Magnetodynamic turbulence at low magnetic Reynolds number

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ABSTRACT

Golitsyn's equations for the study of magnetodynamic turbulence, and in particular ionospheric turbulence, at low magnetic Reynolds number are discussed. It is shown that the hypothesis of small magnetic Reynolds number is not sufficient to justify these equations and appropriate additional conditions are indicated. The simple case of weak turbulence in the presence of a uniform ambient magnetic field is examined in illustration. The law of energy decay in the final period is obtained by the method of steepest descent—first, from the solutions of Golitsyn's equations, and secondly from the more general equations established by Lehnert. With Lehnert's equations, one finds that the magnetic and the kinetic energies are composed of two contributions, one which decays at $t^{-\frac{5}{2}}$ as in non-magnetic turbulence and corresponds to equipartition of energy between velocity and magnetic fields, and another one which decays as t^{-3} and leads to the partition of energy between the two fields in a proportion which depends on the ratio n of the kinetic and magnetic viscosities. With Golitsyn's equations, one obtains only a t^{-3} component. When all the conditions of validity of these equations are satisfied—and not only the condition on Rm—one deduces that the t^{-3} component dominates—although it decays faster—until an inaccessibly large time and the t^{-3} contributions derived from Lehnert's and Golitsyn's equations are found identical. On the contrary, when certain of these conditions are not verified—even if Rm$<$1—the t^{-3} component is initially comparable to the $t^{-\frac{5}{2}}$ component and—as it decays faster—becomes rapidly negligible while energy tends to equipartition. Golitsyn's equations lead then to incorrect results.

Introduction

Various observational data, distortion of meteor trails, scattering or random fading of radio waves, spread F and radio star scintillations as well as observations with rockets and satellites strongly suggest the existence of magnetodynamic turbulence in the ionosphere (Matuura & Nagata, 1962). The study of ionospheric turbulence is made very complicated by the complexity of the medium and a theoretical study is often impossible without more or less justified simplifying hypotheses. One of these hypotheses, propounded by Golitsyn (1960) and presumably valid in the lower regions, is that of low magnetic Reynolds number. With this proviso, Golitsyn derived a system of magnetodynamic equations for the study of homogeneous incompressible[1] turbulence and calculated the fluctuations of the magnetic field induced by the presence of a turbulent velocity field which he further assumed unaffected by the mean magnetic field.

Later, Golitsyn's equations were broadly used in geophysical applications of low magnetic Reynolds number turbulence.

In this paper, Golitsyn's arguments are discussed and it is shown that the hypothesis of small magnetic Reynolds number is not sufficient to justify Golitsyn's equations.

The simple case of weak magnetodynamic turbulence is examined in illustration. The last stages of decay of magnetodynamic turbulence have been treated numerically by Deissler (1963) for values of the ratio $n = \nu/\lambda$ of the kinematic and magnetic viscosities equal to $\frac{1}{2}$, 2 and 10^{-7} respectively and he found that, asymptotically,[2] the energy in the kinetic and

[1] The hypothesis of incompressibility is justified if the velocity of sound is much larger than characteristic turbulent or alfven velocities.

[2] Asymptotic here refers to times sufficiently larger than a time θ—defined previously as the "inhibition time" (Nihoul, 1963)—characteristic of the process of draining of energy from the field of smallest diffusivity by the other field through the coupling due to the uniform ambient magnetic field (θ is usually much smaller than the lifetime of the turbulence).

magnetic modes tended to equipartition, for $n = \frac{1}{2}$ and 2.

In the following, asymptotic solutions of Golitsyn's equations are shown to be in the two cases described above in discordance with the results obtained by Deissler. The discrepancy, it is recognized, must be attributed to the fact that in these cases, the conditions of validity of Golitsyn's equations—apart from that of small magnetic Reynolds number—are not satisfied. An asymptotic analysis using more general equations established by Lehnert (1955) for weak magnetoturbulence exhibits two contributions to the magnetic and kinetic energies. One decays asymptotically as $t^{-\frac{5}{2}}$ and leads to equipartition between magnetic and kinetic modes. The other one decays as t^{-3} and leads to the partition of energy between the two modes in a proportion which depends on the ratio of their diffusivities. This contribution is the only one given by Golitsyn's equations. It is shown in this paper that when all the conditions of validity of these equations are satisfied, the second contribution is dominant—although it decays faster—during most of the life-time of the turbulence and that the first contribution does not become comparable—and the law of decay be $t^{-\frac{5}{2}}$ corresponding to equipartition—before an inaccessibly large time. On the contrary, when certain conditions of validity are not satisfied, even if the magnetic Reynolds number is small, the two contributions are initially comparable and the second one is rapidly negligible. The turbulent energy decays then as $t^{-\frac{5}{2}}$ and energy divides equally between the turbulent magnetic and velocity fields.

