Oscillations in a non-homogeneous viscous fluid

By B. D. DORE, *Department of Mathematics, University of Reading, Reading, England*

(Manuscript received March 8, 1967; revised July 5, 1967)

ABSTRACT

The method of matched asymptotic expansions is employed to derive formulae for viscous corrections occurring in small amplitude oscillatory disturbances in a non-homogeneous fluid. Such formulae are given for quite general variations with depth of the equilibrium density and viscosity distributions in the following cases: (i) two-dimensional disturbances in a fluid of finite depth bounded above by a free surface, (ii) two-dimensional disturbances in fluid bounded above and below by two fixed horizontal planes. The derived expressions are seen to depend on the value of the viscosity at rigid bounding surfaces and not at all on its actual distribution. An additional correction for viscosity is given for case (ii), when the density has an exponential variation with depth and the kinematic viscosity is constant.

1. Introduction

Many investigators have calculated the attenuation with time or distance of small amplitude waves in a homogeneous viscous fluid of finite depth—see, for example, Biesel (1949) and Hunt (1964). Analogous studies to include the effects of non-homogeneity have proved more difficult. Although solutions of the linearised equations can, in principle, be formulated exactly, they are usually of such complexity as to be virtually worthless. Moreover, in many important circumstances, the fluid in question is but slightly viscous. It therefore seems natural to resort to approximate methods of boundary layer type. A suitable modification of the original boundary layer theory has recently been presented by Johns (1967), who illustrated his method by applying it to a two-dimensional problem in which the flow external to the boundary layers is irrotational. A further application has since been carried out by Dore (1967) in studying a case in which the external flow is rotational.

In the present work we adapt Johns' method to consider another example of a rotational external flow, namely that occurring when small amplitude oscillatory disturbances take place in a non-homogeneous viscous fluid which is otherwise at rest. The formulation given here relates to general variations with depth of the equilibrium density and viscosity. For two-dimensional disturbances in a fluid of constant depth bounded above by a free surface, it is found that the derived viscous corrections depend on the value of the viscosity at the bottom and not at all on its actual distribution. When the fluid is bounded above and below by two fixed, horizontal planes the expressions obtained generally depend unequally on the values of the viscosity at these bounding surfaces. By way of illustration the results are considered for a fluid having an exponential variation with depth of its equilibrium density, and the viscous corrections are given graphically for cases when the wave number of the disturbance is prescribed.

2. Formulation of problem

We refer the equations of fluid motion to fixed Cartesian co-ordinates (x, z) for which the z-axis points vertically upwards and the origin lies in the free surface or upper horizontal plane boundary. When the fluid is at rest the pressure, density and viscosity take the forms $\bar{p}(z)$, $\bar{\varrho}(z)$ and $\bar{\mu}(z)$ respectively, where

$$\frac{d\bar{p}}{dz} = -g\bar{\varrho}. \tag{2.1}$$

When there are small amplitude disturbances of the fluid from this equilibrium position, we may consider the equations of motion in linearised form and obtain, as in Yih (1965),

$$\bar{\varrho}\frac{\partial u}{\partial t} = -\frac{\partial p'}{\partial x} + \bar{\mu}\nabla^2 u + \frac{d\bar{\mu}}{dz}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \quad (2.2)$$

$$\bar{\varrho}\frac{\partial w}{\partial t} = -\frac{\partial p'}{\partial z} - g\varrho' + \bar{\mu}\nabla^2 w + 2\frac{d\bar{\mu}}{dz}\frac{\partial w}{\partial z}. \quad (2.3)$$

The horizontal and vertical components of fluid velocity are denoted by u and w, whilst p' and ϱ' are the changes in pressure and density. The linearised equation of incompressibility is

$$\frac{\partial \varrho'}{\partial t} + w\frac{d\bar{\varrho}}{dz} = 0, \quad (2.4)$$

and the (exact) equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \quad (2.5)$$

On defining a stream function χ such that

$$u = \frac{\partial \chi}{\partial z}, \quad w = -\frac{\partial \chi}{\partial x}, \quad (2.6)$$

we find from equations (2.2) to (2.4) that

$$\bar{\varrho}\nabla^2\frac{\partial^2\chi}{\partial t^2} - \bar{\mu}\nabla^4\frac{\partial\chi}{\partial t} = 2\frac{d\bar{\mu}}{dz}\nabla^2\frac{\partial^2\chi}{\partial z\partial t} + \frac{d^2\bar{\mu}}{dz^2}\frac{\partial}{\partial t}\left(\frac{\partial^2\chi}{\partial z^2} - \frac{\partial^2\chi}{\partial x^2}\right) + \frac{d\bar{\varrho}}{dz}\left(g\frac{\partial^2\chi}{\partial x^2} - \frac{\partial^3\chi}{\partial z\partial t^2}\right). \quad (2.7)$$

We shall develop solutions of (2.7) by employing a modification of boundary layer theory given by Johns (1967). Thus, immediately beneath a free surface $z=\zeta$ or adjacent to a rigid boundary $z=0$, there is a viscous layer of thickness δ, whilst adjacent to the rigid boundary $z=-h$, there is a viscous boundary layer of thickness δ. Except within these layers, all velocities and their derivatives are of order unity. We shall write

$$\bar{\varrho}(z) = \varrho^* R(z), \quad \bar{\mu}(z) = \mu^* M(z), \quad (2.8)$$

where ϱ^* and μ^* are reference values of the density and viscosity at the rigid boundary $z=-h$, so that $R(-h)=1=M(-h)$. It will be assumed in the present work that the derivatives of $R(z)$ and $M(z)$ are *everywhere* of order unity. Any boundary conditions associated with oscillating levels will be satisfied at the mean position of such levels, in accordance with the present linearised formulation.

