial x} - \left(\varepsilon\frac{\partial\eta_1}{\partial y} + \varepsilon^2\frac{\partial\eta_2}{\partial y} + \dots\right)\left(\frac{\partial\Psi}{\partial y} - \frac{\partial\varphi}{\partial y}\right) = 0 \tag{7}$$

at $x = \varepsilon\eta_1 + \varepsilon^2\eta_2 + \dots$. We now expand the solution as

$$\Psi' = \Psi_0' + \varepsilon\Psi_1' + \dots, \tag{8a}$$

$$\varphi' = \varphi_0' + \varepsilon\varphi_1' + \dots, \tag{8b}$$

and write down the first two sets of governing equations. The first approximation is

$$\nabla^2\Psi_0' = 0, \tag{9a}$$

$$\nabla^2\varphi_0' = 0, \tag{9b}$$

$$\left.\begin{aligned}\Psi_0' &= y - 1 \quad (10\text{a})\\ \varphi_0' &= -y + 1 \quad (10\text{b})\end{aligned}\right\} \text{ at } x = 0 \quad (0 < y \leqslant 1),$$

$$\frac{\partial\Psi_0'}{\partial x} = \frac{\partial\varphi_0'}{\partial x} \quad \text{at} \quad x = 0 \quad (0 < y \leqslant 1), \tag{11}$$

together with the conditions that Ψ_0' and φ_0' vanish at the latitude boundaris and at $|x| = \infty$.

The solution of this set is well known:

$$\Psi_0' = -\frac{2}{\pi}\sum_1^\infty \frac{e^{n\pi x}}{n}\sin n\pi y = \vartheta - \frac{2}{\pi}\ln(1 - e^{\pi z}), \tag{12a}$$

$$\varphi_0' = \frac{2}{\pi}\sum_1^\infty \frac{e^{-n\pi x}}{n}\sin n\pi y = -\vartheta - \frac{2}{\pi}\ln(1 - e^{-\pi\bar{z}}), \tag{12b}$$

where $z = x + iy$ and $\bar{z} = x - iy$. This is irrotational flow, and represents a symmetric streaming toward the sink from both directions.

The equations of the second approximation are

[1] The Bernoulli Equation then shows that the pressure across the streamline is continuous if we make proper choices of pressure at $(\pm\infty, 0)$.

$$\nabla^2 \Psi_1' + \Psi_0' = 0, \tag{13a}$$

$$\nabla^2 \varphi_1' - \varphi_0' = 0, \tag{13b}$$

$$\left.\begin{aligned} &\frac{\partial \Psi_0'}{\partial x}\eta_1 + \Psi_1' = 0 \quad &(14\,\mathrm{a})\\ &\frac{\partial \varphi_0'}{\partial x}\eta_1 + \varphi_1' = 0 \quad &(14\,\mathrm{b}) \end{aligned}\right\} \text{ at } x = 0 \quad (0 < y \leqslant 1),$$

$$\frac{\partial \Psi_1'}{\partial x} - \frac{\partial \varphi_1'}{\partial x} + \eta_1\left(\frac{\partial^2 \Psi_0'}{\partial x^2} - \frac{\partial^2 \varphi_0'}{\partial x^2}\right) - \frac{\partial \eta_1}{\partial y}\left(-2 + \frac{\partial \Psi_0'}{\partial y} - \frac{\partial \varphi_0'}{\partial y}\right) = 0 \tag{15}$$

at $x = 0$, $0 < y \leqslant 1$. We see from (11) and (14) that $\Psi_1' = \varphi_1'$ at $x = 0$. This condition and Eqs. (13) are satisfied by

$$\Psi_1' = \frac{1}{\pi^2}\sum_1^\infty \frac{x}{n^2} e^{n\pi x} \sin n\pi y + \sum_1^\infty A_n e^{n\pi x} \sin n\pi y, \tag{16a}$$

$$\varphi_1' = -\frac{1}{\pi^2}\sum_1^\infty \frac{x}{n^2} e^{-n\pi x} \sin n\pi y + \sum_1^\infty A_n e^{-n\pi x} \sin n\pi y. \tag{16b}$$

