

The surface boundary layer of a hurricane

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(Manuscript received April 25, 1967)

ABSTRACT

A simple model is developed to investigate some of the features of the surface boundary layer of a hurricane. The flow above the friction layer is represented by a steady cylindrical vortex in which there is gradient flow, specified by suitably choosing the radial pressure profile. It is assumed that the flow in the main vortex is approximately geostrophic at large distances from the centre and the Ekman solution is taken as appropriate for the boundary layer flow at these distances. A momentum integral method is used to follow the boundary layer development to the centre regions of the vortex.

Radial profiles of boundary layer thickness and induced vertical velocity are obtained when a constant eddy viscosity, K_M , is taken as characteristic of the turbulence in the friction layer. Two surface boundary conditions are examined; the no-slip condition and the condition that the surface stress be in the direction of the surface wind. The former of these is found to be the more satisfactory and gives qualitative agreement with observations. The effects of radial and vertical variations of K_M are discussed in relation to the surface condition but an inadequate knowledge of the turbulent structure prevents a more realistic formulation of the layer at this stage.

Introduction

Observational data on hurricanes are still somewhat fragmentary, due primarily to their formation and subsequent motion over parts of the tropical oceans, where recording stations are few. The penetration of aircraft into these storms in recent years has helped to supplement knowledge about their structure (LaSeur, 1957; Cólón, 1961; Riehl & Malkus, 1961; Gray, 1966) and although the detailed mechanisms are still not fully understood, the broad scale features have been identified. Figure 1 shows the main regions of flow in the mature stage of a hurricane. There is an inflow layer adjacent to the ground, extending to a height of a kilometre or two, in which friction plays an important role. The main vortex is a much thicker layer, typically 10 km, containing cloud, with a radial extent of order 100 km, in which there is little mean radial motion (compared to that of the surface layer) and mixing processes take place on the scale of cumulus convective elements, which have horizontal and vertical scales of

order 10 km. The picture is completed by an outflow layer below the level of the tropopause and a core flow with a diameter of about 40–80 km, free from cloud, with light winds and in which there is a downflow of air close to the axis (see e.g. Yanai, 1964). The hurricane may be thought of as part of a larger system, which includes the near environment in which it moves and this will be termed the *outer flow* in this paper. This we suppose to have a radial extent an order of magnitude larger than the main vortex, say 1000 km.

We confine our attention to hurricanes in which the isobars are nearly circular at each level. In their mature stage, these systems show a remarkable degree of symmetry, especially in the main vortex, where azimuthal velocities are high and centrifugal forces tend to preserve roughly circular motion. In our model, the main vortex is represented by a steady, axi-symmetric, potential vortex, which is stationary in a fluid at rest and has a tangential velocity V_{gr} , determined by the gradient wind equation,

$$V_{gr} = -\frac{1}{2} Rf + \left(\frac{1}{2} R^2 f^2 + \frac{R dP}{\rho dR} \right)^{\frac{1}{2}}, \quad (1)$$

where R is the radial distance, f is the coriolis parameter, ρ the density of air and P is the pressure. This vortex is a solution of the Euler equations of motion in a rotating frame of reference, in which there is no radial motion ($U = 0$) and an arbitrary profile of axial flow, $W(R)$. It can thus be taken as the flow at the top of the inflow layer, with the radial variation of W determined from boundary layer theory. Hence, the flow is specified completely by choosing an arbitrary pressure profile $P(R)$, but this is done so as to give a typical velocity profile through Eq. (1).

Variations in f will occur along lines of longitude since $f = 2\Omega \sin \phi$, where Ω is the angular rate of rotation of the Earth about its axis, and ϕ is the latitude. The relative change in f for a small change in latitude δ , is

$$\frac{\delta f}{f} = \cot \phi \cdot \delta \phi.$$

The total horizontal extent of the vortex may be ~ 200 km, although the extent of the vortex and its outer flow region will be considerably larger, say ~ 2000 km. Values of $\delta f/f$ corresponding to these at 30° lat., are roughly .025 and .25 respectively. Thus whilst variations of f across the vortex core are fairly small, variations over the entire flow are not and in practice will result in considerable asymmetry of the outer flow region. Moreover, the interaction between the vortex and the irregular zonal current in which it moves serve to produce further asymmetries in the motion, again especially in the outer flow region. Nevertheless, much can be learned by taking f constant across the entire flow and by ignoring motions of the vortex and its environment. These assumptions are made here.

We shall not concern ourselves with the high-altitude outflow layer or the core flow in this paper.

Equation (1) denotes a balance between the centrifugal and coriolis forces and the radial pressure gradient, in the main vortex. Friction effects near the ground disrupt this balance locally, reducing the azimuthal velocity and hence the centrifugal and coriolis forces, whilst the pressure field of the main vortex is trans-

mitted nearly unchanged through the depth of the friction layer. Thus, there is a net pressure gradient towards the core of the hurricane, which drives a radial inflow in the boundary layer and a corresponding upflow at some inner radii.

The secondary flow, produced by the presence of a rigid boundary perpendicular to the axis of rotation, is common to all atmospheric vortices including tornadoes, dust-devils and waterspouts. Changes in the main vortex produce corresponding changes in the secondary inflow and this provides a main source of fluid for the overlying vortex. This fluid, in the case of a hurricane, is rich in moisture and is undoubtedly a major factor in maintaining the vortex by supplying large amounts of energy in the form of latent heat of condensation at higher levels. Thus, feedback effects of changes in the vortex, due to the presence of a boundary, exert strong constraints on the flow. It is clear, therefore, that a proper treatment of the surface boundary layer is essential to a correct formulation of models for atmospheric vortices and, in particular, for any model of a hurricane. This paper examines some of the features of the boundary layer using a momentum integral method to obtain an approximate solution for the flow.