3. Outer flow

To determine the solution outside the viscous layers we prescribe the stream function Ψ there to have the form

$$\Psi(x, z, t; \varepsilon) = \{\Psi_0(z) + \varepsilon\Psi_1(z) + \varepsilon^2\Psi_2(z) + \dots\}e^{i(kx-\sigma t)}, \quad (3.1)$$

$$\sigma = \sigma_0 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2 + \dots, \quad (3.2)$$

where the non-dimensional parameter ε is assumed small and given by

$$\varepsilon = \left(\frac{\nu^*}{\sigma_0 h^2}\right)^{\frac{1}{2}} < 1, \quad \nu^* = \frac{\mu^*}{\varrho^*}, \quad (3.3)$$

and only real parts of complex quantities have physical significance. The real quantity k is the (given) wave number of the disturbance and viscous effects are represented by $\sigma_1, \sigma_2, \dots$. On substituting equations (3.1) to (3.3) into (2.7) we find that Ψ_0 satisfies

$$L\Psi_0 \equiv \Psi_0'' + \frac{R'}{R}\Psi_0' - k^2\left(1 + \frac{gR'}{\sigma_0^2 R}\right)\Psi_0 = 0, \quad (3.4)$$

(where dashes now denote differentiation with respect to z), and represents the solution for small amplitude oscillations in an inviscid fluid. We express the solution of (3.4) in the form

$$\Psi_0 = A_0 Q(z) + B_0 T(z), \quad (3.5)$$

where $Q(z)$ and $T(z)$ are themselves solutions such that

$$\left.\begin{aligned} Q(-h) &= 1, & T(-h) &= 0, \\ Q'(-h) &= 0, & T'(-h) &= 1. \end{aligned}\right\} \quad (3.6)$$

Similarly, Ψ_1 satisfies

$$L\Psi_1 = -\frac{2\sigma_1}{\sigma_0}\frac{gk^2R'}{\sigma_0^2 R}\Psi_0, \quad (3.7)$$

and so, by use of the method of variation of parameters, has solution

$$\Psi_1 = A_1 Q + B_1 T - \frac{2\sigma_1}{\sigma_0}\frac{gk^2}{\sigma_0^2}\int_{-h}^{z} I(s, z)\,ds, \quad (3.8)$$

where

$$I(s, z) = R'(s)\Psi_0(s)[Q(s)T(z) - T(s)Q(z)]. \quad (3.9)$$

4. Inner flow

4.1. *Solution within the lower viscous layer*

To find the solution in the viscous boundary layer adjacent to the rigid boundary $z = -h$, we first write

$$\varepsilon \mathbf{Z} = z + h, \quad (4.1.1)$$

and

$$\delta = \varepsilon \boldsymbol{\lambda} h < h. \quad (4.1.2)$$

The number $\boldsymbol{\lambda}$ is much larger than unity but is of order unity on the scale of ε and thus represents a measure of the boundary layer thickness. Within the layer we expect

$$u = O(1), \quad w = O(\varepsilon), \quad (4.1.3)$$

and therefore write

$$\boldsymbol{\Psi}(x, z, t; \varepsilon) = \varepsilon\{\boldsymbol{\Psi}_0(\mathbf{Z}) + \varepsilon\boldsymbol{\Psi}_1(\mathbf{Z}) + \dots\}e^{i(kx-\sigma t)}. \quad (4.1.4)$$

On recalling our assumption that the derivatives of $R(z)$ and $M(z)$ are of order unity even within the viscous layers and expanding these functions in Taylor series about $z = -h$, use of (4.1.4) in (2.7) gives

$$\mathbf{L}\boldsymbol{\Psi}_0 \equiv h^2\boldsymbol{\Psi}_0^{iv} + i\boldsymbol{\Psi}_0'' = 0, \quad (4.1.5)$$

of which the physically appropriate solution is

$$\boldsymbol{\Psi}_0 = \mathbf{a}_0 e^{\mathbf{n}\mathbf{Z}} + \mathbf{c}_0 + \mathbf{d}_0\mathbf{Z}, \quad (4.1.6)$$

where

$$h\mathbf{n} = (-1 + i)/\sqrt{2}. \quad (4.1.7)$$

4.2. *Solution within the upper viscous layer*

In the upper viscous layer near $z = 0$ we define

$$\varepsilon Z = z, \quad (4.2.1)$$

and

$$\delta = \varepsilon\lambda h < h. \quad (4.2.2)$$

In the case when the upper surface of the fluid is free, u and w are of order unity within the layer so we write

$$\psi(x, z, t; \varepsilon) = \{\psi_0(Z) + \varepsilon\psi_1(Z) + \varepsilon^2\psi_2(Z) + \dots\}e^{i(kx-\sigma t)}, \quad (4.2.3)$$

with

$$\psi_0 = c_0 = \text{constant}. \quad (4.2.4)$$

Then, substitution of (4.2.3) into (2.7) yields

$$\mathrm{L}\psi_1 \equiv h^2 M(0)\psi_1^{iv} + iR(0)\Psi_1'' = 0, \quad (4.2.5)$$

with required solution

$$\psi_1 = a_1 e^{-nZ} + c_1 + d_1 Z, \quad (4.2.6)$$

where

$$hn = (-1 + i)\sqrt{\frac{R(0)}{2M(0)}}. \quad (4.2.7)$$

Similarly, we find

$$\mathrm{L}\psi_2 = ik^2 R(0)\psi_0\left[1 + \frac{gR'(0)}{\sigma_0^2 R(0)}\right] - Z[h^2 M'(0)\psi_1^{iv} + iR'(0)\psi_1''] + h^2\left[\frac{\sigma_1}{\sigma_0}M(0)\psi_1^{iv} - 2M'(0)\psi_1'''\right] - iR'(0)\psi_1', \quad (4.2.8)$$