Equations (14a) and (15) yield

$$-\eta_1 \cot \frac{\pi y}{2} + \sum_1^\infty A_n \sin n\pi y = 0, \tag{17}$$

$$\frac{2}{\pi^2}\sum_1^\infty \frac{\sin n\pi y}{n^2} + 2\sum_1^\infty A_n n\pi \sin n\pi y = 0. \tag{18}$$

Thus
$$A_n = -\frac{1}{n^3\pi^3}, \tag{19}$$

$$\Psi_1' = \frac{1}{\pi^2}\sum_{n=1}^\infty \frac{x}{n^2} e^{n\pi x} \sin n\pi y - \frac{1}{\pi^3}\sum_{n=1}^\infty \frac{e^{n\pi x}}{n^3} \sin n\pi y \tag{20a}$$

$$\varphi_1' = -\frac{1}{\pi^2}\sum_{n=1}^\infty \frac{x}{n^2} e^{-n\pi x} \sin n\pi y - \frac{1}{\pi^3}\sum_{n=1}^\infty \frac{e^{-n\pi x}}{n^3} \sin n\pi y, \tag{20b}$$

$$\eta_1 = -\frac{1}{\pi^3} \tan \frac{\pi y}{2} \sum_{n=1}^\infty \frac{\sin n\pi y}{n^3}. \tag{21}$$

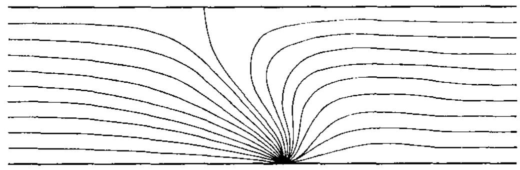

Fig. 2. Flow pattern for strong, perfect-fluid sink. $\varepsilon = .50$.

The total motion to this order is shown in Fig. 2 for a value of $\varepsilon = 0.50$. A basic feature of the rotational motion is that the approaching speeds near the sink and at the latitude of the sink (the equator) are increased in the western sector and decreased in the eastern sector. This is consistent with a qualitative argument, based on the vorticity equation, that flow coming from the west and converging toward the sink will be accompanied by a generation of relative vorticity of the same sign as that in a westerly velocity maximum at the latitude line of the sink. The tilting of the dividing streamline toward the west means that more fluid enters the sink from the western half of the region than from the eastern half. This tendency is a basic feature of the viscous case discussed in the next section.

3. Viscous sink

The perfect-fluid solution of Section 2 applies only for very strong sinks. As the sink at the equator weakens, we may guess that a strong jet begins to form in the western region, and that viscosity will become important as the shears increase. We have little indication as to the behavior of the motion from the eastern region but the total motion will surely be controlled by friction for sufficiently small velocities.[1] Let us therefore investigate the flow into a very weak sink, and we may now permit it to be at an arbitrary latitude. Although we assume that inertia terms are negligible except near the singularity, the solution will also represent motion toward a sink of arbitrary strength at sufficiently great distances from the singularity.

[1] A paper by Stommel, Arons & Faller (1958), contained a study of slow motion in a similar rotating system in the presence of sources and sinks and lateral boundaries. The theoretical discussion neglected viscosity. The experiments showed a strong tendency for zonal jets, as in the present paper.

The appropriate equations, neglecting non-linear terms, are

$$-(s+y)v = -p_x + u_{xx} + u_{yy}, \qquad (22a)$$

$$+(s+y)u = -p_y + v_{xx} + v_{yy}, \qquad (22b)$$

$$u_x + v_y = 0, \qquad (23)$$

where our unit of velocity is $Q(\beta/\nu)^{\frac{1}{2}}$, our unit of length is $(\nu/\beta)^{\frac{1}{2}}$, and

$$s = \frac{f_0}{\nu^{\frac{1}{2}}\beta^{\frac{1}{2}}}.$$

The fluid will be considered unlimited. We may incorporate the effect of the sink into our equations by using, instead of (23), the equation

$$u_x + v_y = -\delta(x, y), \qquad (24)$$

where $\delta(x, y)$ is the delta-function: zero everywhere except at (0, 0), and such that

$$1 = \int_{-\infty}^{\infty}\int \delta(x, y)\, dx\, dy.$$

If we now integrate (24) over the whole plane, we find that there is a sink of strength 1 at the origin.

