On the Dispersion of Planetary Waves in a Barotropic Atmosphere[1]

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Abstract

In this paper an attempt is made to investigate the rate of dispersion of solitary, quasi-horizontal atmospheric disturbances with the aid of a simple barotropic model of the atmosphere. In low latitudes the dispersion is found to be extremely rapid, in good agreement with the observed absence of all but the smallest solitary disturbances. In high latitudes the dispersion is slow, permitting the establishment of persistent solitary circulation patterns. The theory appears so throw some light on the geographic distribution of the interdiurnal variability in the height of the 500 mb surface and on the relatively frequent occurrence of planetary waves with a wave length of 120 longitude degrees. A fuller treatment is now being published by my associate, Dr. T. C. YEH.

In the last year or two a good deal of attention has been given to the problem of formulating the basic equations of atmospheric hydrodynamics in such a manner that they may be integrated numerically with the aid of high-speed computing equipment. The chance of success in such an undertaking depends upon one's ability to introduce appropriate simplifications in the equations of motion, not merely for the sake of reducing the number of operations involved in the integration, but for the equally important purpose of eliminating what J. CHARNEY [2][2] has referred to as undesirable meteorological noise. To achieve this goal various simplified dynamic models have been proposed, among them a barotropic atmosphere in which the initial state of motion would be so chosen as to agree with the observed conditions in the real atmosphere at, say, the 500 mb level. It is evident that this model would be incapable of describing the processes whereby kinetic energy is generated but there are good reasons to believe that it might provide a fairly satisfactory description, for short periods of time, of the redistribution of motion which results from the special dynamic characteristics of the rotating framework in which the atmospheric shell circulates.

A basic problem raised by these attempts at numerical integration concerns itself with the maximum speed at which the "influence" of a given source point is propagated and dispersed into the environment, since this speed obviously determines the size of the area over which initial conditions must be known if numerical predictions are to be made for a prescribed locality. In a previous study [3] of barotropic wave motions the author made a brief reference to the principal factor which appears to limit this speed but no numerical data were given. Since that time it has become increasingly apparent that a better knowledge of the rate of dispersion in the atmosphere may be interesting and useful to meteorologists generally, for instance in the interpretation and analysis of the persistence

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[1] The term planetary wave is here applied to those quasi-horizontal atmospheric wave motions whose shape, wave-length and displacements are controlled by the variation of the Coriolis parameter with latitude. This name agrees well with the term "Planetarische Wirbelwirkung" introduced by V. W. EKMAN [1] to describe the effect of the variable Coriolis parameter upon the configuration of the permanent currents in the ocean.

[2] References to literature cited are given in brackets and listed at the end of the article.

of atmospheric disturbances. Of particular interest at the present moment is the fact that "long-wave" dispersion appears to be one of the more important factors determining the geographic distribution of interdiurnal variability of pressure referred to by NYBERG [4] in his contribution to the present issue of Tellus. For these reasons, a few general comments and numerical data on dispersion in a barotropic atmosphere will be presented below. A comprehensive theoretical study of the subject is now being published by my associate, Dr T. C. YEH [5].

In a barotropic, incompressible atmosphere the motion may be described with the aid of the vorticity equation

$$\frac{d}{dt}\left[\frac{f+\zeta}{D}\right] = 0, \qquad f = 2\Omega \sin\varphi, \qquad (1)$$

in which the operator $\frac{d}{dt}$ represents the individual rate of change with time. ζ is the vertical component of the vorticity of the air motion relative to the earth, f is the Coriolis parameter, D is the depth of the atmosphere, Ω is the angular velocity of the earth and φ the latitude. In such an atmosphere the individual variability of ζ (or f) is likely to be much greater than that of D and hence it would seem permissible to replace (1) with the somewhat simpler equation

$$\frac{d}{dt}(f+\zeta) = 0. \qquad (2)$$

In the study of harmonic waves of the dimensions normally encountered in the atmosphere the two equations (1) and (2) give nearly identical results, but this agreement disappears as soon as one enquires into the maximum rate of dispersion for each system. The second of these is based on the assumption of purely horizontal motion. The corresponding atmospheric model consists of a fluid sheet completely filling the narrow space between two parallel horizontal walls. The pressure is no longer hydrostatic; it is well known that in such a fluid system there exists no other upper limit to the speed with which impulses are propagated than the velocity of sound, which, for incompressible media, becomes infinitely large.

