Summary of a Theoretical Investigation into the Factors Controlling the Instability of Long Waves in Zonal Currents[1]

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Abstract

It is assumed that in horizontally uniform zonal currents certain long waves can grow in amplitude whereby the kinetic energy associated with the growth is drawn from the potential energy of the basic current system. This restriction as to the energy redistribution permits to treat the simultaneous physical changes of state as polytropic, which simplifies considerably the integration of the relevant equations of motion. It is shown that the amplitude growth, at a prescribed zonal dimension of the perturbation(wave-length), is primarily controlled by the value of a non-dimensional quantity, viz. the RICHARDSON number (= static stability divided by the square of the vertical shear of wind), in a similar manner as this number controls the time variation of turbulence to the extent of its being a result of energy redistribution. The derived criterion for instability predicts that all but the very long waves in the middle latitude westerlies are unstable.

The intensity of the absolute circulation, in addition to the vertical shear, has an important bearing on the considered phenomenon of instability. This is discussed with regard to the non-equilibrium position of a cold anticyclonic vortex centered at the pole. The stability criterion is also scrutinized as to its validity at tropical latitudes.

1. Introduction

Considerable attention is at present given to the origin and life-history of the long, undulating upper level disturbances characteristic of the middle latitude belt of westerlies. To the extent that these disturbances may be considered as quasi-horizontal wave motions imposed on autobarotropic currents, their shape, wave-length and displacements are controlled primarily by the variation of the Coriolis parameter with latitude.

In view of recent investigations into the breaking up of zonal motion (BERGGREN, BOLIN and ROSSBY, 1949) one must accept the existence of some mechanism by which the planetary waves are dynamically linked to the minor scale disturbances of the frontal wave type. However, the breaking up of a quasi-zonal current into large scale vortices may be regarded, without reference to the existence of frontal waves, as a phenomenon of long-wave instability on the grounds of a rather marked difference in the horizontal scale of the respective disturbances. Theoretical investigations — HAURWITZ (1940), JAW (1946) and CHARNEY (1947) — have dealt with the dynamics of horizontally uniform, barocline (vertically shearing) zonal currents. These theories demonstrate that the baroclinity and the static stability of the basic current system are the principal factors in the early stages of the life-history of the perturbation.

The object of the investigations summarized below is an attempt to incorporate, in a general manner, the physical process into the theory for long waves (though restricted

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by the same fundamental assumptions as those made by the above authors); to derive, by using relatively simple mathematical procedure, such elementary instability criteria that can be more readily interpreted in the light of the energetical and dimensional aspect of the phenomenon treated here: large-scale turbulence appearing in a flow which initially is large-scale laminar; finally, to develop the elementary criteria with a view to attain more conformity with empirical evidence.

2. A limiting case of energy redistribution] (Introduction of the polytropic invariant)

To the extent that the kinetic energy of the basic zonal current remains unchanged during the growth of the disturbance, the growth phenomenon is merely a manifestation of a release, through the perturbation motion, of potential energy. Evidently, the existence of a barocline field (more specifically of solenoids in the meridional plane) in the basic system is a prerequisite for availability of that part of potential energy which must be converted into kinetic energy of the perturbation motion. These specifications regarding the energetic aspect of the process imply that the current system is thermally inactive and the physical process associated with the motion is characterized by polytropic changes of state. The polytropic process is a reversible thermodynamic path of constant heat capacity c', so that the heat supplied to the system per unit of time to unit of mass is determined by the differential relationship $dQ = c'dT$ where T is the absolute temperature. To each of the polytropic classes (isobaric, isothermal, (dry) adiabatic and isopycnic) one can assign an individual invariant $\tilde{\xi}$ (conservative property) or, in the mathematical formulation, $\frac{d\tilde{\xi}}{dt} = 0$, so that e.g. in the case of dry-adiabatic process $\tilde{\xi}$ is being identified with the potential temperature Θ. It is, however, possible to treat, subject to subsidiary restrictions as to the distribution of water-vapour, a reversible saturated-adiabatic change of state as a quasi-polytropic process so that c' is also in this case very nearly constant. In doing so, allowance can be made for the effect on energy conversion of the formation of extensive cloud masses which may accompany the growth of the baroclinic wave.

