Calculation of the Rossby wave velocities in the Earth's atmosphere

By L. A. DIKY and G. S. GOLITSYN, *Institute of Atmospheric Physics, Academy of Sciences of the USSR, Moscow*

(Manuscript received March 7, 1967)

ABSTRACT

The Rossby wave velocities for several lowest modes were calculated using the Laplace tidal equation. The value of the equivalent dynamical depth of the atmosphere was taken, due to Diky (1965), equal to 10 km. The calculated values of the velocities are in close agreement with those found by Eliasen & Machenhauer (1965), who treated the empirical data.

The extended calculations of the Rossby wave velocities using the empirical data have been recently performed by Eliasen & Machenhauer (1965). The values of velocities determined by them for several low modes of the Rossby waves were somewhat lower than those calculated using the formula of Haurwitz (1940) (see also Blinova, 1943) for the non-divergent barotropic atmosphere

$$\varepsilon = \omega - \frac{2(\omega + \Omega)}{n(n+1)}, \tag{1}$$

where ε is the angular velocity of waves, Ω is the angular velocity of the Earth rotation, ω is the additional angular velocity of the rotation of the atmosphere as a whole. The velocity of the zonal flow is then equal to

$$u = \omega R \sin \theta,$$

where R is the Earth radius, θ is the co-latitude. If $\omega = 0$ then

$$\varepsilon = -\frac{2\Omega}{n(n+1)}. \tag{1'}$$

Note that formula (1) can be obtained from the formula (1′) if one observes the waves propagating in the atmosphere rotating with the velocity $\omega + \Omega = \Omega_0$ while one is standing in the reference system which rotates with the velocity Ω. This remark will be needed later when taking into account the zonal flow of the atmosphere.

Analysing the possible causes of the systematic discrepancy of the empirical values of the velocities with the theoretical ones Eliasen & Machenhauer (1965) used the fact that the compressibility of the atmosphere must reduce the values of the Rossby wave velocities. They considered the divergence effect in the barotropic motion by using the equation

$$\frac{\partial}{\partial t}(\nabla^2 \psi) + \mathbf{v} \cdot \nabla \nabla^2 \psi + \frac{2\Omega}{R^2}\frac{\partial \psi}{\partial \lambda} = \frac{\varkappa}{R^2}\frac{\partial \psi}{\partial t}, \tag{2}$$

where ψ is the stream function, $\mathbf{v}$ is the velocity vector, λ is the longitude and $\varkappa$ is treated as a constant. If $\varkappa$ were equal to 4, then the values of the Rossby wave velocities determined from the equation (2) become close to the empirical ones. However, this value of $\varkappa$ differs essentially from the value used in another problem by Cressman (1958).

The effect of compressibility of the Earth's atmosphere can be rigorously taken into account by using the Laplace tidal equation (see e.g. Hough, 1898)

$$\left[(1-\mu^2)\frac{d}{d\mu} + \frac{m\mu}{f}\right]\frac{1}{f^2-\mu^2}\left[(1-\mu^2)\frac{d}{d\mu} - \frac{m\mu}{f}\right]\chi(\mu) = \left[\frac{m^2}{f^2} - (1-\mu^2)\gamma\right]\chi(\mu), \quad \gamma = \frac{4\Omega_0^2 R^2}{gh}. \tag{3}$$

Here $\mu = \cos\theta$, m is the azimuthal wave number, $f = \sigma/2\Omega_0$ is the dimensionless frequency, i.e.

Table 1

			$T = \frac{1}{2}$ t, days			
			Eq. (4)		Calculations	
n	m	Eq. (1′) $\gamma = 0$	$\gamma = 7.9$	$\gamma = 10$	$\gamma = 7.9$	$\gamma = 10$
1	1	1	1.25	1.35	1.17	1.21
2	1	3	4.86	5.27	4.87	5.25
	2	1.5	1.63	1.67	1.61	1.64
3	1	6	8.12	8.68	8.11	8.65
	2	3	3.70	3.88	3.68	3.87
	3	2	2.08	2.11	2.07	2.09
4	1	10	12.08	12.63	12.10	12.63
	2	5	5.85	6.08	5.84	6.06
	3	10/3	3.70	3.81	3.68	3.78
5	1	15	17.05	17.59	17.10	17.61
	2	7.5	8.41	8.65	8.41	8.66
	3	5	5.48	5.61	5.47	5.60
6	1	21	23.03	23.57	23.04	23.59
	2	10.5	11.44	11.69	11.44	11.68
	3	7	7.54	7.68	7.54	7.68