A local Rossby number Ro , based on the distance from the hurricane centre and the local horizontal velocity at the top of the boundary layer, can be used to divide the flow in this layer into the following three regions:

- I. A distant region ($Ro < 1$) in which the flow is approximately quasi-geostrophic, that is, inertial terms in the equations of motion are small compared to coriolis and frictional terms and can thus be neglected,
- II. A transition region ($Ro \sim 1$) in which the inertial terms are of the same order as coriolis and frictional terms and must be included in the equations.
- III. An inner region ($Ro > 1$) in which coriolis terms are small compared to inertial and frictional terms, which approximately balance in this region.

An exact solution for the boundary layer in the geostrophic region is the well-known Ekman solution. A principal feature of this is that the boundary layer thickness is proportional only to the ratio $(K_M/f)^{\frac{1}{2}}$, where K_M is an eddy visco-

sity characteristic of the turbulence in the boundary layer. If f and K_M are constant, the boundary layer thickness is constant but variations in mainstream velocity produce a vertical flow through the top of the layer, proportional to the vertical component of relative vorticity in the mainstream (Lighthill, 1966).

No exact solution exists in regions II and III where curvature and inertial effects require the non-linear terms to be included in the flow equations. In these regions the boundary layer thickness will generally vary with radial distance from the centre.

Haurwitz (1935) and later, Rosenthal (1962) and Miller (1965) use a perturbation technique to linearize the boundary layer equations in regions II and III. An order of magnitude analysis suggests that these authors have excluded terms which are of the same order as some of those they have retained in their equations.¹ Also they attempt to apply their solutions to the region close to the axis where the boundary layer equations are no longer valid because radial stress terms are then important. Nevertheless, the general validity of a perturbation method of approach is more serious. The secondary flow produced by axial boundaries in vortex flows is not only more than a small perturbation on an outer flow, but is of a comparable magnitude to it. Indeed, the essence of boundary layer theory is to deal with $O(1)$ disruptions of flow at a rigid boundary.

A momentum integral method sacrifices accuracy to a degree, but does not suffer from the need to exclude non-linear terms in the equations. However, it may only be applied with confidence to flows in which the boundary layer thickness varies slowly with radius. Thus, while it is of limited application to the surface interaction boundary layer of "long-thin" vortices, such as tornadoes and dust-devils, one may expect good results when it is applied to a hurricane where the boundary layer is extremely thin in relation to the radial dimension of the vortex, this ratio being of the order 10^{-3} or

smaller. In the treatment which follows, the momentum integral technique is used to continue the Ekman layer solution for the geostrophic region into the two inner regions where inertia terms in the equations cannot be neglected.

The mathematical problem

A system of cylindrical polar co-ordinates (R, θ, Z), with corresponding velocity components (U, V, W), is used and the flow is taken to be steady and axi-symmetric (i.e. derivatives with respect to time and θ are zero).

The Ekman layer equations for the boundary layer flow at large radial distances from the centre of the vortex, are

$$f(V_g - V) = K_M^* \frac{\partial^2 U}{\partial Z^2}, \quad (2)$$

$$fU = K_M^* \frac{\partial^2 V}{\partial Z^2}, \quad (3)$$

$$\text{where} \quad V_g = -\frac{1}{\rho f} \frac{dP}{dR} \quad (4)$$

is the azimuthal (geostrophic) velocity above the boundary layer at these distances and K_M^* is an eddy viscosity appropriate to the Ekman region. (N.B. if dP/dR is small, $V_{gr} \simeq V_g$ from Eqs. (1) and (3), i.e. the gradient flow degenerates into geostrophic flow far out from the storm centre.) For the present, K_M^* is assumed constant but later we shall discuss the effect of vertical variations of this quantity. Equations (2) and (3) are non-dimensionalized with respect to the usual Ekman length and velocity scales, $Z_g (= (K_M^*/f)^{1/2})$ and V_g . Hence, if

$$V = V_g v, \quad U = V_g u, \quad Z = Z_g z,$$

defining u, v and z , we have

$$1 - v = \frac{\partial^2 u}{\partial z^2}, \quad (5)$$

$$u = \frac{\partial^2 v}{\partial z^2}. \quad (6)$$

The full boundary layer equations in a rotating frame of reference are,

¹ In the azimuthal momentum equation, Rosenthal neglects the term $W(\partial v_0/\partial z)$ in comparison with $v_r(\partial v_0/\partial r + v_0/r + f)$. Taking orders of magnitude $\bar{U}$, $\bar{W}$ and $\bar{R}$, $\bar{Z}$ for radial and axial velocity and length scales, the equation of continuity $\partial(v, r)/\partial r + \partial(rw)/\partial z = 0$, is satisfied only if $\bar{U}/\bar{R} \sim \bar{W}/\bar{Z}$. Then $\bar{W} \sim \bar{U}\bar{Z}/\bar{R}$ and $w(\partial v_0/\partial z)$ and $v_r(\partial v_0/\partial r + v_0/r + f)$ have the same order of magnitude $\bar{U}^2/\bar{R}$.

$$U \frac{\partial U}{\partial R} + W \frac{\partial U}{\partial Z} + \frac{V_{gr}^2 - V^2}{R} + f(V_{gr} - V) = K_M \frac{\partial^2 U}{\partial Z^2} \quad (7)$$

$$U \frac{\partial V}{\partial R} + W \frac{\partial V}{\partial Z} + \frac{UV}{R} + fU = K_M \frac{\partial^2 V}{\partial Z^2}. \quad (8)$$

These are derived from the Navier Stokes equations with the assumptions that

$$\frac{\partial^2}{\partial R^2}, \frac{1}{R} \frac{\partial}{\partial R} < \frac{\partial^2}{\partial Z^2} \quad \text{and} \quad P$$

is approximately constant across the boundary layer. These will not be satisfied close to the point of separation, where inflowing fluid meets fluid flowing outwards from the core, in which it has descended.