3. Derivation of a basic frequency equation

A first approximate solution for the phase velocity of plane waves, satisfying the hydrodynamic and all other relevant differential equations as well as a minimum of boundary conditions, yet revealing a fundemental criterion for instability, is obtained in the following way. The theorem for the path variation of the vertical component of vorticity is transformed (and linearized) for "small perturbations" propagating with constant speed in a transversally uniform zonal current of infinite width. Making use of the theorem of geostrophic meridional perturbation velocity[1], a linear differential equation of effectively second order with the dependent variables p (pressure perturbation) and ξ (perturbation of the polytropic individual invariant $\tilde{\xi}$) is derived. Two solutions are found, one establishing a frequency relationship for short waves whose wave-length is of the order of magnitude equalling the depth of the atmosphere, the other yielding an equation for the phase velocity of long (planetary) waves. The waves of the first group are linked with the disturbance of hydrostatic equilibrium and are capable of growing, in certain conditions, if the static stability is negative. The long waves are linked with the disturbance of zonal balance. To understand and measure the extent of validity of the criteria for instability of the latter waves, the reader is asked to follow a few steps in the derivation of the principal equation for the phase velocity.

In order to simplify the mathematical aspect, the effect of compressibility of the air will be disregarded throughout in this summary. Little generality is lost through this expedience as far as the main results are concerned. In view of the restrictions with regard to the energy redistribution, as mentioned earlier

[1] The assumption that the meridional velocity is geostrophic or very nearly geostrophic, involves the exclusion of long gravitational waves from possible solutions of the equations of motion. ROSSBY (1939) has shown that in principle this must be so and later CHARNEY (1947) has given a mathematical proof of this circumstance.

in this paper, it must be expected that the behaviour of the perturbation, i.e. its speed of propagation and its time-rate of growth (determined by the real and imaginary parts of the phase velocity c resp.) depends on a) steady state or "basic" quantities such as the intensity of the zonal circulation $\bar{u}$ or the meridional slope of the isobaric surfaces m_p; the rate of basic baroclinity which in conjunction with m_p is determined by the meridional slope of the constant $\bar{\xi}$-surfaces, m_ξ; b) parameters constituting the structure of the perturbation such as wave-length, L, and phase angle Φ as well as the amplitudes p_0 and ξ_0 of the respective perturbations in the pressure- and $\tilde{\xi}$-field; c) planetary and physical constants as e.g. the geographical latitude $\bar{\varphi}$ (central to the zonal current) and the radius of the earth, a; the acceleration of gravity, g. A general theory of atmospheric waves would have to allow for variations of all these quantities, except a and g, with respect to latitude, longitude and elevation whilst within the limits set by the assumptions made already above, we have to allow for variations of $\bar{u}$ (or m_p), m_ξ, p_0 and ξ_0 as well as the phase angle with elevation. There are no physical or kinematic boundary conditions which impose a restriction on the $\bar{\xi}$-surfaces whose meridional slope may therefore without loss of generality assumed to be uniform. It is well known that the variations of the amplitudes p_0 and ξ_0 with elevation are subject to the boundery conditions and it is this circumstance which makes the integration of the relevant differential equation so difficult. The view is here put forward that among the factors that primarily govern the law of amplitude growth, the height-dependency of the perturbation amplitude is of secondary importance. We shall therefore treat the amplitudes p_0 and ξ_0 as constants for the integration of the transformed vorticity equation.[1] We obtain

$$u^\star = \bar{u} - c = \frac{\beta}{k^2} + \frac{f m_\xi}{H k^2} \frac{\bar{p}}{\bar{\xi}} \frac{\xi_0}{p_0} \cos k\varepsilon \qquad (1)$$