whence

$$\Psi_2 = a_2 e^{-nZ} + c_2 + d_2 Z + \tfrac{1}{2}\left[k^2 c_0 + \frac{R'(0)}{R(0)}\left(\frac{gk^2 c_0}{\sigma_0^2} - d_1\right)\right]Z^2 + \tfrac{1}{4}a_1 Z e^{-nZ}\left[\frac{M'(0)}{M(0)}(1 + nZ) - \frac{R'(0)}{R(0)}(3 + nZ) - 2n\frac{\sigma_1}{\sigma_0}\right]. \quad (4.2.9)$$

If the upper surface of the fluid coincides with a rigid boundary at $z = 0$, we write

$$\hat{\psi}(x, z, t; \varepsilon) = \varepsilon\{\hat{\psi}_0(Z) + \varepsilon\hat{\psi}_1(Z) + \dots\}e^{i(kx-\sigma t)}, \quad (4.2.10)$$

and obtain

$$\hat{\psi}_0 = \hat{a}_0 e^{-nZ} + \hat{c}_0 + \hat{d}_0 Z. \quad (4.2.11)$$

5. Fluid whose upper surface is free

5.1. *Matching of the outer and inner flows*

The solutions for the inner flows must blend smoothly into that for the outer flow at the outer limit of the viscous layers. Matching methods in general are described in the book by van Dyke (1964), whilst Johns (1967) has shown how they may be applied to small oscillations of a viscous fluid of finite depth in the presence of free surfaces. The technique consists of joining the outer and inner solutions for the velocity components u and w at the edges $z = -\delta = -\varepsilon\lambda h$ and $z = -h + \boldsymbol{\delta} = -h + \varepsilon\boldsymbol{\lambda}h$ of the viscous layers and utilises Taylor expansions for the outer solutions about $z = 0$ and $z = -h$. When this procedure is carried out, exponential terms involving λ and $\boldsymbol{\lambda}$, such as $e^{hn\lambda}$ and $e^{h\mathbf{n}\boldsymbol{\lambda}}$ (which arise from the inner solutions), are neglected as is customary in boundary layer theory [see also Johns (1967)]. Further, algebraic terms in the unspecified numbers λ and $\boldsymbol{\lambda}$ cancel so that there does indeed remain a set of relations between the constants of the outer and inner solutions.

When we use this procedure at the edge $z = -h + \boldsymbol{\delta}$ of the lower viscous boundary layer, we obtain for the present problem

$$A_0 = 0, \tag{5.1.1}$$

$$B_0 = \mathbf{d}_0, \tag{5.1.2}$$

$$A_1 = \mathbf{c}_0. \tag{5.1.3}$$

Similarly, we obtain the relations

$$B_0 T(0) = c_0, \tag{5.1.4}$$

$$B_0 T'(0) = d_1, \tag{5.1.5}$$

$$A_1 Q(0) + B_1 T(0) - \frac{2\sigma_1}{\sigma_0}\frac{gk^2}{\sigma_0^2}\int_{-h}^{0} I(s, 0)\, ds = c_1, \tag{5.1.6}$$

$$A_1 Q'(0) + B_1 T'(0) - \frac{2\sigma_1}{\sigma_0}\frac{gk^2}{\sigma_0^2}\int_{-h}^{0} \frac{\partial I(s, 0)}{\partial z}\, ds = d_2, \tag{5.1.7}$$

from use of the above-mentioned matching procedure at the edge $z = -\delta$ of the upper viscous layer.

It further follows that the vorticity matches automatically to the required order at the edges of the upper and lower viscous layers, on making use of the necessary results from equations (5.1).

5.2. *Satisfying the boundary conditions*

In order that the fluid velocity be zero on $z = -h$ we require in equation (4.1.6) that

$$\mathbf{a}_0 \mathbf{n} + \mathbf{d}_0 = 0, \tag{5.2.1}$$

$$\mathbf{a}_0 + \mathbf{c}_0 = 0. \tag{5.2.2}$$

The kinematic condition at the free surface $\zeta = ae^{i(kx-\sigma t)}$ is

$$w = -\frac{\partial\psi}{\partial x} = \frac{\partial\zeta}{\partial t} \quad \text{on} \quad z = 0, \tag{5.2.3}$$

so that, from equations (4.2.4) and (4.2.6),

$$a\sigma_0 = kc_0, \tag{5.2.4}$$

$$a\sigma_1 = k(a_1 + c_1). \tag{5.2.5}$$

Within the order of accuracy of the present linear theory, the viscous stress conditions at the free surface demand

$$\bar{\mu}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0 \quad \text{on} \quad z = 0, \tag{5.2.6}$$

$$p' = \bar{\varrho} g\zeta + 2\bar{\mu}\frac{\partial w}{\partial z} \quad \text{on} \quad z = 0. \tag{5.2.7}$$