If we eliminate the pressure in Eqs. (22), and use (24), we get

$$-(s+y)\delta + v = v_{xxx} + v_{xyy} - u_{yxx} - u_{yyy}. \qquad (25)$$

This and Eq. (24) are the differential equations of the problem.

Let us now form the Fourier integrals for u or v. Assuming that the Fourier transforms U, V exist, we may write

$$u = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int U(l, m)\, e^{ilx + imy}\, dl\, dm, \qquad (26)$$

$$v = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int V(l, m)\, e^{ilx + imy}\, dl\, dm. \qquad (27)$$

Transforming (24) and (25), we obtain

$$ilU + imV = -1, \qquad (28)$$

$$V + il(l^2 + m^2)V - im(l^2 + m^2)U = s. \qquad (29)$$

These determine U and V. We get

$$u = -\frac{1}{4\pi^2}\int\int_{-\infty}^{\infty} \frac{1 + il(m^2 + l^2) + ims}{il - (m^2 + l^2)^2}\, e^{ilx + imy}\, dl\, dm, \qquad (30)$$

$$v = -\frac{1}{4\pi^2}\int\int_{-\infty}^{\infty} \frac{im(l^2 + m^2) - ils}{il - (m^2 + l^2)^2}\, e^{ilx + imy}\, dl\, dm. \qquad (31)$$

Let us consider u first. If we take a complex plane ξ in which the real axis is

$$-\infty \leqslant l \leqslant \infty,$$

we get

$$u = \pm\frac{1}{4\pi^2}\int_{-\infty}^{\infty} e^{imy}\, dm \times \oint \frac{1 + i\xi(m^2 + \xi^2) + ims}{(\xi^2 + m^2)^2 - i\xi}\, e^{i\xi x}\, d\xi, \qquad (32)$$

where the cyclic integral is in the positive sense along the boundary of a large half-disk whose base is along the real axis and whose circular portion is in the upper half of the plane for $x > 0$ and the lower half for $x < 0$. (In these two cases we choose the $+$ sign for $x > 0$ and the minus sign for $x < 0$.) The contributions of the circular portions then vanish as the radius of the circle becomes larger and larger. To evaluate (32) by the method of residues, we must first find the roots of the denominator in the integrand, i.e., roots of

$$\xi^4 + 2\xi^2 m^2 + m^4 - i\xi = 0. \qquad (33)$$

If we denote the four roots by

$$\xi_i = l_i + i\lambda_i \quad (i = 1, 2, 3, 4), \qquad (34)$$

they are given by the *real* roots of

$$l^4 - 6l^2\lambda^2 + \lambda^4 + 2m^2(l^2 - \lambda^2) + m^4 + \lambda = 0, \qquad (35)$$

$$l[4l^2\lambda - 4\lambda^3 + 4m^2\lambda - 1] = 0. \qquad (36)$$

There are two corresponding to $l = 0$, i.e.,

$$\xi_1 = i\lambda_1, \quad \xi_2 = i\lambda_2, \qquad (37)$$

where λ_i are the real roots of

$$(\lambda^2 - m^2)^2 + \lambda = 0. \qquad (38)$$

Obviously λ_1 and λ_2 are both negative, so that they will contribute to the cyclic integral only when $x < 0$. They may be expanded as functions of m in series form:

$$\left.\begin{aligned} \xi_1 &= i\left[-1-\frac{2}{3}m^2+\frac{m^4}{3}-\frac{28}{81}m^6+\dots\right] && \text{small } m, \\ \xi_1 &= i\left[-m-\frac{1}{2m^{\frac{1}{2}}}+\dots\right] && \text{large } m, \end{aligned}\right\} \tag{39}$$

$$\left.\begin{aligned} \xi_2 &= i[-m^4+2m^{10}+\dots] && \text{small } m, \\ \xi_2 &= i\left[-m+\frac{1}{2m^{\frac{1}{2}}}+\dots\right] && \text{large } m. \end{aligned}\right\} \tag{40}$$

