Analogous behavior of rotating and stratified fluids

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ABSTRACT

The present paper continues an investigation on the analogy between fluids which are rotating and fluids which are stratified. The analysis has been extended to include rotation in the stratified system and stratification in the rotating system. Even though the basic equations are the same we differentiate between the two systems on the basis of the forces which drive the motion. System S_Ω corresponds to adding rotation to the stratified system S, in a foregoing paper where the flow is driven by differential heating at the side boundaries. In system Ω_S stratification is added to the rotating fluid, Ω, which is driven by differential vertical vorticity at the top and bottom boundaries. It is shown that an exact analogy continues to hold when the flow is two-dimensional and when certain variables are identified with each other in the two systems. Combining rotation and stratification of the same order inhibits the outflow from the boundary layers which was found to be necessary in the simple systems, and the buoyancy and Ekman boundary layers play no role in two-dimensional flow. The entire fluid is dominated by diffusive processes. In system S_Ω three-dimensional flows have significant buoyancy layers. When the added process (rotation in system S_Ω or stratification in system Ω_S) is introduced as a higher-order correction, it can lead to zero-order changes in two-dimensional flows but has only higher-order effects in the three-dimensional S_Ω system. It is also shown that the rotating, stratified fluid has no boundary layers of the type found by Stewartson (1957) for rotating homogeneous fluids.

1. Introduction

The title of this paper is meant to be ambiguous. One can correctly interpret it as "the analogy between fluids which are stratified and fluids which are rotating". Alternatively one can read the title as applying to fluids which are both rotating and stratified and that too happens to be correct. However, in the latter case the title is somewhat meaningless because an analogy must exist between two separate concepts. The fact is that under certain conditions rotating, stratified fluids which are driven by heating at the sides behave in an analogous fashion to rotating, stratified fluids which are driven by vorticity at the upper and lower boundaries. Hence, this paper is an extension of an investigation (Veronis (1967), hereafter referred to as I) of the analogy between homogeneous, rotating fluids (system Ω) and *stratified non-rotating fluids (system S)*.

In I the exact analogy between two-dimensional, steady, linear flows in systems Ω and S was discussed and particular emphasis was placed on two simple problems in each system. The stratified fluid was forced by means of temperatures imposed at the side boundaries symmetrically and anti-symmetrically. The analogous rotating problem involved corresponding boundary conditions on the zonal velocity at the upper and lower boundaries. The breakdown of the analogy in three-dimensional situations was investigated also.

In the present work we add rotation to the stratified problem (denoting the combined system by S_Ω) and stratification to the rotating problem (system Ω_S). Although strictly speaking, in both systems, S_Ω and Ω_S, we are considering flows in a stratified, rotating fluid which differ only with respect to the applied boundary conditions, it is worth while to keep the description separate in the sense of looking at the effects of rotation on stratified fluids and vice versa. The point is that the analogy, looked at in this fashion, continues to exist and one derives results in one system which may help with the interpretation of the other. These reciprocal possibilities may be especially helpful in more complicated problems where non-linear and/or transient effects may enter and where the analysis may be difficult to carry out without some intuitive aids.

The problems are formulated in the next section where we non-dimensionalize the system in two ways, one appropriate to the system where the forcing is in terms of the zonal velocities (Ω_S) and the other where the system is heated at side boundaries (S_Ω).

In section 3 we discuss the buoyancy boundary layer and the analogous Ekman boundary layer and derive formulae which are useful in the analysis of the complete problems. The following section introduces again the expansion procedure used in I to analyze the flows. Then two situations are considered in each problem. E.g. in system S_Ω we treat the cases where rotation is as strong in its constraining effect as is the stratification and then where the effect of rotation is an order lower.

Section 5 contains the corresponding analyses of the three-dimensional problems in system S_Ω, and the breakdown of the analogy (similar to the breakdown in the three-dimensional problems in S and Ω of I) is pointed out.

The paper ends with a summary of the flows and some concluding remarks in section 6.

