A Wave Pattern Changing with Time as a Solution of the Vorticity Equation for Non-Divergent Motion

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Abstract

A solution of the vorticity equation is obtained for a wave pattern changing with time. It is shown that such a wave pattern is necessarily associated with ridge and trough lines which do not run in the south—north direction.

1. The long wave pattern of the streamlines in the upper troposphere of middle latitudes was investigated first by ROSSBY (1939). Assuming a purely horizontal motion of an inviscid homogeneous incompressible atmosphere, he obtained for a perturbed zonal current of constant velocity U the equation

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) + \beta\frac{\partial \psi}{\partial x} = 0 \quad (1)$$

where ψ is the stream function of the perturbation motion, β is the meridional rate of change of the Coriolis parameter, which is assumed to be constant. The x and y axes are taken in the east and north directions respectively. This equation has been discussed by ROSSBY (1939), and later by HAURWITZ (1940). In the wave models considered by them, the ridge and trough lines are oriented in the south—north direction. Recently MACHTA (1949) obtained a wave solution of equation (1), in which the ridge and trough lines are "tilted", i.e. not running along a meridian. In all these investigations, however, the models contemplated are such that the wave amplitude neither gains nor loses in size with time. In what follows, a more general solution of (1) is obtained which gives rise to a wave pattern having tilted ridge and trough lines and, moreover, undergoing development with time.

2. A solution of equation (1) is sought in the form:

$$\psi = e^{\sigma t}\left[\Phi_1(y)\cos\lambda(x-ct) + \Phi_2(y)\sin\lambda(x-ct)\right] \quad (2)$$

where $2\pi/\lambda$ is the wave-length, c the wave velocity and σ a parameter describing the development of the wave pattern with time. Such a form of solution would lead to a wave having an amplified (or damped) amplitude and such that the ridge and trough lines deviate from the S—N direction.

Substitution of (2) in (1), leads to a system of differential equations in Φ_1 and Φ_2, the general solution of which may be written in the form:

$$\Phi_1(y) = (A\cos\mu y + B\sin\mu y)\,e^{\alpha y} + (C\cos\mu y + D\sin\mu y)\,e^{-\alpha y} \quad (3)$$

$$\Phi_2(y) = (A\sin\mu y - B\cos\mu y)\,e^{\alpha y} + (D\cos\mu y - C\sin\mu y)\,e^{-\alpha y} \quad (4)$$

where A, B, C and D are arbitrary constants, with

$$\sigma = \frac{2\alpha\mu\lambda\beta}{(\lambda^2 + \mu^2 - \alpha^2)^2 + 4\alpha^2\mu^2} \quad (5)$$

and

$$U - c = \frac{\beta(\lambda^2 + \mu^2 - \alpha^2)}{(\lambda^2 + \mu^2 - \alpha^2)^2 + 4\alpha^2\mu^2} \quad (6)$$

It can be seen from (2), (3) and (4) that the perturbation motion is given by a linear combination of the functions

$$e^{\alpha y+\sigma t} \begin{matrix}\cos\\ \sin\end{matrix} \{ \lambda (x - ct) - \mu y \},$$

$$e^{-\alpha y+\sigma t} \begin{matrix}\cos\\ \sin\end{matrix} \{ \lambda (x - ct) + \mu y \}$$

and it is interesting to note that in the wave pattern, corresponding to any of these solutions, the ridge and trough lines are "tilted" straight lines whenever $\sigma \neq 0$, i.e. $\alpha\mu \neq 0$. This means that *for any solution involving a wave pattern changing with time, the ridge and trough lines are necessarily "tilted"*.

The models considered by ROSSBY (1939), HAURWITZ (1940) and MACHTA (1949), can be obtained from the general formulae (2)—(6) by choosing the values of the parameters and constants in the following manner.

ROSSBY: $\alpha = \mu = 0 \quad A + C = 0 \quad D - B = 0,$
HAURWITZ: $\alpha = 0 \quad \mu \neq 0 \quad A = C = 0 \quad B + D = 0,$
MACHTA: $\alpha = 0 \quad \mu \neq 0 \quad A = C = D = 0 \quad B \neq 0.$

The solution of MACHTA (1949) is in fact a linear combination of two solutions of the HAURWITZ (1940) type.

3. In this section the relation between the tilt of the ridge and trough lines and the amplification (or decay) of the waves is discussed. A ridge or a trough line will be called positively tilted when the positive angle it makes with the x-axis is less than a right angle, otherwise it is described as negatively tilted. Also the meridional flux of west momentum and kinetic energy will be examined. These are $\overline{\varrho u v}$ and $U\, \overline{\varrho u v}$ (VAN MIEGHEM 1949), respectively, where ϱ is the density, u and v are respectively the z and y-components of the perturbation velocity, and the bar to denote the average value. In the case under consideration ϱ is constant, and the average value is to be taken with respect to a wave length.