Equations (7) and (8) are non-dimensionalized, again with respect to the Ekman scales and with

$$R = R_g r, \quad V_{gr} = V_g v_{gr}, \quad W = (V_g Z_g / R_g) w,$$

$K = K_M^{*k}$, (allowing K_M to vary radially) to give

$$Ro \left(u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} + \frac{v_{gr} - v^2}{r} \right) + v_{gr} - v_g = k \frac{\partial^2 u}{\partial z^2}, \quad (9)$$

$$Ro \left(u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{uv}{r} \right) + u = k \frac{\partial^2 v}{\partial z^2}, \quad (10)$$

where $Ro = V_g / (R_g f)$ is a Rossby number for the flow above the Ekman region, based on the radius R_g at which the full geostrophic approximation is applied (which we shall call the *geostrophic radius*) and the tangential velocity V_g at this radius. The scale for W is obtained from the equation of continuity (see footnote 1, p. 475) which in non-dimensional form is

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0. \quad (11)$$

As a first approximation we ignore the motion of the sea, which is equivalent to treating the sea surface as a rigid boundary. This is reasonable as wind induced velocities in the sea will be several orders of magnitude less than those in the vortex due to the large difference in density air and water.

Two surface boundary conditions are examined; the no-slip condition, that is, $U = V = 0$ on $Z = 0$ and the condition that the surface stress

be in the direction of the surface wind, that is, $(U, V, 0) = \sigma(\partial/\partial Z)(U, V, 0)$, where $\sigma = K_M^*/(C_D V_{gr})$ and C_D is the surface drag coefficient (see Eliassen & Kleinschmidt, 1957). In terms of non-dimensional variables, these conditions take the form

$$u = v = 0 \quad \text{on} \quad z = 0 \quad (A)$$

$$\text{and} \quad (u, v, 0) = \left(\frac{\alpha k}{v_{gr}} \right) \frac{\partial}{\partial z} (u, v, 0), \quad (B)$$

where $\alpha = K_M^*/(C_D V_g Z_g)$ is a constant. We shall refer to these boundary conditions as A and B respectively and consider condition A first. (N.B. condition B reduces to condition A if $\alpha = 0$.)

The solutions to Eqs. (5) and (6), subject to condition A at the surface are

$$\left. \begin{aligned} u &= -e^{-z/\sqrt{2}} \sin(z/\sqrt{2}), \\ v &= 1 - e^{-z/\sqrt{2}} \cos(z/\sqrt{2}). \end{aligned} \right\} \quad (12)$$

The momentum integral method, due to Karman and Pohlhausen is a procedure for calculating the boundary layer thickness when the mainstream velocity is specified (see, e.g. Schlichting, 1960). The boundary layer equations are integrated across the layer and suitable velocity profiles, satisfying the appropriate boundary conditions, are substituted into these. The method is an approximate one as the profiles, providing they satisfy the boundary conditions, may be chosen arbitrarily. Taylor (1950) applied the technique to the boundary layer flow in a converging nozzle and Mack (1962) used Taylor's method in a study of rotating flows above a finite disk. Mack obtains two ordinary differential equations for the radial variation of scales for the boundary layer thickness and the radial inflow velocity when an outer flow is specified. These equations are solved numerically.

In this treatment we follow Mack, but take into account coriolis forces.

Combining Eq. (11) with Eqs. (9) and (10) gives

$$\begin{aligned} Ro \left[\frac{1}{r} \frac{\partial}{\partial r}(ru^2) + \frac{\partial}{\partial z}(uw) + \frac{v_{gr}^2 - v^2}{r} \right] \\ + v_{gr} - v = k \frac{\partial^2 u}{\partial z^2}, \end{aligned} \quad (13)$$

$$Ro \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 uv) + \frac{\partial}{\partial z} (vw) \right] + u = k \frac{\partial^2 v}{\partial z^2}. \quad (14)$$

These equations are integrated across the boundary layer from $z=0$ to $z=\infty$ subject to the boundary conditions

$$\left. \begin{aligned} z=0; \quad u=v=w=0, \\ z=\infty; \quad u=0, \quad v=v_{gr}, \quad \partial u / \partial z = \partial v / \partial z = 0 \end{aligned} \right\} \quad (15)$$

to give

$$\begin{aligned} Ro \left[\frac{d}{dr} \left(r \int_0^\infty u^2 dz \right) + \int_0^\infty (v_{gr}^2 - v^2) dz \right] \\ + \int_0^\infty r(v_{gr} - v) dz = -kr \left(\frac{\partial u}{\partial z} \right)_{z=0} \end{aligned} \quad (16)$$

$$\begin{aligned} Ro \left[\frac{d}{dr} \left(r^2 \int_0^\infty uv dz \right) + r^2 v_{gr} w_{gr} \right] \\ + \int_0^\infty r^2 u dz = -kr^2 \left(\frac{\partial v}{\partial z} \right)_{z=0}, \end{aligned} \quad (17)$$

$$\text{where} \quad w_{gr} = -\frac{1}{r} \frac{d}{dr} \left(r \int_0^\infty u dz \right) \quad (18)$$

is the non-dimensional vertical outflow velocity at the top of the boundary layer. We now introduce a non-dimensional scale thickness $\delta = \delta(r)$ for the boundary layer and use the Karman T -method described by Mack to represent the velocity components, that is we write,

$$\left. \begin{aligned} u(r, \eta) &= v_{gr}(r) E(r) f(\eta), \\ v(r, \eta) &= v_{gr}(r) g(\eta), \end{aligned} \right\} \quad (19)$$

where $E(r)$ is the amplitude coefficient of the radial velocity, $\eta = z/\delta$ and $f(\eta)$, $g(\eta)$ are the velocity profiles across the flow. For these we take the Ekman profiles with $z/\sqrt{2}$ replaced by z/δ , thus

$$\left. \begin{aligned} f(\eta) &= -e^{-\eta} \sin \eta, \\ g(\eta) &= 1 - e^{-\eta} \cos \eta. \end{aligned} \right\} \quad (20)$$