Conditions of validity of Golitsyn's equations

The fundamental equations of incompressible magnetohydrodynamics may be written (e.g. Sutton & Sherman, 1965).

$$\partial \mathbf{v}/\partial t + \mathbf{v}\cdot\nabla\mathbf{v} + 1/\varrho\nabla p^{\pi} = \mathbf{h}\cdot\nabla\mathbf{h} + \nu\nabla^2\mathbf{v}, \quad (1)$$

$$\partial \mathbf{h}/\partial t + \mathbf{v}\cdot\nabla\mathbf{h} = \mathbf{h}\cdot\nabla\mathbf{v} + \lambda\nabla^2\mathbf{h}, \quad (2)$$

with

$$p^{\pi} = p + \tfrac{1}{2}\varrho\mathbf{h}\cdot\mathbf{h}; \quad \lambda = (\mu\sigma)^{-1}, \quad (3)\ (4)$$

$$\nabla\cdot\mathbf{h} = \nabla\cdot\mathbf{v} = 0. \quad (5)\ (6)$$

p is the pressure, $\mathbf{v}$ the velocity, $\mathbf{h}$ the local Alfvén velocity, ν and λ are respectively the kinematic and the magnetic viscosities (μ is the permeability, σ the conductivity, ϱ the density). If in addition "reduced" electric field and current are introduced by:

$$\mathbf{e} = (\mu\varrho)^{-\frac{1}{2}}\mathbf{E}, \quad \mathbf{j} = (\mu/\varrho)^{\frac{1}{2}}\mathbf{I}, \quad (7)\ (8)$$

one has

$$\mathbf{j} = \nabla\times\mathbf{h} = \lambda^{-1}\ (\mathbf{e} + \mathbf{v}\times\mathbf{h}), \quad (9)\ (10)$$

$$\nabla\times\mathbf{e} = -\frac{\partial\mathbf{h}}{\partial t}; \quad \nabla\cdot\mathbf{e} = -\nabla\cdot(\mathbf{v}\times\mathbf{h}). \quad (11)\ (12)$$

Once $\mathbf{v}$ and $\mathbf{h}$ are determined by (1), (2), (5) and (6), $\mathbf{j}$ is given by (9) and $\mathbf{e}$ by (11) and (12). [Note that (6) is always satisfied if it is at $t = 0$ as a consequence of (11).]

The magnetic field is assumed to be the sum of an induced field and a uniform ambient field, i.e. $\mathbf{h} = \mathbf{b}_0 + \mathbf{b}$.

The essence of Golitsyn's argument is the following: Equation (2) may be regarded as equivalent to the equation of heat conduction in a solid body with a diffusivity λ and a source distribution $\mathbf{h}\cdot\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{h}$. The accomodation time of the field $\mathbf{h}$ to the source distribution can be estimated from the diffusivity λ and the characteristic length L of the problem as being of order $\lambda^{-1}L^2$. Let τ denote the characteristic time of modification of the turbulent field. Whenever $\tau \gg \lambda^{-1}L^2$ we expect the magnetic field to adapt itself within a short time to the turbulence and develop further with the time scale of the "source". Let

$$\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2,$$

where $$\nabla\times\mathbf{e}_1 = 0; \quad \nabla\cdot\mathbf{e}_2 = 0.$$

According to equation (11), we have then

$$e_2 \sim 0\left(\frac{bL}{\tau}\right).$$

According to equation (12)

$$e_1 \sim 0(vb_0),$$

or $$e_1 \sim 0(vb),$$

depending upon which one is the largest.

Going back to equation (10), we estimate the orders of magnitude of the various terms

$$\underbrace{\lambda\nabla\times\mathbf{h}}_{\lambda b/L} = \underbrace{\mathbf{e}_1}_{vb_0,\ vb} + \underbrace{\mathbf{e}_2}_{bL/\tau} + \underbrace{\mathbf{v}\times\mathbf{b}_0}_{vb_0} + \underbrace{\mathbf{v}\times\mathbf{b}}_{vb} \quad (13)$$