where $u^\star$ is thus the zonal wind velocity relative to the moving wave (i.e. the "velocity deficit") and β/k^2 its value in the case of non-divergent waves in a barotropic, homogenous zonal current (ROSSBY 1939). In this equation f denotes the Coriolisparameter, β its rate of change northward, H the height of the homogenous atmosphere whose surface temperature is equal to that at the reference level, $k = 2\pi/L$ the wave number and ε the lag of phase between the $\tilde{\xi}$-wave and the pressure wave.[1] The second term on the r.h.s. gives the combined effect of horizontal divergence and horizontal solenoids in the $\tilde{p}$- and $\tilde{\xi}$-fields on the net acceleration of relative vorticity — this effect expressed as a velocity —, whilst the preceding term, as is well known, gives in the same manner the effect of the planetary or "induced" variation of vorticity.

Equation (1) is as yet underdetermined with respect to $u^\star$ or c, since the amplitude ratio $\frac{\xi_0}{p_0}$ is not expressed in terms of known quantities, and since the phase lag ε is in general not determinable without regard to the vertical tilt of the two component waves, this tilt having been disregarded here so far. From general principles it can be shown that the non-dimensional factor $A = \frac{\xi_0}{p_0} \frac{\bar{p}}{\bar{\xi}} \cos k\varepsilon$ depends upon the meridional horizontal gradient of the conserved property, $\frac{\partial \bar{\xi}}{\partial y}$, and the orbital frequency $\nu^\star$ and thus on wave-length L and velocity deficit $u^\star$, since always $\nu^\star = \frac{2\pi u^\star}{L}$. In particular, in a plane at which the motion is horizontal (nodal plane of vertical motion) the factor A is easily determined from the developed and transformed law of conservation, viz. $\frac{d\tilde{\xi}}{dt} = 0$, which had been

[1] To further simplify the problem we shall, for the time being, disregard the tilt of the component waves in the vertical, in other words, treat also the phase-angle Φ as independent of height. Later on the effect of the inclusion of the tilt will be discussed.

[1] The bar over the quantities p and ξ indicates that the value in the undisturbed state is taken.

discussed earlier in this paper. One finds at a nodal plane

$$A_0 = -\frac{S_\xi^\star H}{u^\star f} \quad [1] \qquad (2)$$

where $S_\xi^\star = \frac{g}{\bar{\xi}}\frac{\partial \bar{\xi}}{\partial y}$, a quantity which, in analogy to the static stability $S = \frac{g}{\bar{\xi}}\frac{\partial \bar{\xi}}{\partial z}$, may be termed "dynamical stability function". Substituting (2) in (1) we obtain for the phase velocity c the equation

$$c = \bar{u}_0 - \frac{\beta}{2k^2}\left[1 \pm \sqrt{1 - \frac{4k^2}{\beta^2}\frac{(S_\xi^\star)^2}{S_\xi}}\right], \qquad (3)$$

where $\bar{u}_0$ is the zonal velocity at the nodal plane. In the derivation of this equation we had assumed that $S_\xi^\star$ and S_ξ are independent of height or, in other words, that the meridional slope of the iso-$\bar{\xi}$-surfaces is uniform throughout. This implies that e.g. in the meteorologically relevant classes of polytropic change of state, viz. the dry- and saturated-adiabatic, ($\bar{\xi} = \bar{\Theta}$ or $\bar{\Theta}_\omega$ resp.) there is a free surface except if the absolute temperature does not vary or increases with height. Eq. (3) gives then the speed of propagation of simple harmonic waves in a zonal current with one nodal plane of vertical motion and one free surface provided the current is horizontally uniform, the vertical and (meridional) horizontal gradients of the quantity $\bar{\xi}$ are constant throughout. The adopted model does therefore not allow for the existence of a tropopause surface at which both $\frac{\partial \bar{\xi}}{\partial y}$ and $\frac{\partial \bar{\xi}}{\partial z}$ possess discontinuities. Neither does it allow for inhomogeneity, e.g. within the troposphere, of the concentration of isentropes in horizontal surfaces which is frequently observed in even otherwise well organized zonal currents. And the variations of the verticallapse-rate of temperature in both, the vertical as well as the (meridional) horizontal direction, which may be of importance, cannot be introduced as significant quantities in eq. (3). Now, disregarding here the effect of the earth's orography as well as that of frictional stresses, the earth's surface is regarded as a nodal plane in the vertical motion, i.e. a horizontal nodal surface. In eq. (3) $\bar{u}_0$ represents then the surface geostrophic wind of the basic current system.