The two roots of (35) and (36) corrresponding to $l \neq 0$ are

$$\left.\begin{aligned} \xi_3 &= \frac{\sqrt{3}}{2}\left[1-\frac{2}{3}m^2+\frac{28}{81}m^6+\dots\right] \\ &\quad + i\left[\frac{1}{2}+\frac{m^2}{3}+\frac{m^4}{3}+\frac{14}{81}m^6+\dots\right] && \text{small } m, \\ \xi_3 &= \frac{1}{2m^{\frac{1}{2}}}\left[1+\frac{1}{32m^3}+\dots\right] \\ &\quad + i\left[m+\frac{1}{128m^5}+\dots\right] && \text{large } m, \end{aligned}\right\} \tag{41}$$

$$\left.\begin{aligned} \xi_4 &= -\frac{\sqrt{3}}{2}\left[1+\frac{2}{3}m^2+\frac{28}{81}m^6+\dots\right] \\ &\quad + i\left[\frac{1}{2}+\frac{m^2}{3}+\frac{m^4}{3}+\frac{14}{81}m^6+\dots\right] && \text{small } m, \\ \xi_4 &= -\frac{1}{2m^{\frac{1}{2}}}\left[1-\frac{1}{32m^3}+\dots\right] \\ &\quad + i\left[m+\frac{1}{128m^5}+\dots\right] && \text{large } m. \end{aligned}\right\} \tag{42}$$

Both of these are in the upper half of the ξ-plane so contribute to the cyclic integral only for $x > 0$.

Let us now evaluate u and v at great distances ($|x| \gg 1$) from the sink when $x < 0$. The velocity integrals are

$$\begin{aligned} u = &-\frac{i}{\pi}\int_0^\infty \\ &\times \frac{e^{-\lambda_1 x}\{[1-\lambda_1(m^2-\lambda_1^2)]\cos my - ms\sin my\}}{(\xi_1-\xi_2)(\xi_1-\xi_3)(\xi_1-\xi_4)}\,dm \\ &-\frac{i}{\pi}\int_0^\infty \\ &\times \frac{e^{-\lambda_2 x}\{[1-\lambda_2(m^2-\lambda_2^2)]\cos my - ms\sin my\}}{(\xi_2-\xi_1)(\xi_2-\xi_3)(\xi_2-\xi_4)}\,dm, \end{aligned} \tag{43}$$

$$\begin{aligned} v = &-\frac{i}{\pi}\int_0^\infty \\ &\times \frac{e^{-\lambda_1 x}\{-m(m^2-\lambda_1^2)\sin my + \lambda_1 s\cos my\}}{(\xi_1-\xi_2)(\xi_1-\xi_3)(\xi_1-\xi_4)}\,dm \\ &-\frac{i}{\pi}\int_0^\infty \\ &\times \frac{e^{-\lambda_2 x}\{-m(m^2-\lambda_2^2)\sin my + \lambda_2 s\cos my\}}{(\xi_2-\xi_1)(\xi_2-\xi_3)(\xi_2-\xi_4)}\,dm. \end{aligned} \tag{44}$$

The first integral in each of these equations may be evaluated by the method of steepest descents. λ_1 is a stationary function of m at $m = 0$ only, so that for large $|x|$, these integrals are of the order of

$$|x|^{-\frac{1}{2}}e^{-|x|}.$$

The integrands in the second integrals tend to zero exponentially as $|x| \to \infty$ except when $m = 0$, $\lambda_2 = 0$. In this vicinity

$$-\lambda_2 = m^4 - 2m^{10} + \dots \tag{45}$$

so that

$$u \to +\frac{1}{\pi}\int_0^\infty \cos my\, e^{m^4 x}\,dm - \frac{s}{\pi}\int_0^\infty m\sin my\, e^{m^4 x}\,dm, \tag{46}$$

$$v \to -\frac{1}{\pi}\int_0^\infty m^3\sin my\, e^{m^4 x}\,dm - \frac{s}{\pi}\int_0^\infty m^4\cos my\, e^{m^4 x}\,dm. \tag{47}$$