The condition $\tau \gg \lambda^{-1}L^2$ entails that $\mathbf{e}_2$ is negligible as compared to $\lambda\nabla\times\mathbf{h}$. The balance of the remaining terms requires then

$$b \sim 0\left(\frac{vL}{\lambda}\,b_0\right) \sim 0\,(R_m b_0).$$

Hence, under the two conditions

$$L^2 < \lambda\tau, \qquad (14)$$

$$R_m < 1, \qquad (15)$$

we may neglect $\mathbf{e}_2$ (the electric field is then a gradient to a very good approximation) and the induced field $\mathbf{b}$ as compared with the applied one. Equation (10) may be written:

$$\lambda\nabla\times\mathbf{h} = \mathbf{e}_1 + \mathbf{v}\times\mathbf{b}_0. \qquad (16)$$

Taking now the curl, we obtain equation (2) under the simple form

$$\mathbf{b}_0\cdot\nabla\mathbf{v} + \lambda\nabla^2\mathbf{b} = 0. \qquad (17)$$

This equation is the same as if the system were stationary. This is not surprising. Since the time scale of turbulence is much larger than the accomodation time of the $\mathbf{b}$ field, the turbulence acts as a "stationary source" on this field. This argument was promoted by Golitsyn (1960). Golitsyn further assumed that the characteristic time of modification of the turbulent field τ is $0(v^{-1}L)$.

In this case, the condition (14) is identical to (15), and Golitsyn's hypothesis reads: whenever $R_m < 1$, one may neglect the induced magnetic field as compared with the applied one in equations (1) and (2) and the time derivative of $\mathbf{b}$ in equation (2). Now the assumption that $\tau \sim 0(v^{-1}L)$ is not generally true.

It presupposes first, that the turbulence is sufficiently evolved for its time scale of modification to be set by the non-linear transfer and not by the viscous dissipation; secondly, that the characteristic time of magnetic intervention in the turbulence —which by comparison of the first terms in the left- and right-hand sides of (1) may be estimated of order $b_0^{-1}b^{-1}vL$—is also much larger than the time of adaptation of the $\mathbf{b}$ *field.* Clearly, indeed, if the coupling between velocity and magnetic fields works fast enough to impose the rate of resistive diffusion on turbulence, the characteristic time of change of the turbulent "source" and the time of magnetic adaptation will be of the same order and the time derivative of $\mathbf{b}$ will have to be retained. This entails the necessity of an additional condition for Golitsyn's equations to be valid,

$$N_m = \frac{b_0 bL}{\lambda v} \sim \frac{b_0^2 L^2}{\lambda^2} \sim \frac{b_0^2}{v^2} R_m^2 < 1. \qquad (18)$$

If the magnetic energy is not prohibitively larger than the kinetic energy, (18) follows from the condition that the magnetic Reynolds number is small. Hence Golitsyn's equations are valid with the provisoes that the turbulence is sufficiently evolved and that the applied field is not too strong. An opposite situation where this second condition doesn't hold and on the contrary an important imposed magnetic field ensures a dominant coupling has recently been studied by Ohji (1964).

We wish to investigate here the case when the first condition is not fulfilled, i.e. the problem of weak turbulence in the final period of decay. In the case of weak turbulence, the conditions (14) and (15) yield:

$$n = \frac{\nu}{\lambda} < 1, \qquad (19)$$

$$R_m = \frac{vL}{\lambda} < 1. \qquad (20)$$

In addition, there is the condition that the time of magnetic intervention in the turbulent velocity field be large as compared with the time of adaptation of the magnetic field to the turbulence. To express this condition in the final period, it is necessary to be a little more specific than previously. Indeed the applied magnetic field has a strong anisotropic influence on the turbulence. This influence is impeded in developed turbulence by the non-linear cascade and redistribution of energy between different eddy sizes and orientations, but it probably prevails in the last stages of decay when the non-linear redistribution has considerably weakened. It is thus advisable now to distinguish between general variations of the $\mathbf{b}$ field and variations along $\mathbf{b}_0$ alone. Let $\varkappa$ be a characteristic wave number such that $\varkappa^{-1} = L$ and let $\mu\varkappa$ be its component along $\mathbf{b}_0$. The characteristic time of adaptation of the magnetic field to the turbulence is $(\lambda\varkappa^2)^{-1}$ while now the characteristic time of magnetic intervention in the turbulent velocity field is $\sim b_0^{-1} b^{-1}(\mu\varkappa)^{-1} v$. Hence (18) must be replaced by:

$$\frac{b_0 b \mu \varkappa}{\lambda \varkappa^2 v} = \frac{b_0 b L}{\lambda v} \mu < 1. \qquad (21)$$