The first of these equations requires

$$a_1 = 0, \tag{5.2.8}$$

and, on using this result, (5.2.7) gives

$$\sigma_0 d_1 = kga, \tag{5.2.9}$$

$$\sigma_0 d_2 = -\sigma_1 d_1. \tag{5.2.10}$$

From equations (5.1) and (5.2) we now find

$$A_0 = 0, \quad B_0 = \frac{a\sigma_0}{kT(0)}, \tag{5.2.11}$$

$$\mathbf{a}_0 = -\frac{\mathbf{d}_0}{\mathbf{n}}, \quad \mathbf{c}_0 = \frac{\mathbf{d}_0}{\mathbf{n}}, \quad \mathbf{d}_0 = \frac{a\sigma_0}{kT(0)}, \tag{5.2.12}$$

$$a_1 = 0, \quad c_1 = \frac{a\sigma_1}{k}, \quad d_1 = \frac{a\sigma_0}{k}\frac{T'(0)}{T(0)} \tag{5.2.13}$$

5.3. *Expressions for σ_0, σ_1*

The normal stress condition (5.2.9) gives

$$\frac{gk^2}{\sigma_0^2} = \frac{T'(0)}{T(0)}. \tag{5.3.1}$$

This relation, in conjunction with (3.4) and (3.6), determines the possible values of σ_0 when the wave number is prescribed and agrees with the result that would be obtained by solving the inviscid problem directly. In general, there will be an infinite number of possible values[1] of σ_0. Further, combined use of equation (5.2.10) and other equations of § 5 which involve A_1, B_1 and d_2, together with the Wronskian $QT' - Q'T = 1/R$, yields

$$4\mathrm{n}\frac{\sigma_1}{\sigma_0}\int_{-h}^{0} R(s)T(s)\,T'(s)\,ds = \frac{T(0)}{T'(0)}, \quad (5.3.2)$$

where $$h\mathrm{n} = (-1+i)/\sqrt{2}. \quad (5.3.3)$$

Equation (5.3.2) thus gives the first order viscous correction for any density distribution and any viscosity distribution (subject to the proviso of § 2), and shows that, as far as the viscosity is concerned, $\varepsilon\sigma_1$ depends only on its value at $z = -h$. In order that $\varepsilon\sigma_1$ yield the main contribution to the viscous correction it is, of course, necessary that kh should not be allowed to become too large (with ν fixed) lest $\varepsilon^2\sigma_2$ become comparable with $\varepsilon\sigma_1$. The condition required in the case of a homogeneous fluid has been considered by Hunt (1964).

We can now use equations (5.3.1) and (5.3.2) to find formulae for σ_0 and σ_1 in a particular case. For example, we might specify the density distribution by

$$R(z) = e^{-\alpha(z+h)}, \quad (5.3.4)$$

while leaving the viscosity distribution $M(z)$ arbitrary. From § 3 we find

$$Q(z) = \frac{-m_2 e^{m_1(z+h)} + m_1 e^{m_2(z+h)}}{\beta},$$

$$T(z) = \frac{e^{m_1(z+h)} - e^{m_2(z+h)}}{\beta}, \quad (5.3.5)$$

where $$m_1 = \tfrac{1}{2}(\alpha+\beta), \quad m_2 = \tfrac{1}{2}(\alpha-\beta), \quad (5.3.6)$$

$$\beta = \sqrt{\alpha^2 + 4k^2\left(1 - \frac{g\alpha}{\sigma_0^2}\right)}. \quad (5.3.7)$$

[1] When any of σ_0, σ_1, ... and T appear in the same equation, and a particular value of σ_0 is under consideration, it is to be understood that the quantities σ_1, ... and T correspond to that same σ_0.

Then (5.3.1) and (5.3.2) become

$$\frac{gk^2}{\sigma_0^2} = \tfrac{1}{2}\left(\alpha + \beta \coth\frac{\beta h}{2}\right), \quad (5.3.1')$$

$$4\mathrm{n}\frac{\sigma_1}{\sigma_0}\left(\frac{\alpha}{\beta}\sinh\beta h - \alpha h + 2\sinh^2\frac{\beta h}{2}\right) = \frac{\beta^2\sigma_0^2}{gk^2}, \quad (5.3.2')$$

respectively. A real value of β is associated with a surface wave. When $\alpha = 0$, so that the density of the fluid is uniform, $\beta = 2k$ and equations (5.3.1′) and (5.3.2′) reduce to the well-known results

$$\frac{gk^2}{\sigma_0^2} = k\coth kh, \qquad \frac{\sigma_1}{\sigma_0} = \frac{-(1+i)\,kh}{\sqrt{2}\sinh 2kh},$$

as given by Biesel (1949).

If β happens to be purely imaginary, $\beta = i\gamma$, corresponding to internal waves,

$$\frac{g\alpha}{\sigma_0^2} > 1 + \frac{\alpha^2}{4k^2},$$

and (5.3.1′) gives

$$\tan\frac{\gamma h}{2} = \frac{2\alpha\gamma}{\gamma^2 - \alpha^2 + 4k^2}, \quad (5.3.8)$$

in agreement with Lamb (1932). If we assume that $|\alpha h|$ is small,

$$|\alpha h| < 1, \quad (5.3.9)$$

which just means that the total change in the equilibrium density between $z = 0$ and $z = -h$ is much less than the average density, we have, approximately, $\gamma h = 2s\pi$ where s is any integer. Equation (5.3.7) now gives

$$\sigma_0^2 = g\alpha\frac{k^2h^2}{s^2\pi^2 + k^2h^2}, \quad (5.3.1'')$$

whilst (5.3.2′)reduces to

$$\frac{\sigma_1}{\sigma_0} = \frac{-(1+i)s^2\pi^2}{\sqrt{2}\,(s^2\pi^2 + k^2h^2)}. \quad (5.3.2'')$$

Equation (5.3.1″) demonstrates clearly that the equilibrium of the fluid is stable or unstable according as αh is positive or negative. It is