This corresponds to a streamfunction

$$\Psi' = -\frac{1}{\pi}\int_0^\infty \frac{1}{m}\sin my\, e^{m^4 x}\,dm - \frac{s}{\pi}\int_0^\infty \cos my\, e^{m^4 x}\,dm. \tag{48}$$

If we put $z = +m^4|x|$ and $\eta = y/|x|^{\frac{1}{4}}$, this may be written

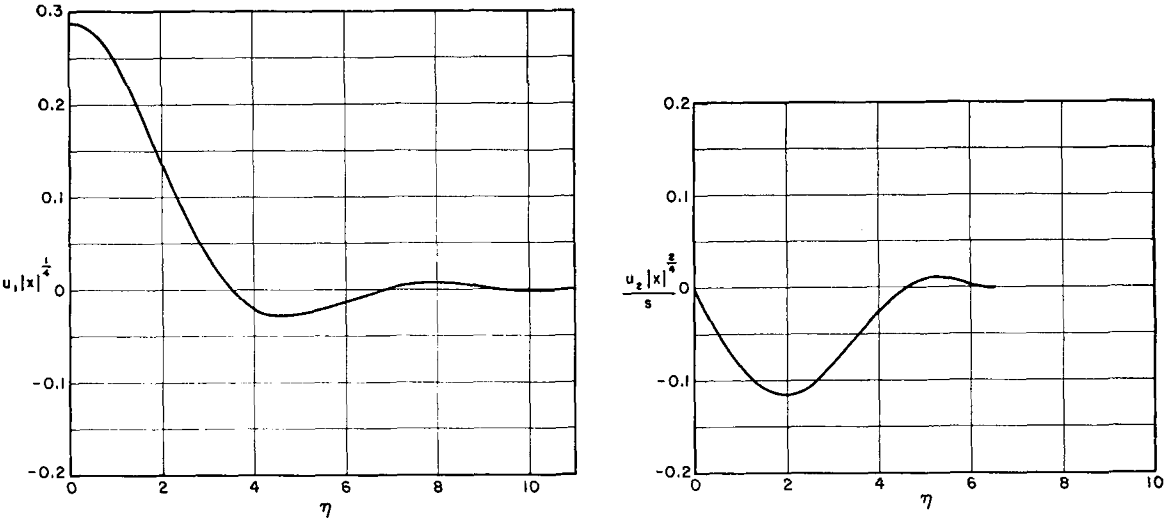

Fig. 3. Graphs of $u_1|x|^{\frac{1}{4}}$ and $\dfrac{u_2|x|^{\frac{2}{4}}}{s}$.

$$\begin{aligned}\Psi &= \Psi_1 + \Psi_2 \\ &= -\frac{1}{4\pi}\int_0^\infty \frac{e^{-z}}{z}\sin \eta z^{\frac{1}{4}}\,dz \\ &\qquad - \frac{s}{4\pi|x|^{\frac{1}{4}}}\int_0^\infty \frac{e^{-z}}{z^{\frac{3}{4}}}\cos \eta z^{\frac{1}{4}}\,dz. \end{aligned} \tag{49}$$

This is a combination of two similarity solutions. It is easy to see that they represent solutions of (24) and (25) with the terms v_{xxx} and u_{yxx} missing. The streamfunction Ψ_1 in (49) represents flow from the west in a jet centered at the latitude of the sink. The width of the jet is

$$\delta \sim \left(\frac{L\nu}{\beta}\right)^{\frac{1}{3}},$$

where L is east-west distance from the singularity. The term Ψ_2 arises from the basic rotation and represents a flattened portion of a vortex occupying the same region. Notice that the jet component dominates at sufficiently great distances toward the west. All of the motion takes place in a boundary layer in which derivatives along the axis are small compared with normal derivatives. The largest neglected terms in the approximation of (46) and (47) are terms involving m^6 times the retained terms of the integrand, so that the ratio of the neglected terms to those retained in (48) is of the order of $|x|^{-\frac{3}{4}}$. This is in accordance with the error in the boundary-layer approximation.