In the two cases $n = \frac{1}{2}$ and $n = 2$, considered by Deissler the condition (19) is not realized and we may forecast that the results obtained with Golitsyn's equations will not check against the numerical calculations of Deissler. In the third case $n = 10^{-7}$ discussed by Deissler, the first condition is satisfied. However, in decaying turbulence, there is an increase of scale together with a decrease of the kinetic energy. This second effect being very likely more important, R_m remains most certainly smaller than 1 if it is initially. The case is different for N_m where b_0 is a constant. At some ultimate stage the condition $N_m < 1$ is necessarily reversed and Golitsyn's equations cease to be valid. However a detailed analysis with Lehnert's equations will show that if n is sufficiently smaller than 1 (condition 19) the time necessary to observe significant changes due to the abandon of the condition on N_m falls largely beyond all experimental limits. This is due, as it will be seen, to the fact that, when $n < 1$, the dominant contribution to the energy comes from wave number vectors perpendicular to the applied magnetic field ($\mu \sim 0$) and for these wave number vectors, the condition (21) is always satisfied.

The final period of decay

The equations for the final period of decay of incompressible magnetoturbulence in the presence of a uniform ambient magnetic field were established by Lehnert. They are, in addition to (15) and (6):

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\varrho} \nabla p^x = \mathbf{b}_0 \cdot \nabla \mathbf{b} + \nu \nabla^2 \mathbf{b}, \qquad (22)$$

$$\frac{\partial \mathbf{b}}{\partial t} = \mathbf{b}_0 \cdot \nabla \mathbf{v} + \lambda \nabla^2 \mathbf{b}. \qquad (23)$$

Introducing a spectral kinetic energy tensor, a spectral magnetic energy tensor and a spectral interaction tensor respectively by:

$$\Omega_{ij}(\boldsymbol{\varkappa}) = \int \tfrac{1}{2} \langle v_i(\mathbf{x}) v_j(\mathbf{x}+\mathbf{r}) \rangle e^{-i\boldsymbol{\varkappa}\cdot\mathbf{r}} d\mathbf{r}, \qquad (24)$$

$$\Lambda_{ij}(\boldsymbol{\varkappa}) = \int \tfrac{1}{2} \langle b_i(\mathbf{x}) b_j(\mathbf{x}+\mathbf{r}) \rangle e^{-i\boldsymbol{\varkappa}\cdot\mathbf{r}} d\mathbf{r}, \qquad (25)$$

$$\gamma_{ij}(\boldsymbol{\varkappa}) = \int \tfrac{1}{2} \langle b_i(\mathbf{x}) v_j(\mathbf{x}+\mathbf{r}) \rangle e^{-i\boldsymbol{\varkappa}\cdot\mathbf{r}} d\mathbf{r} \qquad (26)$$

Lehnert derived and solved the equations for Ω_{ij}, Λ_{ij}, γ_{ij}. Spherical coordinates ($\varkappa$, φ, ψ) being chosen in wave number space and φ being referred to the direction of the applied field, the solutions for the energy densities:

$$\Omega = \frac{1}{2\pi} \int_0^{2\pi} \Omega_{ii} \, d\psi, \qquad (27)$$

$$\Lambda = \frac{1}{2\pi} \int_0^{2\pi} \Lambda_{ii} \, d\psi, \qquad (28)$$

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$$\Omega = \Omega_0 e^{-\beta^2\sigma^2\tau} \left[\frac{\sigma}{[\sigma^2 - \mu^2]^{\frac{1}{2}}} \operatorname{sh} \frac{\alpha\tau}{2} \pm \operatorname{ch} \frac{\alpha\tau}{2} \right]^2, \qquad (29)$$

$$\Lambda = \Omega_0 e^{-\beta^2\sigma^2\tau} \left[\frac{\mu}{[\sigma^2 - \mu^2]^{\frac{1}{2}}} \operatorname{sh} \frac{\alpha\tau}{2} \right]^2, \qquad (30)$$

where the following dimensionless parameters have been introduced for compactness:

$$\beta = \frac{\lambda + \nu}{|\lambda - \nu|}, \qquad (31)$$

$$\sigma = \frac{\varkappa |\lambda - \nu|}{2 b_0}, \qquad (32)$$

$$\alpha = \beta [\sigma^4 - \mu^2 \sigma^2]^{\frac{1}{2}}, \qquad (33)$$

$$\tau = \frac{t}{\theta}, \qquad (34)$$

and where the + and − signs correspond respectively to $\nu < \lambda$ and $\nu > \lambda$. The "inhibition time" θ is given by:

$$\theta = \frac{\lambda + \nu}{4 b_0^2}, \qquad (35)$$

and is characteristic of the process of draining of energy from the field of smallest diffusivity by the other field through the coupling due to the applied magnetic field (Nihoul, 1963).