Table 1. *Values of $\sigma_0^2 h/g$ associated with $[\gamma h]_{\alpha h=0} = 2\pi$ (the first internal mode)*

	$kh=$				
αh	0.5	1.0	2.0	3.0	5.0
−1.0	−0.030	−0.108	−0.315	−0.497	−0.723
−0.75	−0.021	−0.078	−0.232	−0.370	−0.542
−0.5	−0.014	−0.050	−0.151	−0.244	−0.361
−0.3	−0.008	−0.029	−0.089	−0.145	−0.216
−0.1	−0.003	−0.009	−0.029	−0.048	−0.072
0.1	0.002	0.009	0.029	0.047	0.072
0.3	0.007	0.026	0.084	0.141	0.214
0.5	0.011	0.042	0.137	0.231	0.355
0.75	0.016	0.061	0.199	0.341	0.529
1.0	0.020	0.078	0.257	0.446	0.700

readily seen from (5.3.1′) and (5.3.1″) that the dispersive character of the surface wave is different from that of an internal wave, whilst a consideration of (5.3.2′) and (5.3.2″) shows that the decay characteristics differ also.

Some illustration of the analysis of this section will now be given. Values of $\sigma_0^2 h/g$ corresponding to the root of equation (5.3.8) which tends to $2\pi/h$ as $\alpha h \to 0$ are shown in Table 1; similarly, Table 2 refers to that root of (5.3.8) which tends to $4\pi/h$ in the same limit.

By equation (3.3) the analysis will not be valid when $\sigma_0^2 h/g$ is too close to zero. For the range of αh under consideration and for a prescribed wave number we thus require

$$\frac{gh^3}{\nu^{*2}}|\alpha h| \gg 1, \tag{5.3.10}$$

that is, the effect of the buoyancy forces must be much greater than that of the viscous forces. The values of the real (or imaginary) part of $(-\sigma_1/\sigma_0)$ corresponding to those of $\sigma_0^2 h/g$ in Tables 1 and 2 are presented graphically in Figs. 1 and 2 respectively. It is seen that the variation in these values with αh is very small for the shorter wavelengths. In Table 3, values of the real (or imaginary) part of $(\sigma_1^4 h/\sigma_0^2 g)^{\frac{1}{4}}$ as computed from equation (5.3.2′) are given for the first four internal modes in a typical case $\alpha h = 0\cdot 1$.

Since $(2s-1)\pi < \gamma h < (2s+1)\pi$ in the sth internal mode, we have approximately

$$\frac{\sigma_0^2 h}{g} = \alpha h \frac{k^2 h^2}{s^2 \pi^2}, \quad \frac{\sigma_1}{\sigma_0} = \frac{\alpha h}{hn\left[\alpha h + 2\sin^2\dfrac{\gamma h}{2}\right]},$$

when s is a large integer, whence

$$\left(\frac{\sigma_1^4 h}{\sigma_0^2 g}\right)^{\frac{1}{4}} \approx -\frac{(1+i)}{\sqrt{2}}\left(\frac{\alpha h}{\alpha h + 2\sin^2\dfrac{\gamma h}{2}}\right)\left(\alpha h\frac{k^2 h^2}{s^2\pi^2}\right)^{\frac{1}{4}}.$$

Table 2. *Values of $\sigma_0^2 h/g$ associated with $[\gamma h]_{\alpha h=0} = 4\pi$ (the second internal mode)*

	$kh=$				
αh	0.5	1.0	2.0	3.0	5.0
−1.0	−0.007	−0.026	−0.095	−0.191	−0.394
−0.75	−0.005	−0.019	−0.071	−0.142	−0.294
−0.5	−0.003	−0.013	−0.047	−0.094	−0.196
−0.3	−0.002	−0.008	−0.028	−0.056	−0.117
−0.1	−0.001	−0.002	−0.009	−0.019	−0.039
0.1	0.001	0.002	0.009	0.019	0.039
0.3	0.002	0.007	0.027	0.055	0.116
0.5	0.003	0.012	0.045	0.091	0.192
0.75	0.005	0.018	0.067	0.135	0.286
1.0	0.006	0.023	0.088	0.179	0.379

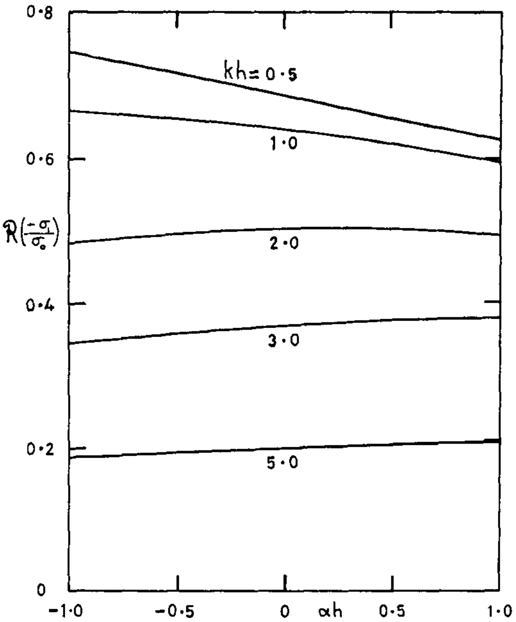

Fig. 1. Variation of the real part of $(-\sigma_1/\sigma_0)$ with αh associated with $\pi < \gamma h < 3\pi$, the first internal mode.