The streamfunction may be evaluated as follows. Expanding the sine and cosine functions, we get

$$\begin{aligned}\Psi = &-\frac{1}{4\pi}\int_0^\infty e^{-z}\left(\eta z^{-\frac{3}{4}} - \frac{\eta^3 z^{-\frac{1}{4}}}{3!} + \frac{\eta^5 z^{\frac{1}{4}}}{5!} - \dots\right)dz \\ &- \frac{s}{4\pi|x|^{\frac{1}{4}}}\int_0^\infty e^{-z} \\ &\times \left(z^{-\frac{3}{4}} - \frac{\eta^2 z^{-\frac{1}{4}}}{2!} + \frac{\eta^4 z^{\frac{1}{4}}}{4!} + \dots\right)dz, \end{aligned} \tag{50}$$

$$\begin{aligned}\Psi = &-\frac{1}{4\pi}\left[\eta\left(-\frac{3}{4}\right)! - \frac{\eta^3(-\frac{1}{4})!}{3!} + \frac{\eta^5(\frac{1}{4})!}{5!} + \dots\right] \\ &- \frac{s}{4\pi|x|^{\frac{1}{4}}}\left[\left(-\frac{3}{4}\right)!\right. \\ &\left. - \frac{\eta^2(-\frac{1}{4})!}{2!} + \frac{\eta^4(\frac{1}{4})!}{4!} - \dots\right]. \end{aligned} \tag{51}$$

Velocity profiles are shown in Figs. 3 and 4.

It is of interest to compute the contribution of the western portion of the region to the flux into the sink. The value of ψ_1 at $\eta = \infty$ may be found by another transformation of Eq. (48). If we let $my = t$,

$$\Psi_1 = -\frac{1}{\pi}\int_0^\infty \exp\left(-\frac{t^4}{\eta^4}\right)\frac{\sin t}{t}\,dt. \tag{52}$$

For infinite η this is

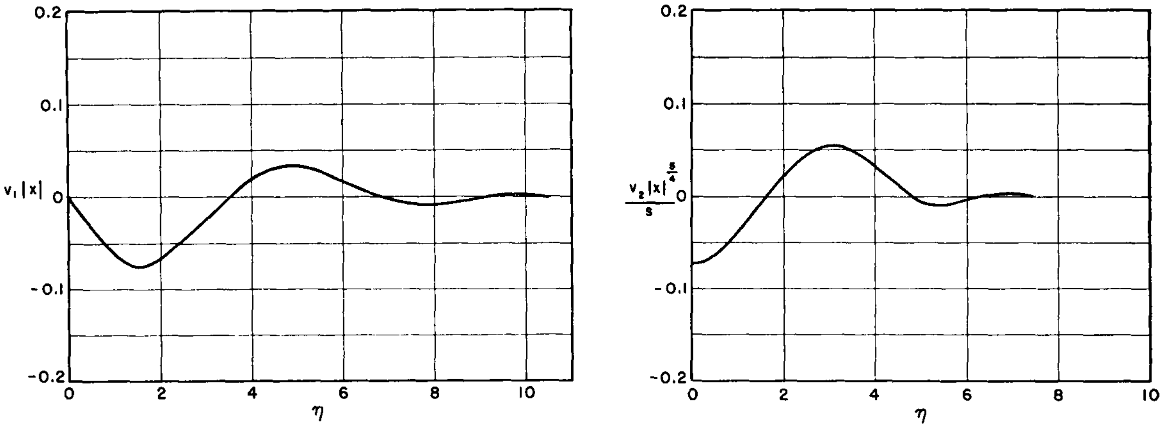

Fig. 4. Graphs of $v_1|x|$ and $\frac{v_2|x|^{\frac{5}{4}}}{s}$.