On substituting Eqs. (17) into Eqs. (14), (15) and (16) we obtain the following ordinary differential equations for E , δ and w_{gr} :

$$\begin{aligned} Ro \left[\frac{d}{dr} (rv_{gr} E^2 \delta) I_1 + v_{gr}^2 \delta I_2 \right] + rv_{gr} \delta I_3 \\ = -\frac{krv_{gr}}{\delta} E f'(0), \end{aligned} \quad (21)$$

$$\begin{aligned} Ro \left[\frac{d}{dr} (r^2 v_{gr}^2 E \delta) I_4 - rv_{gr} \frac{d}{dr} (rv_{gr} E \delta) I_5 \right] \\ + r^2 v_{gr} \delta E I_5 = -\frac{kr^2 v_{gr}}{\delta} g'(0), \end{aligned} \quad (22)$$

$$w_{gr} = -\frac{I_5}{r} \frac{d}{dr} (rv_{gr} E \delta), \quad (23)$$

$$\text{where} \quad I_1 = \int_0^\infty f^2 d\eta, \quad I_2 = \int_0^\infty (1 - g^2) d\eta,$$

$$I_3 = \int_0^\infty (1 - g) d\eta, \quad I_4 = \int_0^\infty f g d\eta, \quad I_5 = \int_0^\infty f d\eta.$$

After some manipulation, Eqs. (21) and (22) are written as two simultaneous first order equations in the variables E^2 and $E\delta^2$, thus

$$\begin{aligned} \frac{1}{E^2} \frac{d}{dr} (E^2) = -\frac{2}{r\tilde{v}^2} \left[\frac{d}{dr} (r\tilde{v}^2) - B\tilde{v} \frac{d}{dr} (r\tilde{v}) \right] \\ - \left(\frac{2A}{r} + \frac{2X}{\tilde{v}} \right) + \frac{2Y}{\tilde{v}} - \frac{2(C + Dk)}{\tilde{v}E\delta^2}, \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{1}{E\delta^2} \frac{d}{dr} (E\delta^2) = \frac{1}{r\tilde{v}} \left[\frac{d}{dr} (r\tilde{v}^2) - 3B\tilde{v} \frac{d}{dr} (r\tilde{v}) \right] \\ + \left(\frac{A}{r} + \frac{X}{\tilde{v}} \right) \frac{1}{E^2} - \frac{3Y}{\tilde{v}} + \frac{C + 3Dk}{\tilde{v}E\delta^2}, \end{aligned} \quad (25)$$

where $\tilde{v} = R_0 v_{gr}$ and

$$A = \frac{I_3}{I_1}, \quad B = \frac{2I_4 - I_5}{I_4 - I_5}, \quad C = \frac{f'(0)}{I_1},$$

$$D = \frac{g'(0)}{I_5 - I_4}, \quad X = \frac{I_3}{I_1}, \quad Y = \frac{I_4}{I_4 - I_5}.$$

Also, Eq. (23) reduces to

$$w_{gr} = -\frac{I_5}{Ro} \left[\frac{kD}{\delta} + E\delta \left(\frac{(1-B)}{r} \frac{d}{dr} (r\tilde{v}) - Y \right) \right]. \quad (26)$$

With profiles given by Eqs. (20), $f'(0) = -1$,

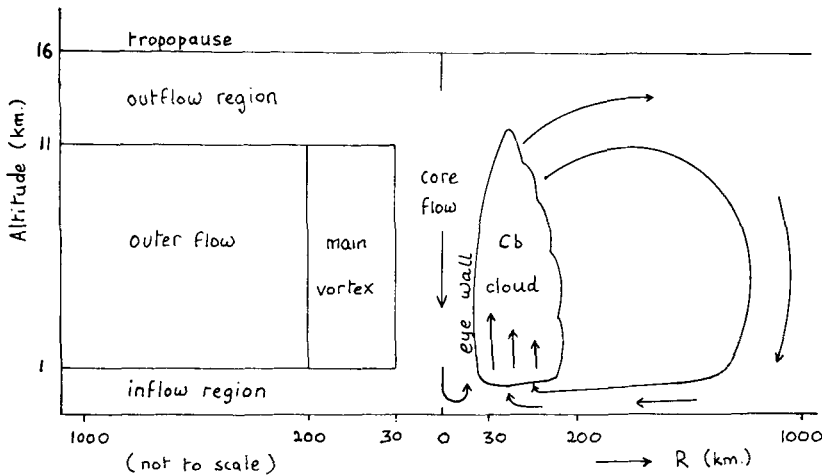


Fig. 1. The main regions of flow in a mature hurricane (see text). Arrows indicate the vertical circulation of air in the system.

$g'(0) = 1$ and $I_1 = \frac{1}{8}$, $I_2 = \frac{1}{8}$, $I_3 = \frac{1}{2}$, $I_4 = -\frac{1}{8}$, $I_5 = -\frac{1}{2}$.