The same initial conditions as Deissler's have been assumed, i.e.[1]

$$(\Omega)_{t=0} = \Omega_0 = C\varkappa^2 \quad (C \text{ constant}), \tag{36}$$

$$(\Lambda)^{t=0} = (\gamma)^{t=0} = 0. \qquad (37)\ (38)$$

Deissler (1963) derived independently the solutions (29) and (30). To obtain the *total* kinetic and magnetic energies, he then integrated numerically over all wave number space for values of $n = \nu/\lambda$ equal to 10^{-7}, $\frac{1}{2}$ and 2. He found that, for t sufficiently larger than θ, in the last two cases, the situation tended to equipartition while, in the first case, equipartition was not attained before the turbulence was damped out.

Now, if we accept Golitsyn's argument whenever $R_m < 1$—disregarding the discussion of the preceding section—Lehnert's equations may be further simplified and the time derivative of $\mathfrak{b}$ may be neglected in (23). The solutions for Ω and Λ are then (with the same notations as previously):

$$\Omega = \Omega_0 e^{-\beta(\beta\mp 1)\sigma^2\tau} \exp\left(-\beta\mu^2\tau/\beta\pm 1\right), \tag{39}$$

$$\Lambda = \Omega \frac{\mu^2}{(\beta\pm 1)^2\sigma^2}, \tag{40}$$

the + and − signs corresponding respectively to $\nu < \lambda$ and $\nu > \lambda$.

The total kinetic energy (K, say) and the total magnetic energy (M, say) are obtained by integration of (29) and (30) or (39) and (40) over all wave number space. We seek here the asymptotic form of these integrals for large τ by the method of steepest descent. The advantage of this analytical method is that it gives the asymptotic law of decay, a result beyond the reach of Deissler's numerical approach and that it permits a detailed comparison of the results obtained by Lehnert's and Golitsyn's equations respectively.

I. According to Golitsyn's equations, one has, using (31) and (36) and calculating the contribution from the saddle point $\mu = 0$ by Laplace's method (Erdelyi, 1956, p. 37).

[1] These conditions correspond to isotropic turbulence and zero magnetic fluctuations at the initial time.

$$K_g = 2\pi \left[\frac{2b_0}{|\lambda-\nu|}\right]^5 C \int_0^\infty \sigma^4 e^{-\beta(\beta\mp 1)\sigma^2\tau} d\sigma \times \int_{-1}^{1} e^{-\beta/(\beta\pm 1)\mu^2\tau} d\mu$$

$$\sim 2\pi C \left[\frac{2b_0}{|\lambda-\nu|}\right]^5 \left[\frac{\beta\pm 1}{\beta\tau}\right]^{\frac{1}{2}} \Gamma(\tfrac{1}{2}) \int_0^\infty \sigma^4 e^{-\beta(\beta\mp 1)\sigma^2\tau} d\sigma$$

$$\sim \frac{\pi}{4} C \left[\frac{2b_0}{|\lambda-\nu|}\right]^5 \left[\frac{2}{\beta\tau}\right]^{\frac{1}{2}} \frac{3\pi/2}{[\beta\tau(\beta\mp 1)]^{\frac{5}{2}}} [2(\beta\pm 1)]^{\frac{1}{2}}, \tag{41}$$

$$M_g = 2\pi C \left[\frac{2b_0}{|\lambda-\nu|}\right]^5 \int_0^\infty \sigma^2 e^{-\beta(\beta\mp 1)\sigma^2\tau} d\sigma \times \int_{-1}^{1} \frac{\mu^2}{(\beta\pm 1)^2} e^{-\beta/(\beta\pm 1)\mu^2\tau} d\mu$$

$$\sim 2\pi C \left[\frac{2b_0}{|\lambda-\nu|}\right]^5 \left[\frac{\beta\pm 1}{\beta\tau}\right]^{\frac{3}{2}} \frac{\Gamma(\frac{3}{2})}{(\beta\pm 1)^2} \times \int_0^\infty \sigma^2 e^{-\beta(\beta\mp 1)\sigma^2\tau} d\sigma$$

$$\sim \frac{\pi}{4} C \left[\frac{2b_0}{|\lambda-\nu|}\right]^5 \left[\frac{2}{\beta\tau}\right]^{\frac{1}{2}} \frac{\pi/2}{[\beta\tau(\beta\mp 1)]^{\frac{3}{2}}} \left[\frac{2(\beta\mp 1)^2}{\beta\pm 1}\right]^{\frac{1}{2}}. \tag{42}$$