The streamlines for the total motion, obtained by the superimposition of the perturbation and the mean motion are given by $\psi_1 =$ constant, where

$$\psi_1 = - U y + \psi \tag{7}$$

ψ being given by (2), (3) and (4). To have an idea of the orientation of the ridge and trough lines, it will be sufficient to find the loci of the points where $v = 0$, i.e. where $\frac{\partial \psi_1}{\partial x} = 0$. Differentiating the expression so obtained we find that the slope, dy/dx, of a ridge or a trough line is given by:

$$\lambda \frac{dx}{dy} = \frac{\Phi_2' \Phi_1 - \Phi_1' \Phi_2}{\Phi_1^2} \cos^2 \lambda (x - ct) \tag{8}$$

We note also that

$$\overline{u v} = \frac{\lambda}{2} e^{\sigma t} [\Phi_2' \Phi_1 - \Phi_1' \Phi_2] \tag{9}$$

which shows that $\overline{uv}$ is of the same sign as dy/dx.

It is easy to verify, from (3), (4) and (8), that if $A = B = 0$, or $C = D = 0$, the ridge and trough lines are straight. It follows that, in this case, $\overline{uv}$ will be of constant sign; and it can be seen that west momentum and kinetic energy are transported northward for positively, southward for negatively, tilted (straight) ridge and trough lines; a result pointed out by MACHTA (1949) when $\sigma = 0$. The amplification, or decay, of the waves depends, by (5), on the sign of $(\alpha\mu)$. It thus becomes evident that with straight ridge and trough lines an amplified wave pattern is associated with northward increase of amplitude if it is positively tilted, and a southward increase if it is negatively tilted.

For a decaying wave, on the other hand, the amplitude increases to the south in the case of a positive tilt, and to the north in case of a negative tilt. The meridional flux of kinetic energy and west momentum is towards the region of increasing or diminishing amplitude, according as the wave amplitude is gaining or losing in size with time.

We examine now the more general case $A^2 + B^2 \neq 0$ and $C^2 + D^2 \neq 0$. By (8), (3) and (4), we find after some reduction that

$$\lambda \frac{dx}{dy} = \frac{\mu \sinh (2 \alpha y + 2 k) - \alpha \sin (2 \mu y + \delta)}{\cosh (2 \alpha y + 2 k) - \cos (2 \mu y + \delta)} \tag{10}$$

where $\qquad exp\,(2k) = \sqrt{\dfrac{A^2 + B^2}{C^2 + D^2}}$

and $\qquad \tan \delta = \dfrac{BD - AC}{CB + AD}.$

The denominator of the fraction on the right hand side of (10) is always positive and when it vanishes, it can be seen that the limit of the fraction is zero Thus the sign of dx/dy which indicates the orientation of a ridge or a trough line at any point, is determined by the sign of the numerator. Two cases are considered:

(i) Amplified Waves $\sigma > 0$, ($\alpha\mu > 0$).

It is seen from (10) that when y is sufficiently large and positive, then dx/dy is positive, and when it is sufficiently large and negative dx/dy is negative. This means that sufficiently far to the north the ridge and trough lines are positively tilted, while sufficiently far to the south they are negatively tilted. Since the sign of $\overline{uv}$ is the same as that of dx/dy, it follows that $\overline{uv}$ is positive far enough to the north and negative far enough to the south; and hence there is an outflow of west momentum and kinetic energy from the central part of the stream for waves which are gaining in size with time.

(ii) Damped Waves $\sigma < 0$, ($\alpha\mu < 0$).

Similarly as in (i), it can be seen that the ridge and through lines are negatively oriented in the northern part of the stream, and positively oriented in the southern part. Thus $\overline{uv}$ is negative far enough to the north and positive far enough to the south, corresponding to an inflow of kinetic energy and west momentum in the central part of the flow.

It thus appears that *a wave pattern changing with time is always associated with a tilt of the ridge and trough lines*. Furthermore, when the configuration of these lines is as described in (i) or (ii) above, amplification or decay of the perturbation motion takes place respectively, accompanied by an outflow or inflow of west momentum and kinetic energy in the central part of the stream. This last remark is in agreement with a result obtained by VAN MIEGHEM (1949). He in fact has shown, from momentum and energy considerations, that with the ridges and troughs oriented SE—NW to the north of the zonal current, and SW—NE to the south of the stream, there is accumulation of west momentum and kinetic energy which would increase the strength of the zonal westerly current. From the above, it is seen that this corresponds to the case of a damped perturbation, which agrees with a result obtained by KUO (1949).

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