If boundary condition B is taken at the sea surface, the profiles for u and v are

$$\left. \begin{aligned} f(\eta) &= ce^{-\eta}(a_1 \sin \eta + a_2 \cos \eta), \\ g(\eta) &= 1 - ce^{-\eta}(a_1 \cos \eta + a_2 \sin \eta), \end{aligned} \right\} \quad (23)$$

where $c = 1/(1 + \sqrt{2}\gamma + \gamma^2)$, $a_1 = 1 + \gamma/\sqrt{2}$, $a_2 = \gamma/\sqrt{2}$ and $\gamma = v_{gr}/\alpha k$.

With these profiles,

$$\begin{aligned} f'(0) &= c(a_2 - a_1), \quad g'(0) = c(a_1 - a_2) \text{ and} \\ I_1 &= \frac{1}{8} c^2 (a_1^2 + 2a_1a_2 + 3a_2^2), \\ I_2 &= c(a_1 + a_2) - \frac{1}{8} c^2 (3a_1^2 + 2a_1a_2 + a_2^2), \\ I_3 &= \frac{1}{2} c(a_1 + a_2), \\ I_4 &= \frac{1}{8} c^2 (a_1^2 + 4a_1a_2 + a_2^2) - \frac{1}{2} c(a_1 + a_2), \\ I_5 &= -\frac{1}{2} c(a_1 + a_2). \end{aligned}$$

In this case, the quantities A , B , C , D , X , Y in Eqs. (24) and (25) are functions of r .

Pressure and velocity profiles

It remains to choose a realistic profile for the radial pressure variation. In any two (or multi)-layer hurricane model, the pressure profile $p(r)$ would be given by that in the flow region

(assumed cylindrical) just above the boundary layer, detaching the behaviour of the boundary layer from that of the hurricane as a whole. This is justified only in the steady problem described here. In fact, the flow in the main vortex is strongly coupled with the flow in the boundary layer for reasons discussed in the introduction. Thus the behaviour of the flow in the inflow layer may be expected to play an important role in determining the structure of the entire hurricane during its time development. It is sufficient to take a profile which has the general form of those observed in typical hurricanes, that is, the pressure decreases outwards from the centre and the pressure gradient increases rapidly from zero to a maximum at a radius just outside the core, then decreases more slowly with R until a geostrophic balance is attained at large radii, typically 1000 km or more. Thus the swirling component of velocity in the main vortex increases from zero at the centre to large values slightly beyond the radius of maximum pressure gradient and then decreases gradually with increasing R .

Guided by this form we take

$$P(R) = P_c + (P_g - P_c) e^{\alpha b - R_m/R} \quad (28)$$

where P_c is the pressure at the centre, P_g the pressure at the geostrophic radius R_g , α is a constant chosen to make the azimuthal velocity above the boundary layer a maximum at $R = R_m$, and $b = R_m/R_g$. This form is essentially

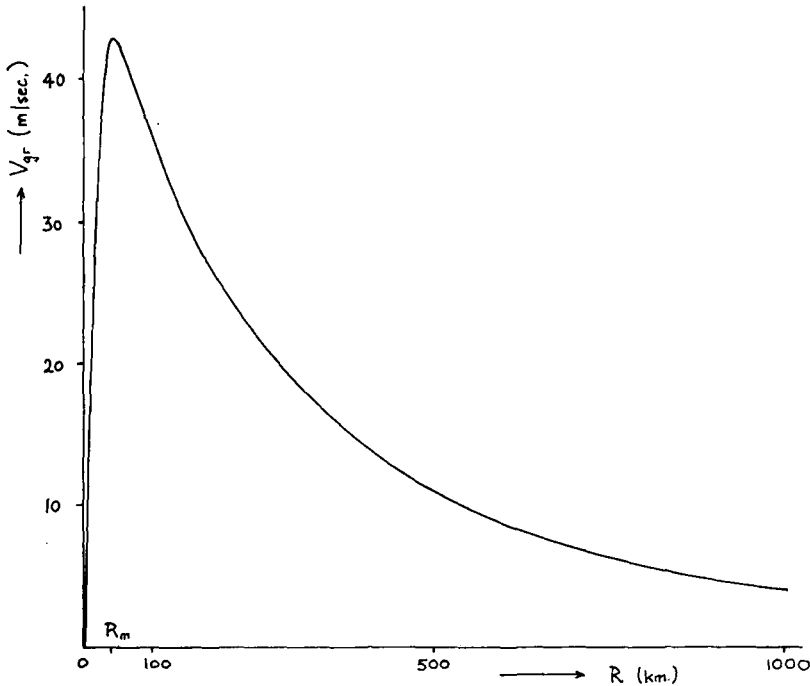


Fig. 2. The profile of swirl velocity, V_{gr} , at the top of the boundary layer.

the one found by Rosenthal (1962) to give a good representation of the radial variation of the gradient wind V_{gr} , obtained from Eq. (1). Making Eq. (1) non-dimensional with respect to the scales used in the last section, and using Eq. (28) to obtain dP/dr , we have

$$\tilde{v} = R_0 v_{gr} = -\frac{1}{2} r^2 + \left(\frac{1}{4} r^2 + \frac{m\kappa b}{r} e^{\kappa b(1-(1/r))} \right)^{\frac{1}{2}}, \quad (29)$$

where $m = ((P_g - P_c)/\rho)/R_g^2 f^2$. But, by the choice of scales, $v_{gr} = 1$ when $r = 1$ so that

$$R_0 = -\frac{1}{2} + \left(\frac{1}{4} + m\kappa b \right)^{\frac{1}{2}}. \quad (30)$$

If $dv_{gr}/dr = 0$ at $r = b$, then it is easy to show that κ satisfies the equation

$$m\kappa(\kappa - 1)^{\frac{1}{2}} e^{\kappa(b-1)} - (2 - \kappa)b^{\frac{1}{2}} = 0 \quad (31)$$

and $1 < \kappa < 2$.