[1] For a barotropic current this relation, after some transformation, becomes identic with ROSSBY's formula (1942) for the ratio between the amplitude of isotherms (iso-lines of conserved property) and the amplitude of streamlines, $\frac{A_T}{A} = \frac{\bar{u}}{u^\star}$.

4. The elementary instability criterion and its validity in the middle latitude westerlies

An analysis of the equation (3) relating the phase velocity c to the intensity of the zonal circulation at the ground $\bar{u}_0$, the wavelength L (through k), the geographical latitude $\bar{\varphi}$ (through β) and the stability functions $S_\xi^\star$ and S_ξ (henceforth in application to dry-adiabatic processes denoted by simply $S^\star$ and S) shows briefly the following. A *necessary* condition for amplitude growth (existence of an imaginary component c_i) is that the static stability $S \geq 0$, higher static stability values involving less rapid, lower stability values involving more rapid amplitude growth. This dependency on static stability appears physically quite in order, since an existing rate of static stability must damp out those vertical components of the wave motion which occur everywhere above the nodal plane and are a consequence of the requirement of mass continuity. In the limiting case when the static stability vanishes, i.e. at so-called "indifferent vertical stratification" of the fluid, the solution eq. (3) breaks down except when simultaneously the meridional inequalities in the $\bar{\xi}$-field (potential temperature- or entropy field) vanish as well.[1] Because of this break-down at the singularity $S = 0$ ($S^\star \neq 0$), it cannot be expected that in the negative regime of S the relationship holds valid for one and the same phenomenon as it does in the positive regime. In the latter regime we have to take the first of the solutions of

[1] In this particular case we have homogeneity in $\bar{\xi}$ and eq. (3) reduces to the well known Rossby formula for planetary waves in an autobarotropic zonal current.

the differential equation, mentioned in § 3 which yields a relationship for very short waves only, when the relevant atmospheric values for $S^\star$ and S are being introduced. We shall not here discuss the properties of these waves. However, it is worth mentioning that in the normal case of static stability being present ($S > 0$), these short waves can be identified with the phenomenon of billowing clouds for which HAURWITZ (1941) has given an adequate theory. One has then to consider the limiting case when the baroclinity (vertical shear) is concentrated in an infinitely thin horizontal layer of air and the static stability in the adjoining infinitely deep strata is disregarded.

In the following it will be assumed that the static stability is positive. (It is, however, noteworthy that to a vertical lapse-rate of potential temperature or entropy which in meteorological practice is regarded as "very small", corresponds a value of S such as to yield, at prescribed L (k) and $S^\star$, still values for c_r or c_i of meteorologically reasonable dimension). With this specification the requirement for instability may be expressed by the inequality:

$$\frac{|S^\star|}{\sqrt{S}} > \frac{\beta}{2k} = \frac{\Omega \cos\bar{\varphi}}{2\pi}\,\frac{L}{a} \tag{4}$$

or, if it is desirable to express this condition through a critical (maximum) wavelength L_c, by the equation

$$L_c = \frac{2\pi a}{\Omega \cos\bar{\varphi}}\,\frac{|S^\star|}{\sqrt{S}} \tag{4 a}$$

or, finally, if the critical (minimum) number n_c of waves around the circumference is preferred, by the equation

$$n_c = \Omega \cos^2 \bar{\varphi}\,\frac{\sqrt{S}}{|S^\star|} \tag{4 b}$$

In these relationships Ω is the angular velocity of the earth rotation, a the earth's radius and $\bar{\varphi}$ the latitude which is representative for the zonal current under consideration.