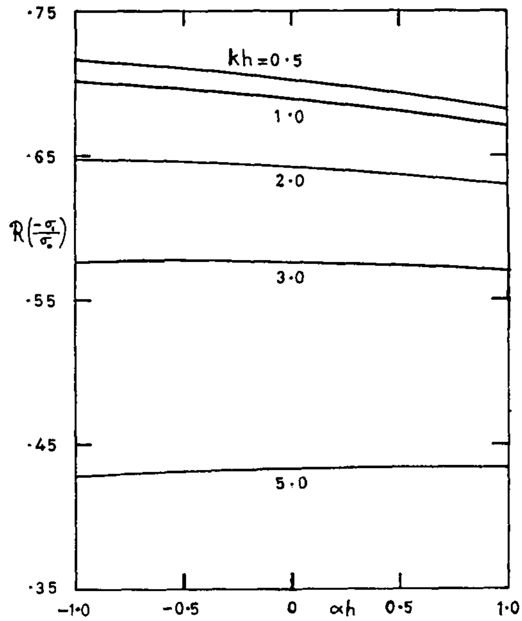

Fig. 2. Variation of the real part of $(-\sigma_1/\sigma_0)$ with αh associated with $3\pi < \gamma h < 5\pi$, the second internal mode.

Provided s is not so large that the condition (3.3) is violated, the slower internal modes can certainly decay less rapidly than the surface wave. In fact, by Table 3, when $kh = 0 \cdot 5$, $\alpha h = 0 \cdot 1$, all the internal modes (for which the analysis is valid) have a slower rate of decay than the surface wave.

5.4. *Case of given angular frequency*

The analysis can be repeated when the quantity σ is prescribed as the angular frequency of the disturbance and k expanded in the form

$$k = k_0 + \varepsilon k_1 + \varepsilon^2 k_2 + \dots, \tag{5.4.1}$$

where now

$$\varepsilon = \left(\frac{\nu^*}{\sigma h^2}\right)^{\frac{1}{2}}. \tag{5.4.2}$$

The result is

$$\frac{gk_0^2}{\sigma^2} = \frac{T'(0)}{T(0)}, \tag{5.4.3}$$

$$2\mathrm{n}k_0 k_1 \int_{-h}^{0} RT\left(T - \frac{2g}{\sigma^2}T'\right) ds = 1. \tag{5.4.4}$$

For the particular density distribution of equation (5.3.4), but for arbitrary viscosity distributions, we find

$$\frac{gk_0^2}{\sigma^2} = \frac{\beta}{2}\coth\frac{\beta h}{2} + \frac{\alpha}{2}, \tag{5.4.3'}$$

$$-\frac{4\mathrm{n}k_0 k_1}{\beta^3}\left[\left(1 - \frac{g\alpha}{\sigma^2}\right)(\beta h + \sinh \beta h) + \frac{\alpha^2}{k_0^2}\sinh^2\frac{\beta h}{2}\left(\frac{\beta}{\alpha} + \coth\frac{\beta h}{2}\right)\right] = 1. \tag{5.4.4'}$$

Table 3. *Values of the real part of* $(\sigma_1^4 h/\sigma_0^2 g)^{\frac{1}{2}}$ *for the first four internal modes in the case when* $\alpha h = 0 \cdot 1$.

Internal mode	$kh =$ 0.5	1.0	2.0	3.0	5.0
1	−0.152	−0.197	−0.207	−0.173	−0.104
2	−0.111	−0.153	−0.198	−0.212	−0.192
3	−0.091	−0.128	−0.173	−0.199	−0.212
4	−0.079	−0.111	−0.154	−0.181	−0.208

When $\alpha = 0$, equation (5.4.4') reduces to

$$\frac{k_1}{k_0} = \frac{\sqrt{2}(1+i)k_0h}{2k_0h + \sinh 2k_0h},$$

in agreement with the well-known result for a fluid of uniform density. If, as in (5.3), we take $\beta = i\gamma$ and $|\alpha h| < 1$, we have, approximately, $\gamma h = 2s\pi$ and

$$k_0^2h^2 = \frac{s^2\pi^2}{\dfrac{g\alpha}{\sigma^2} - 1}, \qquad (5.4.3'')$$

$$k_1h = \frac{(1+i)s\pi}{\sqrt{2\left(\dfrac{g\alpha}{\sigma^2} - 1\right)}} = \frac{(1+i)k_0h}{\sqrt{2}} \qquad (5.4.4'')$$

6. Fluid contained between plane rigid boundaries

6.1. *Boundary and matching conditions*

We first consider the case when the wave number k is regarded as given. Boundary and matching conditions associated with the lower viscous boundary layer are as in § 5, that is equations (5.1.1) to (5.1.3), (5.2.1) and (5.2.2) hold. Those associated with the boundary layer adjacent to the rigid boundary $z = 0$ are

$$B_0T(0) = 0, \qquad (6.1.1)$$

$$B_0T'(0) = \hat{d}_0, \qquad (6.1.2)$$

$$A_1Q(0) + B_1T(0) - \frac{2\sigma_1}{\sigma_0}\frac{gk^2}{\sigma_0^2}\int_{-h}^{0} I(s, 0)\,ds = \hat{c}_0, \qquad (6.1.3)$$

$$-\hat{a}_0n + \hat{d}_0 = 0, \qquad (6.1.4)$$

$$\hat{a}_0 + \hat{c}_0 = 0, \qquad (6.1.5)$$

on using equation (4.2.11). In consequence, σ_0 will be obtained by imposing the requirement

$$T(0) = 0, \qquad (6.1.6)$$

and then σ_1 is given by

$$4\frac{\sigma_1}{\sigma_0}\frac{gk^2}{\sigma_0^2}\int_{-h}^{0} RTT'ds = \frac{1}{n}\frac{T'(0)}{Q(0)} + \frac{1}{\mathfrak{n}}, \qquad (6.1.7)$$

$$hn = (-1+i)\sqrt{\frac{R(0)}{2M(0)}}, \quad h\mathfrak{n} = (-1+i)/\sqrt{2}. \qquad (6.1.8)$$

Thus, the first order correction depends on the value of the viscosity at the rigid boundaries only.