Boundary conditions and method of solution

Taking the Ekman solution (Eq. (12)) at $r = 1$ (i.e. $R = R_g$) gives starting values $E = 1$

and $\delta = \sqrt{2}$. Since, however, Eqs. (24) and (23) include inertial terms, the derivatives of E^2 and $E\delta^2$ are not zero with these values, as would be the case for strict geostrophic flow; indeed their absolute values are quite large. The explanation lies in the fact that each is the difference of two large terms (i.e. the viscous and coriolis terms) and this makes the equations somewhat unstable in the starting region. A remedy can be found by choosing the derivatives to be zero at $r = 1$ and calculating the corresponding values of E^2 and $E\delta^2$ from Eqs. (21) and (22). (This gives sensible results since $dv_{gr}/dr < 1$ when $r = 1$ in the cases considered.) These turn out to be of the form $E^2 = 1 + \epsilon_1$, $E\delta^2 = 2 + \epsilon_2$ where $\epsilon_1, \epsilon_2 < 1$. These values are then used as starting conditions at $r = 1$ and the solution of the two differential equations (21) and (22) is advanced radially inwards by a stepwise integration routine on the Manchester Atlas Computer.

Discussion

The azimuthal velocity profile (Fig. 2) is computed for a hurricane vortex defined by Eq. (28) with $P_c = 940$ mb, $P_g = 1000$ mb, $R_g = 1000$

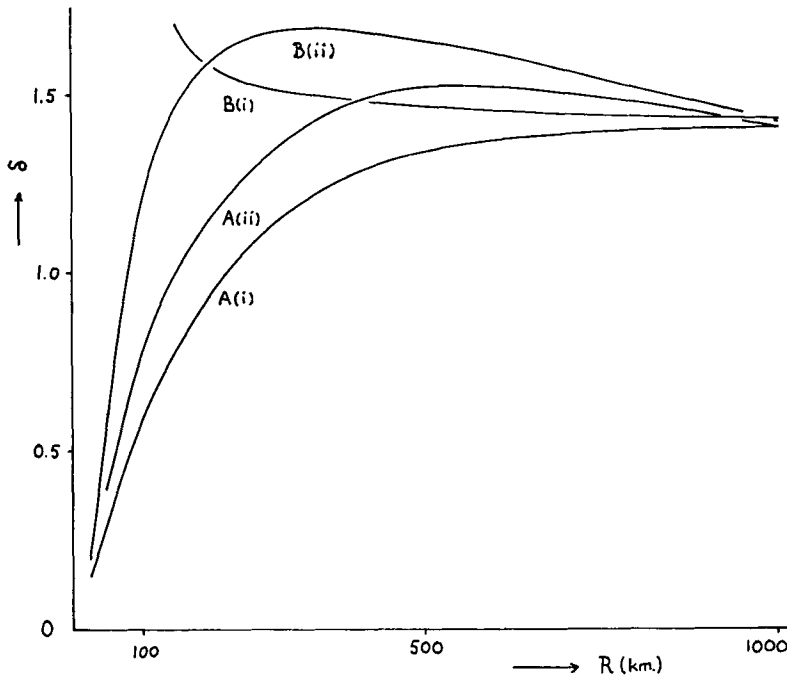


Fig. 3. Profiles of non-dimensional scale boundary layer thickness, δ , in the four cases: A (i), A (ii), B (i), B (ii). A = "no-slip" at the surface; B = surface stress in direction of surface wind; (i) eddy viscosity, K_M , radially constant; (ii) K_M increases inwards by a factor of two between 1000 km and 40 km radii.

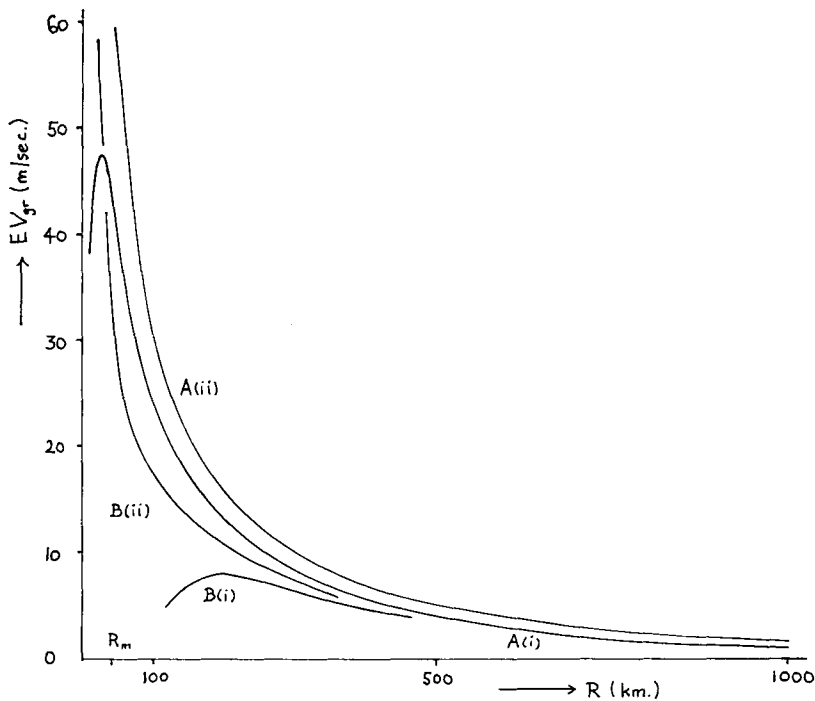


Fig. 4. Profiles of scale inflow velocity, EV_{gr} , in the boundary layer. Legend as for Fig. 3.