Both the kinetic and magnetic energies are found to decay as τ^{-3} and:

$$\frac{K_g}{M_g} \sim 3\frac{\lambda}{\nu}. \tag{43}$$

II. The expressions of the total energies obtained from the solutions of Lehnert's equations are conveniently written as the sum of two components. The first of these components is independent of μ. In the second component, the integral over μ is dominated for large τ by the terms in $e^{\alpha\tau}$. Hence:

$$K_L = K_1 + K_2, \tag{44}$$

$$M_L = M_1 + M_2, \tag{45}$$

with

$$K_1 = M_1 = 2\pi \left[\frac{2b_0}{|\lambda-\nu|}\right]^5 C \int_0^\infty \sigma^4 e^{-\beta^2\sigma^2\tau} d\sigma = \frac{\pi}{4} \left[\frac{2b_0}{|\lambda-\nu|}\right]^5 \frac{3\pi^{\frac{1}{2}}}{[\beta^2\tau]^{\frac{5}{2}}} C, \tag{46}$$

$$K_2 \sim \pi \left[\frac{2b_0}{|\lambda-\nu|}\right]^5 C \int_0^\infty \sigma^4 e^{-\beta^2\sigma^2\tau} d\sigma \times \int_0^1 \left[1 \pm \frac{\sigma}{[\sigma^2-\mu^2]^{\frac{1}{2}}}\right]^2 e^{\alpha\tau} d\mu, \tag{47}$$

$$M_2 \sim \pi \left[\frac{2b_0}{|\lambda-\nu|}\right]^5 C \int_0^\infty \sigma^4 e^{-\beta^2\sigma^2\tau} d\sigma \times \int_0^1 \frac{\mu^2}{\sigma^2-\mu^2} e^{\alpha\tau} d\mu. \tag{48}$$

The dominant contribution to these integrals comes from the saddle point $\mu = 0$ (where $\alpha = \beta\sigma^2$, $\alpha' = 0$, $\alpha'' = -\beta < 0$) and may be calculated by Laplace's method (Erdelyi 1956) after expanding the functions $[1 \pm \sigma/[\sigma^2-\mu^2]^{\frac{1}{2}}]$ and $\mu^2/(\sigma^2-\mu^2)$ in power series of μ/σ.

Let: $$\left[1 \pm \frac{\sigma}{[\sigma^2-\mu^2]^{\frac{1}{2}}}\right]^2 = \sum_0^\infty A_n \left(\frac{\mu}{\sigma}\right)^{2n}, \tag{49}$$

$$\frac{\mu^2}{\sigma^2-\mu^2} = \sum_0^\infty \left(\frac{\mu}{\sigma}\right)^{2n+2}, \tag{50}$$

we have:

$$\int_0^1 \left[1 \pm \frac{\sigma}{[\sigma^2-\mu^2]^{\frac{1}{2}}}\right]^2 e^{\alpha\tau} d\mu = \sum_0^\infty \frac{A_n}{\sigma^{2n}} \int_0^1 \mu^{2n} e^{\alpha\tau} d\mu \sim \sum_0^\infty \frac{A_n}{\sigma^{2n}} \tfrac{1}{2}\Gamma(n+\tfrac{1}{2}) \left[\frac{2}{\beta\tau}\right]^{n+\frac{1}{2}} e^{\beta\sigma^2\tau},$$

$$\int_0^1 \frac{\mu^2}{\sigma^2-\mu^2} e^{\alpha\tau} d\mu = \sum_0^\infty \frac{1}{\sigma^{2n+2}} \int_0^1 \mu^{2n+2} e^{\alpha\tau} d\mu \sim \sum_0^\infty \frac{1}{\sigma^{2n+2}} \tfrac{1}{2}\Gamma(n+\tfrac{3}{2}) \left[\frac{2}{\beta\tau}\right]^{n+\frac{3}{2}} e^{\beta\sigma^2\tau}.$$