$$\Psi_1 = -\frac{1}{\pi}\int_0^\infty \frac{\sin t}{t}\,dt = -\frac{1}{2}. \tag{53}$$

Since Ψ_2 contributes no net flux, the total flux from west to east in the western portion of the region is 1 or, in dimensional terms Q. Thus, *all of the fluid supplied to the sink comes from the west in the form of an intense jet.*

When $x > 0$, the velocity integrals are

$$u = \frac{i}{2\pi} \times \int_{-\infty}^{\infty} e^{-\lambda_3 x}\left\{\frac{1 + ims + i\xi_3(m^2+\xi_3^2)}{(\xi_3-\xi_1)(\xi_3-\xi_2)(\xi_3-\xi_4)}e^{i(my+l_3x)} + \frac{1+ims+i\xi_4(m^2+\xi_4^2)}{(\xi_4-\xi_1)(\xi_4-\xi_2)(\xi_4-\xi_3)}e^{i(my-l_3x)}\right\}dm, \tag{54}$$

$$v = \frac{i}{2\pi} \times \int_{-\infty}^{\infty} e^{-\lambda_3 x}\left\{\frac{im(m^2+\xi_3^2)+i\xi_3 s}{(\xi_3-\xi_1)(\xi_3-\xi_2)(\xi_3-\xi_4)}e^{i(my+l_3x)} + \frac{im(m^2+\xi_4^2)-i\xi_3 s}{(\xi_4-\xi_1)(\xi_4-\xi_2)(\xi_4-\xi_3)}e^{i(my-l_3x)}\right\}dm. \tag{55}$$

Since λ_3 is positive (minimum of $\frac{1}{2}$) for all m, we see that the motion drops off exponentially with distance in the eastern sector, so that the flow will be extremely weak at great distances. This is to be expected since, as we have already seen, the entire flux is from the west.

4. Summary

This paper presents a study of two-dimensional flow toward sinks on a β-earth. When there is a strong sink at the equator, there is a slight tendency to draw more strongly from the west at the latitude of the sink and less strongly from the east at this latitude. In addition, there is a slight tendency for the majority of streamlines to approach the sink from the west, as the streamline dividing the two sectors tilts westward. These effects are a maximum for a weak, viscous sink at the equator. Then, there is no net motion from the east, and all the fluid is supplied by an intense westerly jet whose thickness increases gradually as the one-fourth power of distance.[1] When the sink is at an arbitrary latitude, there is an additional component of motion arising from the basic rotation. It is a flattened vortex in the western sector occupying the same area as the jet. The jet motion tends to dominate at sufficiently great distances, however. Both velocity fields drop off exponentially toward the east.

[1] This solution may be more relevant to the equatorial undercurrents in the oceans (Cromwell, Montgomery & Stroup, 1954), than an earlier solution by the author for a constant-momentum-transfer jet on a β-earth (Long, 1960). The latter had no net transfer of fluid.

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ИСТОЧНИКИ И СТОКИ НА β-ЗЕМЛЕ

Рассматриваются стоки в потоке жидкости на β-земле. Решение для идеальной жидкости в случае сильного стока на экваторе, посередине между двумя твердыми кругами, показывает, что линия тока, разделяющая западный и восточный потоки, является прямой линией, направленной с севера на юг, если пренебречь β-эффектом, однако, она наклонена на запад при наличии этого эффекта. β-эффект также заставляет сток действовать сильнее в области к западу от стока и слабее к востоку от стока. Эро согласуется с качественными соображениями, основанными на уравнении вихря.

Также найдены решения для слабого вязкого стока на произвольной широте. Приближенные аналитические выражения, действительные далеко от начала координат, указывают на значительные различия характера потоков в западном и восточном секторах. В первом случае движение разделено на две компоненты. Одна из них представляет собой струю на широте стока, несущую весь свой поток к стоку. Другая соответствует движению, вызываемому основным вращением и является сглаженной частью вихря в области, занятой западной струей. Струйная компонента преобладает на довольно большом расстоянии. Восточный поток экспоненциально затухает с расстоянием и имеет волнообразный характер.