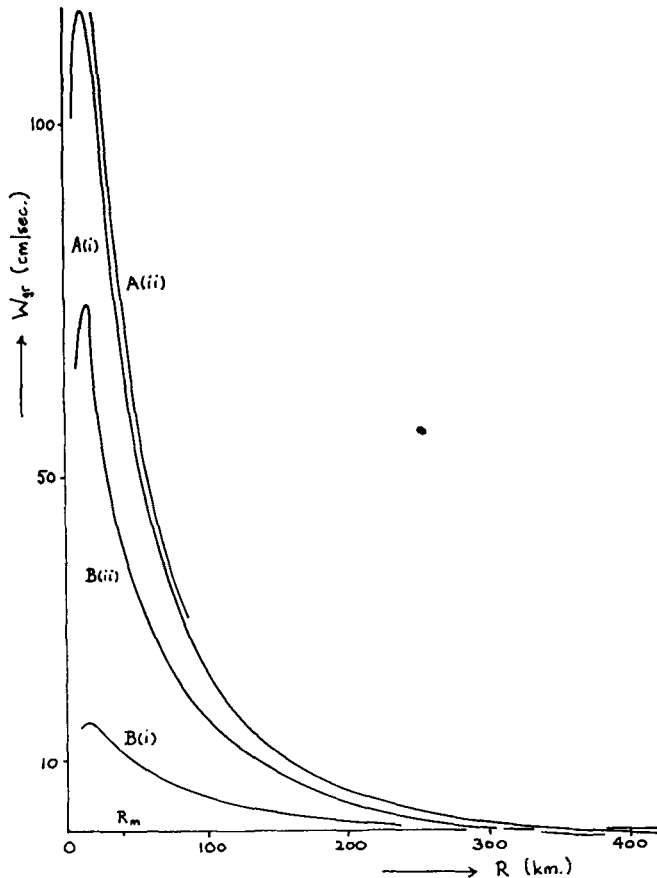


Fig. 5. Profiles of vertical velocity W_{gr} through the top of the boundary layer. Legend as for Fig. 3.

km, $R_m = 40$ km, $\rho = .0012$ gm/cc at 21°C and $f = 5 \times 10^{-5}$ (typical of about 20° lat.). The velocity increases rapidly from zero at the centre to a maximum of 43 m/sec at 40 km and then decreases more slowly to about 4 m/sec at 1000 km.

A Rossby number for the entire flow, given by Eq. (30), is $Ro = 7.8 \times 10^{-2}$.

Figures 3, 4 and 5 show the variation of non-dimensional scale boundary layer thickness δ , scale inflow velocity EV_{gr} and upflow velocity W_{gr} , respectively. The four curves in each figure correspond to the solution, with one of the surface boundary conditions, A or B and with;

- (i) the eddy viscosity K_M , a constant, equal to 50 m²/sec, or,
- (ii) K_M increasing linearly from 25 m²/sec at the geostrophic radius to 50 m²/sec at the radius of maximum velocity.

The value of C_D in boundary condition (B) is taken as 2×10^{-2} . These orders of magnitude for K_M and C_D seem to be supported in the literature and references are given by Rosenthal (1962).

The solution for the case A (i), in which the no-slip condition at the surface is combined with a constant eddy viscosity, is equivalent to that for a laminar viscous boundary layer, produced by a rigid boundary perpendicular to the vortex axis. In the latter, the kinematic viscosity ν , is replaced by the eddy viscosity K_M and the only difference between the laminar and turbulent flow is one of scale. The boundary layer thickness decreases towards the centre (Fig. 3), slowly at first, but more rapidly as the swirling velocity gradient above the layer becomes larger, until it is about one fifth of its geostrophic value at the maximum swirl radius. The scale

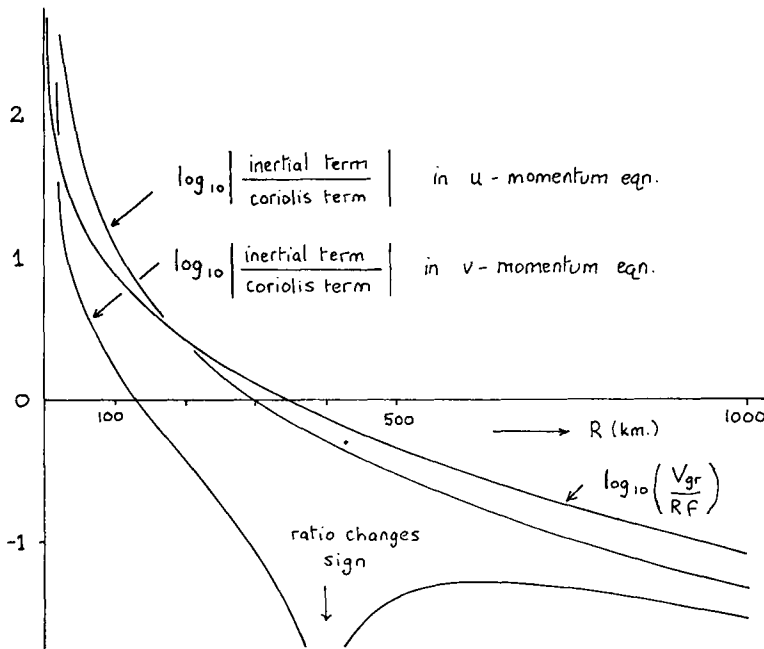


Fig. 6. Logarithmic profiles of the ratio [inertial term/Coriolis term] in the radial (u) and transverse (v) momentum equations compared with the logarithmic profile of the local Rossby number.