It is immediately seen that when the isentropic surfaces are horizontal ($S^\star = 0$), even the shortest waves would be stable, whilst when the static stability is reduced to zero, even the longest waves would be unstable. To which extent these elementary instability criteria conform with empirical data, could be checked (though only very roughly), since $S^\star$ and S could be computed and mapped for discrete zonal belts over long periods of time. However, it is preferable to make first use (as is usually done) of the gradientwind equation which relates the dynamic stability function $S^\star$ to the surface (zonal) wind, the vertical shear of the (zonal) wind, the so-called thermal wind, and to the static stability. Equation (4 b) may then be written as

$$n_c = \frac{1 - \sin^2\bar{\varphi}}{2\sin\bar{\varphi}}\;\frac{\sqrt{S}\,/\,|\varDelta|}{\left|1 - \dfrac{\bar{u}_0\, S}{g\,|\varDelta|}\right|} \tag{5}$$

where the symbol $\varDelta$ denotes the vertical windshear. It is noteworthy that the non-dimensional quantity $\sqrt{S}\,/\,|\varDelta|$ is the square root of the Richardson number (see BRUNT, 1934) which plays a similar rôle in the theory of energy redistribution in turbulent currents in which shearing stresses are operating (RICHARDSON, 1920). According to various investigators, this number (R) is close to unity. The condition that turbulence should increase, i.e. the rate of supply of energy by the shearing stresses be at least as great as the work which has to be done to maintain the turbulence against gravity, is expressed by the inequality $R < 1$, or in analogy to eq. (5)

$$R_c^{\frac{1}{2}} \approx 1 \tag{6}$$

In order to develop further the analogy between these criteria for turbulent redistribution of energy — the breaking up of organized zonal currents into vortices has been since long regarded as a phenomenon of large scale turbulence — we shall for the time being neglect the term $\bar{u}_0\, S/g\,|\varDelta|$ in the denominator on the r.h.s. of eq. (5), as being small compared with unity. In fact this is permissible in all "normal" (statistical average) conditions in the belt of middle latitude westerlies. Important exceptions occur when normal or high static stability ($S \geq 1.3 \; 10^{-4} \; sec^{-2}$) is coupled with weak shear ($\Delta \leq 5 \cdot 10^{-4} \; sec^{-1}$) and/or normal to high zonal velocity at the ground ($\bar{u}_0 = 10$ to 50 m/sec). The above criterion (eq. 5) may be given the form:

$$R_c = \left(\frac{2 \, n \, \sin \bar{\varphi}}{1 - \sin^2 \bar{\varphi}}\right)^2 \qquad (5 \, a)$$

In the middle latitude belt of westerlies, between 45° and 65° latitude, one observes frequently 4 to 6 deep troughs in the streamline pattern which — for the sake of providing a point of argument — may be looked upon as the result of a redistribution of energy in the limited sense as earlier specified. With the values $n = 5$ and $\bar{\varphi} = 55°$ the quantity R is in the neighborhood of 600. In the diagram fig. 1 the relationship between R, n and $\bar{\varphi}$ is represented. In "normal" conditions in the belt of westerlies the vertical shear is of the order of magnitude 1 m/sec. per kilometer elevation and the static stability about $1.3 \, . \, 10^{-4}$ sec^{-2} (corresponding to a temperature lapse-rate 0.65°/100 m). With these values we find $R \approx 100$. On the other hand, it is seen from the diagram that the critical Richardson number is for all but the very longest waves considerably above the value 100 (north of 45°). Thus if the criterion, derived with the use of many simplifying assumptions, is expected to give some part of the truth, extremely long planetary waves only should, north of latitude 45°, exhibit a large degree of persistency with regard to shape. To some degree this persistency is confirmed by synoptic evidence but naturally other factors not having been considered here, must have a great and frequently overruling effect.