If $R(z) = e^{-\alpha(z+h)}$, equations (5.3.5) and (6.1.6) require $e^{m_1h} = e^{m_2h}$ so that

$$\beta h = 2is\pi, \qquad (6.1.9)$$

and then, by (5.3.7),

$$\frac{g\alpha}{\sigma_0^2} = 1 + \frac{s^2\pi^2}{k^2h^2} + \frac{1}{4}\frac{\alpha^2}{k^2}, \qquad (6.1.10)$$

in agreement with Lamb (1932). Thus, in the outer flow,

$$\Psi_0(z) = Ce^{\frac{1}{2}\alpha(z+h)}\sin\frac{s\pi(z+h)}{h}, \qquad (6.1.11)$$

where C is a constant. We also find from (6.1.7) that

$$\frac{\sigma_1}{\sigma_0} = \left(\frac{1}{hn} + \frac{1}{h\mathfrak{n}}\right)\frac{s^2\pi^2}{k^2h^2 + s^2\pi^2 + \frac{1}{4}\alpha^2h^2}, \qquad (6.1.7')$$

$$hn = (-1+i)\sqrt{\frac{e^{-\alpha h}}{2M(0)}}, \quad h\mathfrak{n} = (-1+i)/\sqrt{2}. \qquad (6.1.8')$$

Thus, for the exponential variation of density the first order viscous correction is such that

$$\varepsilon\frac{\sigma_1}{\sigma_1} \propto \sqrt{\bar{\nu}(-h)} + \sqrt{\bar{\nu}(0)}, \quad \bar{\nu}(z) = \frac{\bar{\mu}(z)}{\bar{\varrho}(z)}, \qquad (6.1.12)$$

although, from (6.1.7), this is not true of general variations of density.

The above analysis can be repeated for the case when σ is prescribed. As a result, k_0 must be obtained by fulfilling

$$T(0) = 0, \qquad (6.1.13)$$

and k_1 satisfies

$$2k_0k_1\int_{-h}^{0} RT\left(T - \frac{2g}{\sigma^2}T'\right)ds = \frac{1}{n}\frac{T'(0)}{Q(0)} + \frac{1}{\mathfrak{n}}. \qquad (6.1.14)$$

For the exponential distribution of density, equation (5.3.4),

$$k_0^2 h^2 = \frac{4s^2\pi^2 + \alpha^2 h^2}{4\left(\dfrac{g\alpha}{\sigma^2} - 1\right)}, \tag{6.1.15}$$

$$\frac{k_1}{k_0} = -\left(\frac{1}{hn} + \frac{1}{h\mathbf{n}}\right)\frac{4s^2\pi^2}{4s^2\pi^2 + \alpha^2 h^2}, \tag{6.1.14'}$$

and so $$\varepsilon\frac{k_1}{k_0} \propto \sqrt{\bar{\nu}(-h)} + \sqrt{\bar{\nu}(0)}. \tag{6.1.16}$$

6.2. *Second order viscous correction*

To determine σ^2, the second order correction for viscosity, we need the following additional results. For the outer flow, (2.7) gives

$$L\Psi_2 = -\frac{2\sigma_1}{\sigma_0}\frac{gk^2R'}{\sigma_0^2 R}\Psi_1 + \frac{gk^2R'}{\sigma_0^2 R}\left(\frac{3\sigma_1^2}{\sigma_0^2} - \frac{2\sigma_2}{\sigma_0}\right)\Psi_0$$

$$+\frac{ih^2}{R}\left[M\left(\frac{d^2}{dz^2} - k^2\right)^2 + 2M'\frac{d}{dz}\left(\frac{d^2}{dz^2} - k^2\right)\right.$$

$$\left. + M''\left(\frac{d^2}{dz^2} + k^2\right)\right]\Psi_0 = f(z), \quad \text{say}, \tag{6.2.1}$$

with solution

$$\Psi_2 = A_2 Q + B_2 T + \int_{-h}^{z} J(s,z)\,ds, \tag{6.2.2}$$

$$J(s,z) = f(s)R(s)[Q(s)T(z) - T(s)Q(z)]. \tag{6.2.3}$$

For the inner flow in the boundary layer adjacent to $z = -h$, we find

$$\mathbf{L}\boldsymbol{\Psi}_1 = -[h^2M'(-h)\boldsymbol{\Psi}_0^{iv} + iR'(-h)\boldsymbol{\Psi}_0'']\mathbf{Z}$$

$$-2h^2M'(-h)\boldsymbol{\Psi}_0''' - iR'(-h)\boldsymbol{\Psi}_0' + h^2\frac{\sigma_1}{\sigma_0}\boldsymbol{\Psi}_0^{iv}, \tag{6.2.4}$$

whence

$$\boldsymbol{\Psi}_1 = \mathbf{a}_1 e^{\mathbf{nZ}} + \mathbf{c}_1 + \mathbf{d}_1\mathbf{Z} - \tfrac{1}{2}R'(-h)\mathbf{d}_0\mathbf{Z}^2$$

$$+\tfrac{1}{4}\mathbf{a}_0\mathbf{Z}e^{\mathbf{nZ}}\left[(1-\mathbf{nZ})M'(-h)\right.$$

$$\left. - (3-\mathbf{nZ})R'(-h) + 2\mathbf{n}\frac{\sigma_1}{\sigma_0}\right]. \tag{6.2.5}$$