Hence, setting $\varrho = \beta(\beta-1)\sigma^2\tau$, we get:

$$K_2 \sim \frac{\pi}{4} \left[\frac{2b_0}{|\lambda-\nu|}\right]^5 C \frac{A\pi}{[\beta(\beta-1)\tau]^{\frac{5}{2}}} \left[\frac{2}{\beta\tau}\right]^{\frac{1}{2}},$$

$$M_2 \sim \frac{\pi}{4} \left[\frac{2b_0}{|\lambda-\nu|}\right]^5 C \frac{B\pi}{[\beta(\beta-1)\tau]^{\frac{5}{2}}} \left[\frac{2}{\beta\tau}\right]^{\frac{1}{2}},$$

with

$$\begin{aligned} A &= \frac{1}{\pi} \sum_0^\infty A_n \Gamma(\tfrac{5}{2}-n) \Gamma(n+\tfrac{1}{2}) [2(\beta-1)]^n \\ &= 2[2(\beta-1)]^2[1+2(\beta-1)]^{-3} \\ &\quad \pm \tfrac{3}{2}[2(\beta-1)]^2[1+2(\beta-1)]^{-\frac{5}{2}} \\ &\quad + 2(\beta-1)[1+2(\beta-1)]^{-2} \\ &\quad \pm 2(\beta-1)[1+2(\beta-1)]^{-\frac{3}{2}} \\ &\quad + \tfrac{3}{4}[1 \pm (2\beta-1)^{-\frac{1}{2}}]^2, \end{aligned}$$

$$\begin{aligned} B &= \frac{1}{\pi} \sum_0^\infty \Gamma(\tfrac{3}{2}-n) \Gamma(\tfrac{3}{2}+n) [2(\beta-1)]^{n+1} \\ &= -2[2(\beta-1)]^3[1+2(\beta-1)]^{-3} \\ &\quad + [2(\beta-1)]^2[1+2(\beta-1)]^{-2} \\ &\quad + \tfrac{1}{4}[2(\beta-1)][1+2(\beta-1)]^{-1}. \end{aligned}$$

There are thus two contributions to the total energy calculated from the solutions of Lehnert's equations: one which decays as $t^{-\frac{5}{2}}$ as in ordinary turbulence and corresponds to equipartition of energy (see equation 46), one which decays as t^{-3} and leads to a partition of the energy in a ratio which is a function of β, i.e. in second instance of the ratio of the two diffusivities. *To determine which of these composants is the* most important let us compare the total (kinetic + magnetic) energy in the two components. We have:

$$\frac{K_1+M_1}{K_2+M_2} \sim (\beta\tau)^{\frac{1}{2}} \left(\frac{\beta-1}{\beta}\right)^{\frac{5}{2}} \frac{3}{A+B} \left[\frac{2}{\pi}\right]^{\frac{1}{2}},$$

and thus the first component will dominate after a time given by:

$$\tau_1 = \frac{\pi(A+B)^2}{18} \frac{\beta^4}{(\beta-1)^5}.$$

Let us examine separately the cases $n < 1$ ($n = 10^{-7}$, say), and $n \sim 1$ ($n = \frac{1}{2}$, 2, say). In the first of these cases we expect the solutions obtained with Lehnert's and Golitsyn's equations to coincide; at least up to a certain time when possibly the condition on N_m reverses (see the discussion in the preceding section). Simple calculations give:

$$n < 1, \quad \beta - 1 < 1, \quad \beta \sim 1,$$

$$\tau_1 \sim \frac{\pi}{2}(\beta-1)^{-5} (\sim 10^{34} \text{ for } n \sim 10^{-7}),$$

$$n \sim 1, \quad \beta - 1 \sim 1, \quad \beta \sim 1, \quad \tau_1 \sim 0(1).$$

Hence:

In the case $n < 1$—

1. The t^{-3} component dominates up to a time so large that the turbulence is practically damped out. Taking for instance $b_0 \sim 10$, $\lambda \sim 1$, $\nu \sim 10^{-7}$ in *MKS* units,[1] one finds $\theta \sim 10^{-2}$ and

[1] These values correspond to laboratory experiments with mercury.

the effective law of decay is t^{-3} for all time such that $10^{-2} < t < 10^{32}$, i.e. practically the life-time of the turbulence.

2. In the t^{-3} component, the ratio of kinetic to magnetic energy is

$$\frac{K_2}{M_2} \sim \frac{A}{B} \sim \frac{6}{\beta - 1} \sim 3\frac{\lambda}{\nu}.$$

The energy is partitioned between the turbulent velocity and magnetic fields in the inverse ratio of their respective diffusivities[2] and equipartition is not realized before a time inaccessibly large when the law of decay reverts to $t^{-\frac{5}{2}}$. This agrees with Deissler's results.