In the diagram fig. 2 the above criterion of instability is given as a relation between the circumferential wave number, the vertical shear and the latitude for a prescribed "normal" rate of static stability. For a number $n \geq 5$ ($L \leq 72°$ longitude) there is good agreement between this elementary criterion and that derived by CHARNEY (1947) who carried out a complete integration of the hypergeometric confluent differential equation which is obtained through taking into account the variation of the perturbation amplitude with height.[1] This agreement indicates that

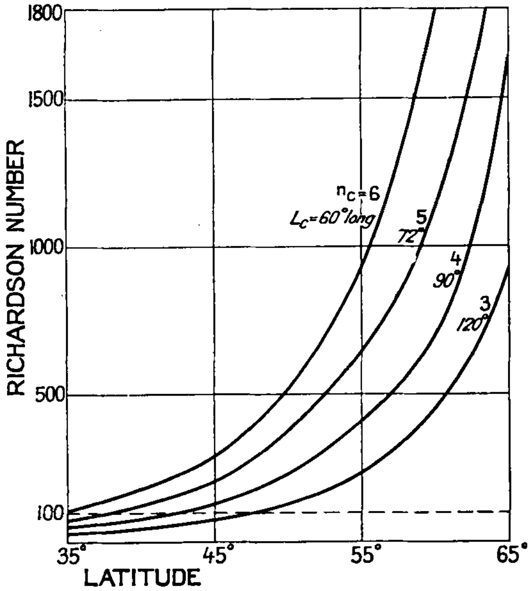

Fig. 1. Diagram illustrating the dependence of the critical wave-length L_c (or circumferential wave-number n_c) upon the value of the non-dimensional quantity $R = S/\Delta^2$ (RICHARDSON number) and the latitude $\bar{\varphi}$, through four selected isopleths for n_c.

one does not introduce a really essential factor in regarding the physical upper boundary conditions.

5. Remarks on zonal currents in tropical latitudes

In the derivation of (3) and of the ensuing instability criteria it was not essential that the zonal current at the nodal plane be directed from west to east and the vertical shear be positive. Therefore these relations are, at least in principle, applicable to easterly winds and/or negative shear. To the north and south of the equator there are extensive belts of low-tropospheric easterlies in which the temperature decreases but slightly northwards. In view of the low latitude this weak solenoidal field is associated with a strong positive vertical shear, so that the east-west circulation is superimposed by a west-east circulation. The static stability in this current system is on the whole rather low which

[1] CHARNEY's numerical criterion applies to a prescribed (normal) static stability and a representative latitude 45°,

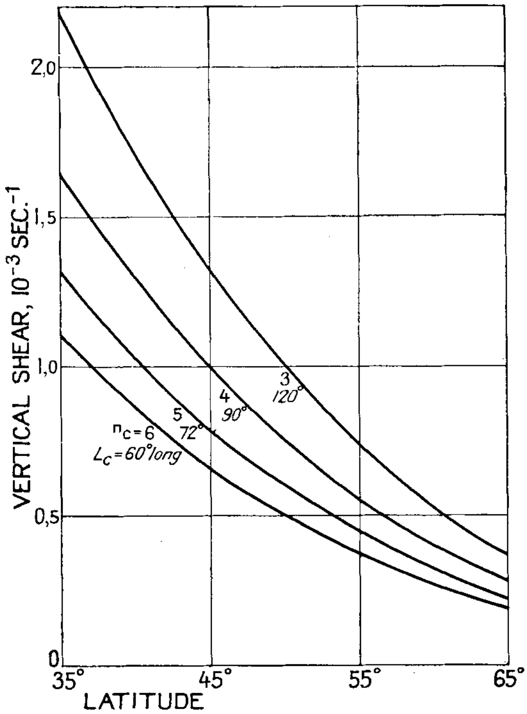

Fig. 2. Diagram showing the dependence of the critical wave-length L_c upon the vertical shear of the zonal wind Δ and the latitude $\bar{\varphi}$, for a prescribed, normal value of the static stability S.