radial velocity EV_{gr} , (Fig. 4) increases from about one third of the local azimuthal component V_{gr} , to a comparable magnitude, over this range. There is a small downflow surrounding the main vortex, with mean velocities of the order $\frac{1}{2}$ cm/sec and an upflow at smaller radii, reaching a maximum of just over 1 m/sec at about 13 km from the centre. Observations suggest that the maximum upflow occurs near the maximum swirl radius (Gray, 1966), but the discrepancy of our model is not surprising for two reasons. Firstly, in our representation of the hurricane vortex, we have taken no account of the core flow, in which there is a small downflow and consequently, a small outflow near the surface. Thus, there must be a small reversed pressure gradient in the "eye" and some "stagnation radius", at which incoming boundary layer fluid meets outflowing fluid from the "eye". We may conclude, therefore, that the core flow serves as an obstacle for the incoming fluid and this would cause the radius of maximum upflow to be displaced outwards. Secondly, care must be taken not to apply the solution too near the stagnation radius, or the centre, where radial stress terms become important and must be included in the equations. Indeed, it is for

this reason that the curves in Figs. 3-6 have not been drawn in a neighbourhood of the axis.

The inflow velocities computed in the case A (i) seem too large compared with those in actual hurricanes (Miller, 1958). It is likely that this is a result of taking the eddy viscosity constant with height as well as with radius. In a turbulent boundary layer, K_M will in general vary with position and to a first approximation, the layer may be divided into two sublayers; a relatively thin layer adjacent to the surface where vertical gradients of velocity are large and K_M is small and increases linearly with height; a much thicker layer where velocity gradients are small and K_M is large and may be treated as a constant. In the atmosphere, the lowest layer extends upwards to a few tens of metres whilst the upper layer may be up to a kilometre in depth (see, for example, Rohl, 1965). By taking K_M large and using the no-slip condition, we are overestimating the momentum transport to the surface. As a result, we overestimate the disruption of the net pressure field near the boundary and obtain too much inflow.

An attempt to overcome the problem of the lower sublayer is made by taking the less restrictive condition (B) at the surface, or more pre-

cisely at the top of this sublayer. This condition is used by Rosenthal (1961). Profiles for the case B (i) are shown in Figs. 3–5. The boundary layer “erupts” before it reaches the radius of maximum tangential wind; that is, the boundary layer starts to increase very rapidly (Fig. 3); the maximum inflow is attained at 120 km from the centre and this, together with the upflow is far weaker than in the case A (i). This “eruption” of the boundary layer so far from the centre is not observed. Its occurrence in the model B (i) is, however, not surprising. Although the boundary condition (B), together with a constant K_M , serves a useful representation in parallel flows when there is no transverse pressure gradient, it may be of limited application to flows in which transverse pressure gradients are strong, as is the case of vortex flows. The largest distortion of the net pressure field, due to the presence of the boundary, occurs in the region of largest velocity shear and this is in the sublayer. Thus, by taking a model for the turbulence above this layer we will obtain a considerable underestimate of the inflow.

It is clear that a more sophisticated description of the vertical structure of the turbulence is essential to a satisfactory formulation of the inflow layer. This can only come after many more observations have been made in this region.

It seems unlikely that K_M may be regarded as a constant over the whole extent of the storm and its environment. Nevertheless, little is known about the radial variation of K_M in actual hurricanes. Miller (1965) takes K_M to increase towards the storm centre and we have examined

the simplest case here, in which K_M increases linearly by a factor of two between the geostrophic radius and the radius of maximum tangential wind. The effect in case A (ii) is to increase the inflow considerably in the inner region, and consequently the upflow, compared with the case A (i). In the case B (ii), we obtain reasonable profiles of radial and vertical velocity, but the remarks of the above paragraph concerning this boundary condition still hold. Whilst no real inferences can be made at this stage, it is interesting to note the ease with which a radial varying K_M may be incorporated in this model and the latter should be useful in testing theories of the turbulent structure of the inflow layer, when these can be better compared with observations.

Figure 6 shows the relative importance of inertial to coriolis terms in the two momentum equations (21) and (22) and compares this ratio with the local Rossby number V_{gr}/Rf , in the case A (i). With these curves, one may easily identify the extent of the three regions of flow in the boundary layer, I, II and III, which are described in the Introduction. Moreover, comparison of the ratio (inertial term/coriolis term) in the radial momentum equation, with the local Rossby number, demonstrates the useful scaling properties of this quantity.

Acknowledgement

The author is grateful to Professor B. R. Morton and Mr. E. J. Watson for their many helpful suggestions and stimulating discussions during the course of this work.

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ПОГРАНИЧНЫЙ СЛОЙ НА ПОВЕРХНОСТИ УРАГАНА

Развивается простая модель для исследования некоторых особенностей пограничного слоя на поверхности урагана. Течение над слоем трения представляется циклоническим вихрем, в котором течение является градиентным, определяемым подходящим выбором радиального профиля давления. Предполагается, что поток в главном вихре является приблизительно геострофическим на больших расстояниях от центра вихря и для пограничного слоя на этих расстояниях берется решение Экмана. Для прослеживания развития пограничного слоя до центральных областей вихря используется интегральный метод импульса.

Получены радиальные профили толщины пограничного слоя и индуцированные верти-

кальные скорости для случая, когда в качестве характеристики турбулентного слоя трения взята постоянная турбулентная вязкость K_M . Исследованы два граничных условия на поверхности: условие прилипания и условие, что напряжение на поверхности совпадает по направлению с ветром у поверхности. Найдено, что первое условие более удовлетворительно и дает качественное согласие с наблюдениями. В связи с условиями на поверхности обсуждаются эффекты радиальных и вертикальных вариаций K_M , однако, недостаточное знание структуры турбулентности на данном этапе не позволяет дать более реалистического описания структуры пограничного слоя.