the frequency divided by the double angular velocity of the atmosphere. The function $\chi(\mu)$ may be the three-dimensional velocity divergence, the pressure or the two-dimensional velocity potential (for the two-dimensional motion, e.g. for the motion in the Rossby waves). The influence of the compressibility is described by the value of the dimensionless parameter γ, where g is the acceleration of gravity, h is the so-called equivalent dynamical depth of the atmosphere. For the determination of the value of h one has to consider the equation describing the vertical structure of the oscillations (see Diky, 1965) together with the equation (3). For the long period oscillations with the periods more than 20 minutes (Diky, 1965) the "vertical" equation does not depend on the frequency but only on the value of h. Therefore all Rossby waves have the same value of h. For the non-divergent atmosphere $\gamma = 0$, i.e. $h \to \infty$. For the barotropic isothermal atmosphere $gh = c_T^2$, where c_T is the isothermal sound velocity; for the baroclinic (isothermal) atmosphere $gh = c^2$, where c is the adiabatic sound velocity (see e.g. Diky, 1961). According to computations by Diky (1965) the value of h for the Rossby waves in the Earth's atmosphere with the vertical stratification, close to the real one (the temperature profile was taken from the model CIRA 1961), is equal to 10 km. It is this value of h which we shall use. This value of h corresponds to $\gamma = 8.8$ if $\omega = 0$ (no zonal flow). If one considers, according to Eliasen & Machenhauer (1965),

$$\omega = 0.0225\ \Omega, \quad \text{then} \quad \gamma = 9.2.$$

A method of determination of the eigen values and eigen functions (Hough's functions) of the Laplace tidal equation has been described by Diky (1965). As has been shown by computations (see e.g. Golitsyn & Diky, 1966), the values of the eigen periods of the Rossby waves even at $\gamma \approx 10$ are with the great accuracy described by (Hough, 1898) the following expression

$$2m\,T_n^m = \frac{m}{f} = n(n+1) + \frac{(n^2-1)^2\,(n^2-m^2)}{(4n^2-1)\left[\dfrac{(n-1)^2 n^2}{\gamma} + \dfrac{2m^2}{n(n+1)^2}\right]} + \frac{n^2(n+2)^2(n-m+1)(n+m+1)}{(2n+1)(2n+3)\left[\dfrac{(n+1)^2(n+2)^2}{\gamma} - \dfrac{2m^2}{n^2(n+1)}\right]}, \tag{4}$$

where T_n^m is the period in days for the mode with the indices (m, n). The results of numerical computations of eigen periods of several low modes of Rossby waves are given in the Table 1 together with the corresponding periods calculated with the expression[1] (4) and with Haurwitz's formula (1′).

As one can see from this table the accuracy of Hough's formula increases rapidly with the increasing of m and n. Therefore as a rule it is not necessary to make numerical computations by using the Laplace tidal equation (3) to obtain the periods of the Rossby waves except very first few modes because one may use Hough's formula (4). Note also that the values of the periods depend rather slightly on the value of γ in the range considered here.

Let us turn to the calculation of the velocities. The velocity of a Rossby wave in degrees of longitude per day for an observer standing on the Earth surface is expressed by

$$\varepsilon = 360^\circ \left(\omega - \frac{\omega + \Omega}{mT}\right). \tag{5}$$

In Table 2 (columns 2 and 3) we present the values of the Rossby waves velocities for the

[1] A formula, similar to (4), was obtained by Blinova, in 1949 (see Blinova, 1960). For (m, n) not very small (nr 3) her formula also gives good results.

Table 2

(m, n)	ε, eq. (1′)	ε_{RH}, eq. (1)	$\varepsilon_{\text{calc}}$	ε_{obs}
(1.2)	−120	−115	−64.0	−70
(2.3)	−60	−53	−40.3	−40
(3.4)	−36	−28	−24.7	−20
(1.4)	−36	−28	−21.5	−20
(2.5)	−24	−16	−13.4	−12
(3.6)	−17.1	−9	−8.0	−8

non-divergent atmosphere with no zonal flow (eq. (1′)) and for the non-divergent atmosphere rotating with the velocity $\Omega_0 = \omega + \Omega$ (eq. (1)). The values of velocities computed on the base of the Laplace tidal equation with $\gamma = 9.2$ (compressible atmosphere rotating with the velocity $\Omega_0 = 1.0225\ \Omega$) are given in the fourth column and in the last column the empirical values of the velocities found by Eliasen & Machenhauer (1965) are presented. The comparison of the two last columns of this table shows their close agreement. The greatest discrepancy observed for the modes (1.4) and (2.5) in less than 20 per cent. Note also that the compressibility removes "the degeneracy" from the modes (3.4) and (1.4) and their theoretical velocities begin to differ.