circumstance together with the strong vertical shear permits to disregard the term $\frac{u_0 S}{g|\Delta|}$ on the r.h.s. of (5). We may thus regard the elementary criterion (6) to have at least the same degree of validity in the latitude belt 10°—35° as in the middle latitude belt. As is evident (see fig. 1) the critical Richardson number is in tropical latitudes much smaller than 100 (and would e.g. be reduced to unity at latitude 9° for a circumferential wave number 3). On the other hand the rates of static stability and vertical shear prevailing in "normal" conditions in the low latitude belts of surface easterlies (trade winds) are such that the corresponding value of R exceeds for even relatively short waves considerably the critical value, though it is, in such conditions, smaller than in middle latitude westerlies. Thus the kind of instability considered here is not likely to be realised for any reasonable perturbation dimensions. However, in extreme conditions the actual value of R may fall below the critical value and then instability is possible. The present theory may possibly be of some use for the understanding of the coupling between the storm systems forming in or near the trade wind zone and the long waves in the near-equatorial upper westerlies which has come to light in recent investigations (RIEHL 1948).

It should be emphasized that in applying the above criterion to quasi-zonal currents with an east-west motion at the ground it is essential that the reversal to opposite motion (westwind) should take place at rather short distance from the ground. In average conditions, in the regions just to the north and south of the equator, this condition is not at all satisfield. However, as e.g. the observations from the Finnish aerological expedition of the year 1939 show (see also VUORELA, 1948), the average flow pattern in the belt 5°—15° south during the period October—November 1939 is featured by a rather shallow layer with light easterlies components and by uniformily increasing westerly components in the troposphere above, the average vertical shear amounting to as much as 2.5 m/sec per km. The static stability had an average value of $1.0 \cdot 10^{-4}$. These values bring the number R down to a low value of 16. According to our criterion this value requires (at 9° latitude) a circumferential wave number 12, i.e. a wavelength of 30° longitude.

6. The significance of the absolute circulation

In order to test the validity of the basic equation (5) which had been derived for an extremely simplified atmospheric model and under many restrictive assumptions, it is of importance to find out whether computed velocities of propagation (of the perturbation) and time-rates of its growth bring out some of the characteristics of middle latitude disturbances. In the diagram fig. 3 the velocity deficit $\bar{u}_0 - c$ is represented as function of the surface zonal wind $\bar{u}_0$ for different rates of baroclinity (vertical shear Δ) at a prescribed latitude (45°) and wavelength (51° longitude), the static stability being assumed to have the normal value. The characteristic time-rate of growth, τ_i, taken as that interval during which the amplitude would increase to e times its initial value, is also given as function of the surface zonal wind for different shear values.

It is seen that for moderate or strong surfaces westerlies ($\bar{u}_0 > 0$), the disturbance moves eastwards with a speed that differs little from the surface velocity, when the shear is considerable, and this correspondence is the better the closer the surface wind is to its critical value (indicated by a circle) where instability would set in. This circumstance implies that when the westwind circulation is strong aloft (and not too weak at the ground) the "neutral" and unstable waves move approximately with the speed of the surface zonal current (a similar result has also been obtained by CHARNEY 1947). We find here a verification of a well known empirical rule frequently applied in forecasting: a disturbance, which does not move quickly "through" the upper air flow pattern (long wave), propagates eastward with approximately half the velocity of the westwind at ca. 5 km height. Now, the surface zonal (geostrophic) wind is in such cases indeed about one half of the middle tropospheric wind. As soon as the initial disturbance has assumed semi-mature dimensions, the results, derived from a theory of "small perturbations", are not applicable.

For weak surface westerlies or surface easterlies (decreasing upwards) the theory is not in agreement with the observations. As far as the sense and speed of propagation is concerned it breaks completely down. However, it brings out a perhaps not uncharacteristic other feature. It is seen that a given shear rate may be sub-critical (not involving instability) when the surface circulation is from west to east, but be super-critical when it is from east to west. For example at a shear 0.6 m/sec per kilometer elevation, a value of $\bar{u}_0 = +10$ m/sec would predict stability whilst with the same shear rate at a moderately strong surface east wind ($\bar{u}_0 = -10$ m/sec) we are well within the region of instability. The significant time-rate of growth, as defined above, is, in the latter case, 58 hours. (See fig. 3.)