In the real atmosphere the pressure is nearly always hydrostatic and local pressure changes must be brought about through addition or removal of mass, i.e. through processes which take a finite time for their completion. Under those circumstances it is not surprising that a barotropic atmosphere in hydrostatic equilibrium and with a free surface, such as the one described by (1), possesses an upper limit to the speed with which it can transmit impulses.

Among the many types of motion described by (1) are also certain long gravity waves which are of little interest meteorologically. It has been shown by CHARNEY (l. c.) that these may be eliminated from (1) by substituting for ζ the vorticity of the geostrophic wind. If this is done, (1) reduces to a differential equation for D which is of the first order and linear, with respect to the term containing the time derivative. Very little progress has been made with the study of two-dimensional solutions to this equation. However, it is reasonable to assume that even such simple one-dimensional solutions as plane waves of small amplitudes might shed some light on the dispersion problem. We shall, therefore, consider the special case of a constant westerly wind current U, upon which are superimposed plane waves travelling in the x-direction (eastward). With those assumptions (1) reduces to the form

$$\frac{\partial^3 D}{\partial x^2 \partial t} + U\frac{\partial^3 D}{\partial x^3} + \beta\frac{\partial D}{\partial x} - q^2\frac{\partial D}{\partial t} = 0, \qquad (3)$$

where the constant q has the dimension of a wave number and is given by

$$q = \frac{1}{\lambda} = \frac{f}{\sqrt{gD_0}}, \qquad (4)$$

g being the acceleration of gravity, and D_0 the mean depth of the homogeneous atmosphere. The numerical value of $\sqrt{gD_0}$ is about 280 m.p.s. In (3) β represents the rate of change of the Coriolis parameter northward and has the value

$$\beta = \frac{2\Omega \cos\varphi}{a}, \qquad (5)$$

a being the radius of the earth.

The frequency equation corresponding to (3) is of the form

$$c = \frac{\nu}{k} = \frac{Uk^2 - \beta}{k^2 + q^2}, \qquad (6)$$

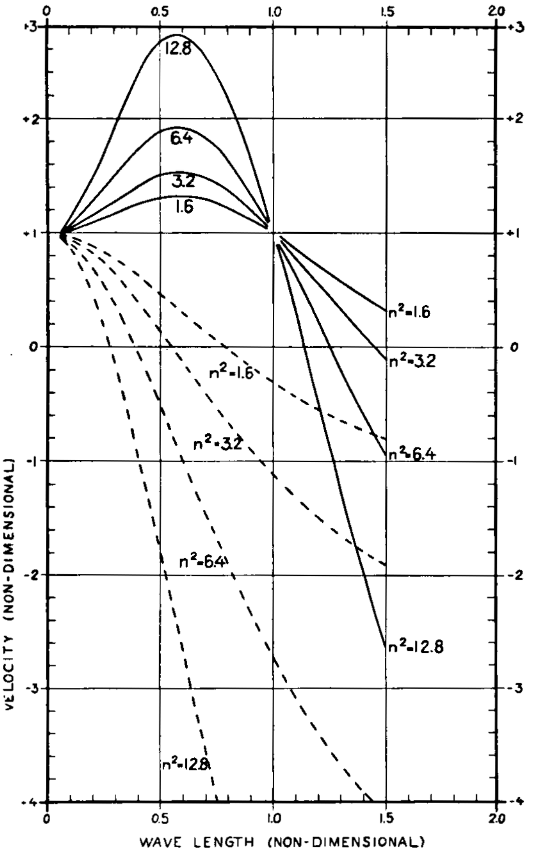

Fig. 1. Curves for the phase velocity c and the group velocity c_g, expressed as fractions of the zonal wind U, and plotted as function of the non-dimensional wave length $\frac{L}{2\pi\lambda}$, for different values of the parameter $n^2 = \frac{\beta}{q^2 U}$. The phase velocity is represented by dashed lines, the group velocity by full lines.