3. The solutions obtained from Lehnert's and Golitsyn's equations are identical. We have indeed (for $\beta \sim 1$):

$$K_g \sim \frac{\pi}{4} C \left[\frac{2b_0}{|\lambda - \nu|}\right]^5 \left[\frac{2}{\beta\tau}\right]^{\frac{1}{2}} \times \frac{3\pi}{[\beta\tau(\beta - 1)]^{\frac{5}{2}}} \sim K_2 \sim K_L,$$

$$M_g \sim \frac{\pi}{4} C \left[\frac{2b_0}{|\lambda - \nu|}\right]^5 \left[\frac{2}{\beta\tau}\right]^{\frac{1}{2}} \times \frac{(\beta - 1)\pi/2}{[\beta\tau(\beta - 1)]^{\frac{5}{2}}} \sim M_2 \sim M_L.$$

In the case $n \sim 1$—

1. The $t^{-\frac{5}{2}}$ component and the t^{-3} component are initially comparable and, since it decays faster, the second one becomes rapidly negligible.

2. In the $t^{-\frac{5}{2}}$ component, energy is partitioned equally between kinetic and magnetic fields in agreement with Deissler's numerical calculations.

3. Golitsyn's equations give only the negligible t^{-3} component and thus lead to incorrect results as predicted by our discussion above.

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REFERENCES

Deissler, R. G. 1963. Magneto-fluid dynamic turbulence with a uniform imposed magnetic field. *The Phys. of Fluids, 6,* 9, 1250–59.

Golitsyn, G. S. 1960. Fluctuations du champ magnétique et de la densité de courant dans la turbulence d'un fluide faiblement conducteur. *Dokl. Akad. Nauk SSSR 132,* 2, 315–18.

Lehnert, B. 1955. The decay of magneto-turbulence in the presence of a magnetic field and Coriolis force. *Quart. Appl. Math., 12,* 4, 321–41.

Matuura, N. & Nagata, T. 1962. Turbulent conditions in the upper atmosphere. *Report of Ionosphere and Space Research in Japan, 16,* 2, 185–255.

Nihoul, J. C. J. 1963. Note sur la turbulence magnétodynamique. *J. de Mécanique, II,* 3, 251–62.

Ohji, M. 1964. Effect of a very strong uniform magnetic field on magneto-fluid dynamic turbulence. *J. Phys. Soc. Japan, 19,* 8, 1430–39.

Sutton, G. W. & Sherman, A. 1965. *Engineering Magnetohydrodynamics.* McGraw-Hill.

[2] The t^{-3} component is indeed feeded by the fluctuating field which has the smallest direct dissipation and, prolonging the fluctuations of the other field, maintains there an energy which is a small fraction of its own.

МАГНИТОГИДРОДИНАМИЧЕСКАЯ ТУРБУЛЕНТНОСТЬ ПРИ МАЛЫХ ЧИСЛАХ РЕЙНОЛЬДСА

Обсуждаются уравнения Голицына для изучения магнитогидродинамической турбулентности и, в частности, турбулентности в ионосфере. Показано, что гипотеза о малости магнитного числа Рейнольдса недостаточна для справедливости этих уравнений и указаны соответствующие дополнительные условия. В качестве иллюстрации рассмотрен простой случай слабой турбулентности в присутствии однородного внешнего магнитного поля. Методом перевала получен закон затухания энергии на последней стадии вырождения турбулентности сначала из решений уравнений Голицына, а затем из более общих уравнений Ленерта. Из уравнений Ленерта следует, что магнитная и кинетическая энергии пульсаций состоят из двух частей, одна из которых затухает как $t^{-5/2}$, как и в немагнитной турбулентности, и соответствует равнораспределению энергии между полями флуктуаций скорости и магнитного поля, и другой части, затухающей как t^{-3}, и ведущей к распределению энергии между двумя полями в пропорции, которая зависит от отношения n между кинетической и магнитной вязкостями. С помощью уравнений Голицына получается лишь компонента, пропорциональная t^{-3}. Когда все условия справедливости этих уравнений удовлетворены — не только условие малости числа Rm — найдено, что компонента t^{-3} превалирует вплоть до недостижимо больших времен, при этом t^{-3} компоненты, полученные из уравнений Ленерта и Голицына, эквивалентны. С другой стороны, когда некоторые из найденных условий не выполняются, даже если $\mathrm{Rm} \ll 1$, то t^{-3} компонента, первоначально сравнимая с $t^{-5/2}$ компонентой, поскольку она затухает быстрее, быстро становится пренебрежимо малой, в то время как энергия стремится к равнораспределению. В этом случае уравнения Голицына ведут к неправильным результатам.