Similarly, for the inner flow in the boundary layer adjacent to $z = 0$,

$$\hat{\psi}_1 = \hat{a}_1 e^{-nZ} + \hat{c}_1 + \hat{d}_1 Z - \tfrac{1}{2}\frac{R'(0)}{R(0)}\hat{d}_0 Z^2$$

$$+\tfrac{1}{4}\hat{a}_0 Z e^{-nZ}\left[\frac{M'(0)}{M(0)}(1 + nZ)\right.$$

$$\left. - \frac{R'(0)}{R(0)}(3 + nZ) - 2n\frac{\sigma_1}{\sigma_0}\right]. \tag{6.2.6}$$

Then, by imposing the necessary boundary and matching conditions on these higher order quantities Ψ_2, $\boldsymbol{\Psi}_1$ and $\hat{\psi}_1$, $\boldsymbol{\Psi}_1$, we obtain a set of relations between the various constants, which thus enables us to derive a general formula for σ_2. This formula is necessarily of a complex nature, and we shall be content to give the result for the special case

$$R(z) = e^{-\alpha(z+h)}, \quad M(z) = e^{-\alpha(z+h)}, \tag{6.2.7}$$

so that the kinematic viscosity $\bar{\nu}(z)$ is constant throughout the fluid. The result is

$$4\frac{\sigma_2}{\sigma_0} = 6\frac{\sigma_1^2}{\sigma_0^2}\left(\frac{\alpha^2 + 4k^2}{\beta^2}\right) + \tfrac{1}{2}\frac{\sigma_1}{\sigma_0}\frac{\beta^2\sigma_0^2}{gk^2\alpha h}\left(\frac{1}{n} + \frac{1}{\mathbf{n}}\right)$$

$$-2ih^2\alpha^2\left(\frac{gk^2}{\sigma_0^2\alpha} + \frac{\sigma_0^2}{g\alpha}\right), \tag{6.2.8}$$

where now $hn = (-1+i)/\sqrt{2} = h\mathbf{n}$, and no use has been made in (6.2.8) of equation (6.1.7′) for σ_1/σ_0.

6.3. *Comparison with Hide's result*

A study of the behaviour of a viscous fluid contained between two plane rigid boundaries and subjected to small amplitude oscillatory disturbances has been made by Hide (1955) for the special case when the density and viscosity distribution are as in equation (6.2.7). His results are further confined to instances when $|\alpha h| < 1$ for which he finds (in the present notation)

$$\sigma^2\left(1 + \frac{s^2\pi^2}{k^2h^2}\right) + i\sigma\nu^* k^2\left(1 + \frac{s^2\pi^2}{k^2h^2}\right)^2 - g\alpha = 0. \tag{6.3.1}$$

The method by which this equation was obtained makes use of a variational principle involving the vertical component of velocity, w. Hide used the inviscid form of w, whose z-dependence is $\sin[s\pi(z+h)/h]$, which can be seen by setting $\alpha h = 0$ in equation (6.1.11). Consequently, the full condition of zero fluid velocity on the rigid boundaries is not satisfied and Hide would find $\sigma_1 = 0$, $k_1 = 0$, as is clear from the structure of (6.3.1). For the second order term, equation (6.3.1) yields

$$\frac{\sigma_2}{\sigma_0} = -\tfrac{1}{2} i k^2 h^2 \frac{g\alpha}{\sigma_0^2}, \qquad (6.3.2)$$

and we note that (to the required order) this agrees with the result (6.2.8) of the present analysis if, in the latter, σ_1 is put equal to zero.

REFERENCES

Biesel, F. 1949. Calcul de l'amortissement d'une houle dans un liquide visqueux de profondeur finie. *La Houille Blanche 4*, 630–634.

Dore, B. D. 1967. *Viscous damping effects in long waves on the rotating earth Quart. J. Mech. Appl. Math.* In press.

Hide, R. 1955. The character of the equilibrium of an incompressible heavy viscous fluid of variable density: an approximate theory. *Proc. Camb. Phil. Soc. 51*, 179–201.

Hunt, J. N. 1964. The viscous damping of gravity waves in shallow water. *La Houille Blanche 19*, 685–691.

Johns, B. 1967. A boundary layer method for the determination of the viscous damping of small amplitude gravity waves. *Quart. J. Mech. Appl. Math.*

Lamb, H. 1932. *Hydrodynamics* (sixth edition). Cambridge University Press, pp. 378–380.

Van Dyke, M. 1964. *Perturbation methods in fluid mechanics.* Academic Press.

Yih, C.-S. 1965. *Dynamics of nonhomogeneous fluids.* New York, Macmillan, pp. 145–146.

КОЛЕБАНИЯ В НЕОДНОРОДНОЙ ВЯЗКОЙ ЖИДКОСТИ

Метод асимптотических разложений применяется к выводу формул для вязких поправок, имеющих место в случае осциллирующих возмущений малой амплитуды в неоднородной жидкости. Такие формулы даются для весьма общих распределений с глубиной плотности в состоянии равновесия и вязкости в следующих случаях: (1) двумерные возмущения в жидкости конечной глубины, ограниченной сверху свободной поверхностью, (2) двумерные возмущения в жидкости, ограниченной сверху и снизу двумя фиксированными горизонтальными поверхностями. Выведенные выражения зависят от величины вязкости на твердых ограничивающих поверхностях, а не от ее фактического распределения. Дополнительная поправка на вязкость дана для случая (2), когда плотность экспоненциально изменяется с глубиной, а кинематическая вязкость постоянна.