SHORTER CONTRIBUTION

On the linear balance equation in terms of spherical harmonics

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The linear or geostrophic balance equation is given as

$$f\nabla^2\psi + \nabla f \cdot \nabla\psi = g\nabla^2 Z, \tag{1}$$

where f is the Coriolis parameter, g is the acceleration due to gravity, Z is the geopotential height of a constant pressure surface, and ψ is the stream function for horizontal flow. This note is concerned with the solution of equation (1) in terms of spherical harmonics, and the implicit conditions on the geopotential height field which are necessary to produce a convergent solution.

Eliasen & Machenhauer (1965) have derived the equations relating the coefficients of the expansion of the stream function to the coefficients of the height field using the linear balance equation. These are

$$n(n+2)\varepsilon_{n+1}^{m}\begin{bmatrix}\alpha_{n+1}^{m}\\ \beta_{n+1}^{m}\end{bmatrix} + (n-1)(n+1)\varepsilon_n^m\begin{bmatrix}\alpha_{n-1}^{m}\\ \beta_{n-1}^{m}\end{bmatrix} = n(n+1)\begin{bmatrix}A_n^m\\ B_n^m\end{bmatrix}, \tag{2}$$

where $\varepsilon_n^m = (n^2-m^2)^{\frac{1}{2}}(4n^2-1)^{-\frac{1}{2}}$ and the coefficients are defined by the following expansions:

$$\left.\begin{aligned}\psi &= g/2\Omega \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}(\alpha_n^m \cos m\lambda + \beta_n^m \sin m\lambda)P_n^m,\\ Z &= \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}(A_n^m \cos m\lambda + B_n^m \sin m\lambda)P_n^m.\end{aligned}\right\} \tag{3}$$

In terms of these coefficients the mean horizontal kinetic energy is given by

$$\bar{E} = \frac{1}{4\pi}\int_s \frac{1}{2}(\mathbf{V}\cdot\mathbf{V})\,ds = \frac{1}{2}\left(\frac{g}{2\Omega a}\right)^2 \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} K_{m,n}, \tag{4}$$

where $K_{m,n} = \begin{cases} 2n(n+1)\alpha_n^{02} & \text{for } m=0,\\ n(n+1)(\alpha_n^{m2}+\beta_n^{m2}) & \text{for } m \neq 0.\end{cases}$

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Given the values of A_n^m, B_n^m equations (2) then permit the evaluation of α_n^m, β_n^m according to a recurrence relationship. The difficulty is that the solution so obtained *does not* define the kinetic energy in the sense that the series (4) *diverges* unless some condition is imposed on the height field. In order to simplify the demonstration of this property we will consider only the purely zonal case. The same result holds for the non-zonal case. However, the physical interpretation is not clear.

By equation (2) the antisymmetric components of Z are related to the symmetric components of ψ, and vice versa. Thus specializing these equations for zonal flow, we may determine α_2^0 from A_1^0; α_4^0 from α_2^0 and A_3^0; etc. That is, the recurrence relation is self-starting for symmetric stream function components. It is not, however, self-starting for antisymmetric stream function components. This latter problem will be treated later. Considering the antisymmetric height field components, let us suppose that the expansion of the height field is terminated at some point, i.e.

$$A_{2k+1}^0 \equiv 0, \quad \text{for } k > K. \tag{5}$$

This does not imply that $\alpha_{2k+2}^0 = 0$ for $k > K$ as can be seen from equations (1). Rather for $k > K$ we have

$$\frac{\alpha_{2k+4}^0}{\alpha_{2k+2}^0} = \frac{-(2k+2)(2k+4)}{(2k+3)(2k+5)}\,\frac{\varepsilon_{2k+3}^0}{\varepsilon_{2k+4}^0}. \tag{6}$$

At first sight this does not seem to be a problem since the ratio $\alpha_{2k+k}^0/\alpha_{2k+2}^0$ is less than one. However it can be shown that under these circumstances the series defining the kinetic energy diverges. The series under question (apart from constants and ignoring those terms before which (6) applies) is of the form

$$\cdots + n(n+1)\alpha_n^{0*} + (n+2(n+3)\alpha_{n+2}^{0*} + (n+4) \times (n+5)\alpha_{n+4}^{0*} + \cdots; \quad n = \text{even}. \tag{7}$$

Using equation (6), the ratio of adjacent terms in the above series may be written as

$$b_n = \frac{(n+1)(n-1)(2n+3)}{n(n+2)(2n-1)}; \quad n = \text{even}. \tag{8}$$

This may be compared to the series

$$\frac{2}{2} + \frac{2}{4} + \frac{2}{6} \cdots \frac{2}{n}, \quad n = \text{even}, \tag{9}$$

whose corresponding ratio is $r_n = n/n+2$. If we assume that $b_n > r_n$ we have

$$(n+1)(n-1)(2n+3) > n^2(2n-1)$$

or, after some algebra

$$4n^2 - 2n - 3 > 0,$$

which is true for all $n > 2$. Since the series defined by the ratio r_n diverges, then series (7) diverges.