c, ν and k representing phase velocity, frequency and wave number respectively. In Fig. 1 the dotted lines represent $\frac{c}{U}$ as functions of $\frac{q}{k}$, or $\frac{L}{2\pi\lambda}$, L being the wave length, for a few values of the non-dimensional parameter $\frac{\beta}{q^2 U}$.

The same figure contains the group velocity

$$c_g = \frac{d\nu}{dk} = \frac{Uk^4 + (3Uq^2 + \beta)k^2 - \beta q^2}{k^2 + q^2)^2} \tag{7}$$

in the form of curves for $\frac{c_g}{U}$ as functions of $\frac{q}{k}$ for selected values of $\frac{\beta}{q^2 U}$ (full lines). A comparison between (6) and (7) shows that the group velocity always is in excess of the corresponding phase velocity. For very large values of L ($k \to 0$), both c and c_g approach the limiting value

$$c = c_g = -\frac{\beta}{q^2} = -\beta\lambda^2. \tag{8}$$

The group velocity given by (7) reaches its maximum value when

$$k = k_m = q\sqrt{3}, \; L = L_m = \frac{2\pi\lambda}{\sqrt{3}}. \tag{9}$$

Then

$$c_g = \frac{9U}{8} + \frac{\beta}{8q^2} = \frac{9U}{8} + \frac{\beta\lambda^2}{8} \tag{10}$$

and the corresponding value of c is given by

$$c = \frac{3U}{4} - \frac{\beta}{4q^2} = \frac{3}{4}\left(U - \frac{\beta}{k_m{}^2}\right). \tag{11}$$

It should be noticed that for all values of L which are less than $2\pi\lambda$, or for all values of c_g which are greater than U, each group velocity is associated with two separate wave lengths or phase velocities.

Fig. 2 represents $\frac{\beta}{q^2 U}$ as a function of the zonal wind U and the latitude. The rate of dispersion in the atmosphere is measured by

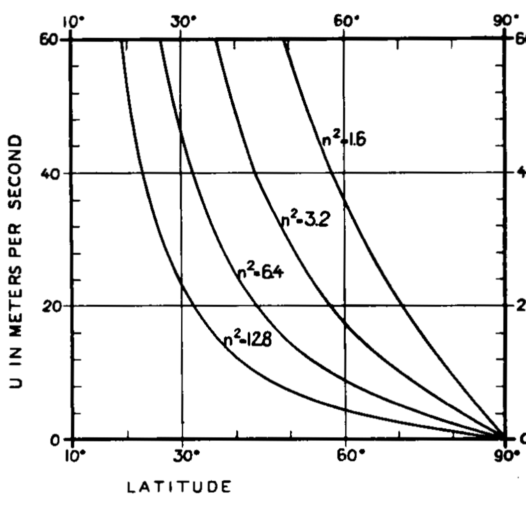

Fig. 2. Diagram illustrating the dependence of $n^2 = \frac{\beta}{q^2 U}$ upon the strength of the basic current U and the latitude φ, through four selected isopleths for n^2.

this quantity; the diagram shows quite clearly the rapid decrease of dispersion with increasing latitude.