The program of computations described by Diky (1965) and used here gives also the coefficients of the expansion of Hough's functions in the series of the normalized associated Legendre polynomials

$$Q_n^m(\mu) = \sum_{k=m}^{\infty}{}' c_k P_k^m(\mu). \tag{6}$$

The prime on the summation sign means that only those values of k enter the summation for which $k-n$ is an odd integer. Having the coefficients c_k one can calculate Hough's functions $Q_n^m(\mu)$ and also stream functions for the Rossby waves

$$\psi_n^m(\mu) = \sum_{k=m}^{\infty}{}' a_k P_k^m(\mu)\,(k-n \text{ even}). \tag{7}$$

Here the following relationship between coefficients c_k and a_k has to be used (see e.g. Diky, 1961, 1965):

$$[m - k(k+1)f]a_k = \frac{(k+1)(k-m)}{k(2k+1)} c_{k-1} + \frac{k(k+m+1)}{(k+1)(2k+3)} c_{k+1}. \tag{8}$$

From expressions (6)–(8) one can see that if Hough's function is even, the corresponding stream function must be odd and vice versa. Hough's functions $Q_n^m(\mu)$ are a system of the orthonormal functions so $\sum_{k=m}^{\infty}{}' c_k^2 = 1$ and $\int_{-1}^{1}[Q_n^m(\mu)]^2 d\mu = 1$. For the case of the Earth's atmosphere the coefficients c_k decrease rather rapidly as k increases. The same behaviour is shown by the coefficients a_k. For a non-divergent atmosphere $\psi_n^m(\mu) = P_n^m(\mu)$, but if one considers a compressible atmosphere, then it is necessary to determine the coefficients of the expansion in the series (7). Hough (1898) presented some approximate expressions for the calculations of the coefficients a_k also, but we found that they are not as accurate as his expression (4) for the periods. For the case $m \leqslant n \leqslant 3$ such a comparison has been done earlier (see Golitsyn & Diky, 1966). We present now several first terms of the expansions at $\gamma = 9.2$ of the stream functions for the modes presented in Table 2. These stream functions are normalized in the manner that $\sum_{k=m}^{\infty}{}' a_k^2 = 1$. These expansions are as follows:

$$\psi_2^1 = 0.993P_2^1 - 0.110P_4^1 + 0.009P_6^1 + \dots$$

$$\psi_3^2 = 0.997P_3^2 - 0.075P_5^2 + 0.004P_7^2 + \dots$$

$$\psi_4^3 = 0.999P_4^3 - 0.035P_6^3 + 0.002P_8^3 + \dots$$

$$\psi_4^1 = 0.147P_2^1 + 0.986P_4^1 - 0.087P_6^1 + 0.004P_8^1 + \dots$$

$$\psi_5^2 = 0.123P_3^2 + 0.992P_5^2 - 0.053P_7^2 + 0.002P_9^2 + \dots$$

$$\psi_6^3 = 0.131P_4^3 + 0.990P_6^3 - 0.043P_8^3 - 0.002P_{10}^3 + \dots$$

These expansions show that even at $\gamma \approx 10$ the shape of the stream functions for the lower modes of Rossby waves does not differ greatly from the shape of those in the non-divergent atmosphere ($\gamma = 0$) when $\psi_n^m = P_n^m$.

Acknowledgement

The authors wish to express their sincere gratitude to Prof. E. Eliasen and Dr. B. Machenhauer for their proposal to carry out this study and helpful discussions on this subject.

REFERENCES

Blinova, E. N. 1943. Hydrodynamic theory of pressure waves, temperature waves, and atmospheric action centers. *Doklady Akademii Nauk SSSR 39*, 284–287.

Blinova, E. N. 1960. On the theory of the long-range weather forecast taking into account vertical velocities and tropopause level oscillations. Problems of dynamical meteorology, 12–20. Izdatel'stro Akademii Nauk SSSR.

Cressman, G. P. 1958. Barotropic divergence and very long atmospheric waves. *Mon. Wea. Rev. 86*, 293–297.

Diky, L. A. 1961. Natural oscillation of a baroclinic atmosphere above a spherical earth. *Izvestija Akademii Nauk SSSR, Geophysics*, Series No. 5, 756–765.

Diky, L. A. 1965. The terrestrial atmosphere as an oscillating system. *Izvestija Akademii Nauk SSSR, Atmospheric and Oceanic Physics 1*, 469–489.

Eliasen, E. & Machenhauer, B. 1965. A study of the fluctuations of the atmospheric planetary flow patterns represented by spherical harmonics. *Tellus 17*, 220–238.

Golitsyn, G. S. & Diky, L. A. 1966. Oscillations of planetary atmospheres as a function of the rotational speed of the planet. *Izvestija Akademii Nauk SSSR, Atmospheric and Oceanic Physics 2*, 225–235.

Haurwitz, B. 1940. The motion of atmospheric disturbances on the spherical earth. *J. Marine Res. 3*, 254–267.

Hough, S. S. 1898. On the application of harmonic analysis to the dynamical theory of tides, pt. II. On the general integration of Laplace's dynamical equation, *Phil. Trans. Roy. Soc.* (A) 191, 139–185.

ВЫЧИСЛЕНИЕ СКОРОСТЕЙ ВОЛН РОССБИ В АТМОСФЕРЕ ЗЕМЛИ

Для ряда первых мод волн Россби проведены вычисления их скоростей на основе использования приливного уравнения Лапласа. Согласно Дикому (1965), величина динамически эквивалентной глубины атмосферы бралась равной 10 км. Вычисленные значения скоростей оказались в хорошем согласии с результатами эмпирических вычислений Элиасена и Махенхауэра (1965).