Thus some condition must be imposed in the height field such that if equation (5) applies, then

$$\alpha_{2K+2}^0 = 0. \tag{10}$$

Since $A_{2k+1}^0 \equiv 0$ for $k > K$, and because of (10) then α_{2K+4}^0, α_{2K+6}^0 etc. are all identically zero and the kinetic energy is well defined. It can be shown that the necessary and sufficient condition that equation (10) be valid is that the derivative of the height field with respect to latitude vanishes at the equator. This does not mean that only symmetric components of the height field are permissible, but rather that the antisymmetric components are not linearly independent. That is, they satisfy a relationship of the form

$$\sum_{i=0}^{K} k_i A_{2i+1}^0 = 0, \tag{11}$$

where k_i is determined by equations (2).

The condition that $dZ/d\phi = 0$ at the equator may be written as

$$\left.\frac{dZ}{d\phi}\right|_{\phi=0} = \sum_{i=0}^{K} A_{2i+1}^0 \left.\frac{dP_{2i+1}^0}{d\phi}\right|_{\phi=0} = 0, \tag{12}$$

since the derivatives of symmetric Legendre functions vanish at the equator. By evaluating $dP_{2i+1}^0/d\phi$ at $\phi = 0$ we may show that it is proportional to k_i, independent of i but dependent on K, so that the condition applies no matter what the level of truncation of the height field expansion.

The above consideration concerning the convergence of the series defining the kinetic energy may be equally applied to the recurrence relation defining the antisymmetric stream function components. It may then be shown that the condition under which the series converges is that

$$\alpha_{2K+1}^0 = 0. \tag{13}$$

This in turn implies that the mean angular momentum (which is measured by the component α_1^0) be equal to the mean geostrophic angular momentum, i.e.

$$\int_{-\pi/2}^{\pi/2} au \cos^2 \phi d\phi = \int_{-\pi/2}^{\pi/2} au_g \cos^2 \phi d\phi. \tag{14}$$

This is the only starting condition which will produce a convergent series for the kinetic energy. (This is precisely the method by which Eliasen & Machenhauer circumvented the problem of the non-starting recurrence relation.)

These conditions on the solution in terms of spherical harmonics may be obtained for the original balance equation. If we consider the zonal case, and define $\Phi = g/2\Omega Z$ then the linear balance equation (1) becomes

$$\sin\phi \frac{\partial}{\partial\phi}\left(\cos\phi \frac{\partial\psi}{\partial\phi}\right) + \cos^2\phi \frac{\partial\psi}{\partial\phi} = \frac{\partial}{\partial\phi}\left(\cos\phi \frac{\partial\Phi}{\partial\phi}\right). \tag{15}$$

Equation (15) may be transformed into

$$\frac{\partial}{\partial\phi}(\sin\phi M_\psi) = \frac{\partial X}{\partial\phi}, \tag{16}$$

where $M_\psi = -\cos\phi\, \partial\psi/\partial\phi$; $X = -\cos\phi\, \partial\Phi/\partial\phi$.

Thus we have

$$\sin\phi M_\psi = X - X_E, \tag{17}$$

where $X_E = X(\phi = 0)$.

Now $M_\psi \to 0$ as $\phi \to \pm\pi/2$ and $x \to 0$ as $\phi \to \pm\pi/2$, therefore it follows at $X_E = 0$; i.e.

$$\cos\phi \frac{\partial\Phi}{\partial\phi} = 0, \quad \text{at } \phi = 0. \tag{18}$$

Now the mean angular momentum $\overline{M}$ is given by

$$\overline{M} = \int_{-\frac{\pi}{2}}^{\pi/2} M_{\psi} \cos \phi d\phi = \int_{-\frac{\pi}{2}}^{\pi/2} \frac{X \cos \phi}{\sin \phi} d\phi$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} aug \cos \phi d\phi. \quad (19)$$

Thus the two conditions on the height field which are necessary for a non-singular solution are the same as those which produce a well-defined kinetic energy series in terms of spherical harmonics.

The author (1965) has applied the above system of solving the linear balance equation to a global field of 500 millibar height data for the month of September 1957. Generally, the results indicate that the system produces reasonable wind fields over the globe. The high values of kinetic energy that one might expect in the equatorial regions fail to appear because of the implicit condition on the geopotential height field. The interpretation of the resulting harmonic components as global entities is open to serious doubt however, mainly because of their behaviour as a function of time (Merilees, 1966).

In summary, the solution of the linear balance equation in terms of spherical harmonics requires implicit conditions on the given geopotential height field. In the zonal case the conditions are (1) the derivative of the height field vanishes at the equator (2) the mean angular momentum is equal to the mean geostrophic angular momentum. The physical interpretations of the conditions in the non-zonal case awaits further analysis.

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