The above results show how important for the breakdown of zonal motion the absolute circulation appears to be. At subpolar latitudes, the concurrence of surface easterlies with upper westerlies is a common feature of the general circulation. The horizontally uniform

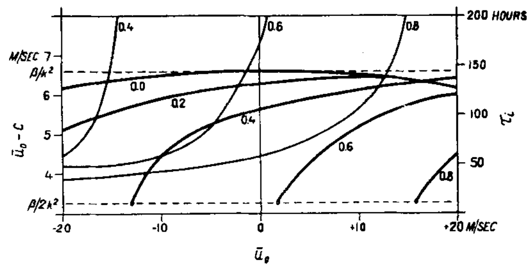

Fig. 3. Curves for the velocity deficit $\bar{u}_0 - c$ (heavy lines) and for the characteristic time of amplitude growth τ_i (thin lines), plotted as functions of the surface zonal wind $\bar{u}_0$, for different values of vertical shear Δ. The circles at the lower dashed line indicate that the selected shear value is critical. — These curves are evaluated for prescribed values of latitude $\bar{\varphi}$ ($= 45°$), of circumferential wave number $n (= 7)$, and of static stability S ($= 1.3 \cdot 10^{-4}$ sec^{-2}, normal value).

zonal current systems we have treated here, constitute an integral part of an anticyclonic cold polar vortex superimposed by a cyclonic vortex. By computing the kinetic and potential energy of similar current systems, ROSSBY (1949) has demonstrated that an even barotropic anticyclonic vortex centered at the pole is in a state of dynamic inequilibrium. Whether the (cold) vortex as a whole is displaced equatorwards or whether the transport of the cold air towards lower latitudes is realised by a breaking up of the zonal motion, seems to be dependent upon the dimensional properties as well as the shape of the cold dome centered at, or near, the pole.

In order to clarify the matter of the dependency of the instability on the intensity of the absolute circulation, we should add that this is, in the present analysis, merely a consequence of including in the (perturbation) vorticity equation the horizontal solenoid term. Now, the existence of p, Θ solenoids in the (meridional) *vertical* plane is absolutely necessary to bring about the phenomenon of instability considered here, because the vertical shear (necessarily asscociated with the vertical solenoids) controls the inequalities, in the vertical, of vorticity transport: to this extent this solenoid field controls the deformation of the flow pattern, in conjunction with the effect of the earth's rotation. On the other hand, the *horizontal* solenoid field appears immediately once the perturbation has set in, but it only *modifies* the nature of the deformation of the streamlines and is not a necessary condition for amplitude growth.

In conclusion a very brief summary of the results of a development of the above elementary criteria may be given. Through including into the simplified model of a polytropic atmosphere the effect of the vertical tilt of the pressure wave — assuming in conformity with empirical evidence the temperature wave to have no such tilt —, one can deal with certain types of advection (horizontal mass transport). These are associated with critical wave-lengths which are longer or shorter than those calculated on the basis of the elementary criterion. For just the type of advection which is most commonly observed during the initial stages of a deepening large-scale disturbance (see also NYBERG, 1949) the critical wave-length is shorter than that evaluated from the elementary criterion. The stability curve is then in still better agreement with the one derived by CHARNEY (1947). There is perfect agreement between these curves when the height of the polytropic atmosphere is replaced by the height of an equivalent polytropic atmosphere equal to the mean height of the tropopause in the middle latitude westerlies. — The inclusion of the above mentioned tilt permits finally to evaluate the phase lag, between the pressure wave and the temperature wave, for which at prescribed values of the Richardson number the growth-rate of the disturbance has a maximum. This phase lag corresponds to an extremely high rate of the relevant advection.

After completion of this investigation a paper by Dr. E. T. EADY has been published in this journal (Tellus, **1**, 3, p. 33), wherein the author assigns to the value of the Richardson number a fundamental rôle for the behaviour of cyclone waves and long waves.

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