Now assume that the equilibrium of this barotropic atmosphere is disturbed, at the time $t = 0$, by a line source perturbation applied at the longitude $x = 0$. It was shown in [3] that the group velocity distribution in the x,t-plane corresponding to such a perturbation is given by

$$c_g = \frac{x}{t}. \qquad (12)$$

It follows that the effect of the line source will spread eastward and westward, the maximum rate of propagation eastward being given by

$$x = \left(\frac{9U}{8} + \frac{\beta}{8q^2}\right) t \qquad (13)$$

and the maximum rate of propagation westward (retrogression) by

$$x = -\frac{\beta}{q^2} t. \qquad (14)$$

It likewise follows, from the character of the group velocity curves in Fig. 1, that double wave trains must appear in the region for which

$$U < \frac{x}{t} < \frac{9U}{8} + \frac{\beta}{8q^2}. \qquad (15)$$

Disregarding the advective effect of the zonal current one may say that the dispersion is confined to the area

$$-\frac{\beta}{q^2} < \frac{x}{t} < \frac{\beta}{8q^2}. \qquad (16)$$

The rapid propagation westward which is indicated by the lower limit in (16) is effectively reduced by the fact that L cannot exceed the circumference of the earth[1]. Thus

$$\frac{q}{k} = \frac{L}{2\pi\lambda} \leqslant \frac{a\Omega \sin 2\varphi}{\sqrt{gD_0}} \leqslant \frac{a\Omega}{\sqrt{gD_0}} = 1.6 \qquad (17)$$

With the aid of (16) it is possible to express the maximum rate of propagation eastward in any particular latitude as a fraction of the earth's linear speed at that latitude. One finds then

$$\frac{c_{g\,\max}}{a\Omega \cos \varphi} = \frac{gD_0}{16\, a^2 \Omega^2 \sin^2 \varphi}. \qquad (18)$$

The corresponding wave length is given by (9), or

$$L = \frac{\pi}{\sqrt{3}} \frac{\sqrt{gD_0}}{\Omega \sin \varphi}. \qquad (19)$$

Expressed as a fraction of the circumference of the earth in that latitude this becomes

$$\frac{L}{2\pi a \cos \varphi} = \frac{1}{\sqrt{3}} \frac{\sqrt{gD_0}}{a\Omega} \frac{1}{\sin 2\varphi} = \frac{0.35}{\sin 2\varphi}. \qquad (20)$$

To the same extent as the barotropic model underlying these calculations may be used to represent the behaviour of large scale motions in the atmosphere the following conclusions appear permissible:

(A) Dispersion increases rapidly with increasing dimensions of the systems considered, up to a certain maximum wave length. Hence solitary perturbations of small dimensions are much more likely to show persistence than those of larger dimensions.

(B) In equatorial latitudes dispersion is so rapid that no solitary disturbances except those of extremely small dimensions can survive for long periods of time.

(C) North of latitude 60° even large-scale solitary disturbances are capable of showing considerable persistence, for time intervals of perhaps as much as a month.

(D) A solitary source of perturbations at a prescribed longitude is likely to set off "resonance" effects in other longitudes farther downstream, with a spacing which corresponds to the wave length of maximum group velocity. For the zone of maximum activity in the northern hemisphere (between 35° N and 55° N) this wave-length is, according to (20), approximately $^1/_3$ of the circumference of the earth. NYBERG finds three centers of maximum variability in the free atmosphere (south of Berings Strait, Labrador and European Russia, with intensities decreasing in this order). It is not unreasonable to suggest that the last of these centers may represent a "re-

[1] The author is indebted to Drs. J. CHARNEY and G. PLATZMAN for this comment.

sonance" effect of the other two, whose location would be determined primarily by geographic factors.

(E) The maximum rate of propagation eastward (relative to the basic current) in the zone of maximum activity (lat. 45° N) may be computed from (13) and has a numerical value of about 16 m. p. s.

It should perhaps be stated explicitly that the barotropic dispersion waves discussed above would be of minor significance were it not for the fact that they, at least occasionally, appear to be able to release some of the potential energy available in the barocline mass distribution, as described by CRESSMAN [6]. They thus seem to perform the role of "primäre Druckstörungen" in the sense used by VON FICKER in his theory of the structure of atmospheric disturbances [7]. It is finally appropriate to mention that synoptic evidence for the dispersive processes analyzed above has been brought out, independent of theory, by various investigators, for instance EVJEN [8] and especially NAMIAS [9].

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