The present work is the extension of the analysis in I and the latter was developed from some lectures which were presented by me at the Woods Hole Oceanographic Institution during the summer of 1966. Although this work is self-contained, an earlier manuscript by Barcilon & Pedlosky (1966) develops the equations for a rotating, stratified fluid with the same expansion as that which is used here. Barcilon had previously given a similar expansion in a paper presented at the Rotating Fluids Conference in La Jolla, California, in March, 1966. The procedure is developed again in this paper because the non-dimensionalizations differ from that of Barcilon and Pedlosky; the present ones, being outgrowths of those in I, are oriented toward the analogous behavior of the two systems. Also, the emphasis in the entire work here is on the analogy. Barcilon & Pedlosky have attemted a general development for problems of rotating, stratified fluids for linear steady flows but with an emphasis on the change which stratification brings about in the theory of rotating homogeneous fluids. Their notation and the one used here, an overbar for variables in the buoyancy layer and a tilde for variables in the Ekman layer, are the same.

2. Equations and non-dimensionalization

We start with the Boussinesq form of the Navier-Stokes equations for a stably stratified fluid rotating with angular velocity Ω about the vertical axis:

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} + 2\mathbf{\Omega}\times\mathbf{v} = -\frac{1}{\varrho}\nabla p + g\alpha T\hat{\mathbf{k}} + \alpha T\nabla\left[\frac{\Omega^2}{2}(x^2+y^2)\right] + \nu\nabla^2\mathbf{v}, \tag{2.1}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{2.2}$$

$$T_t + \mathbf{v}\cdot\nabla T + \frac{4\Delta T}{L}w = \varkappa\nabla^2 T, \tag{2.3}$$

$$\varrho' = \varrho\alpha T, \tag{2.4}$$

where $\mathbf{v}$ is the velocity with components (u, v, w) in the respective directions (x, y, z), $\hat{\mathbf{k}}$ is the unit vector in the z direction, ν and $\varkappa$ are kinematic viscosity and thermal diffusivity, p is pressure, ϱ (= constant) is the mean density of the fluid, ϱ' is the perturbation density expressed as a linear function of temperature in accord with the Boussinesq approximation, g is gravity, and α is the coefficient of thermal expansion. Equation (2.3) contains the stable linear temperature gradient $\partial T/\partial z = 4\Delta T/L$ written explicitly as the last term on the left hand side. L is a characteristic scale.

We shall suppose that the horizontal extent of the fluid is sufficiently small so that the centrifugal force term $\alpha T\nabla[\Omega^2(x^2+y^2)/2]$ can be neglected relative to the gravitational term. Formally the condition can be expressed as

$$\frac{\Omega^2 H}{g} < 1, \tag{2.5}$$

where H is the largest distance from the axis of rotation to a side boundary. This condition may be stated alternatively as follows: It is not possible for a rotating stratified fluid to be in a state of zero relative motion if diffusive processes are present and the stratification is brought about by conditions imposed at the boundaries. We shall assume that the motions brought about by the combined effects of rotation and stratification are much smaller than those forced by the applied variations in the boundary conditions which we treat explicitly.

We shall non-dimensionalize the equations in two different ways. The first is based on the assumption that the flow is driven by boundary temperatures with amplitudes ΔT_c. Then choosing parameters so that the hydrostatic relation will result naturally and also that the Rayleigh number appears as a significant parameter, we write

$$\partial_t = \sqrt{\frac{g\alpha\Delta T}{L}}\,\partial_{t'}, \quad \mathbf{v} = \sqrt{\frac{g\alpha L}{\Delta T}}\,\Delta T_c \mathbf{v}',$$

$$T = 2\Delta T_c \sqrt{\sigma}\, T', \quad p = g\alpha\Delta T_c \varrho L \sqrt{\sigma}\, p',$$

$$\nabla = \frac{1}{L}\nabla' \tag{2.9}$$

to derive (dropping the primes)

System S_Ω

$$\sigma^{-\frac{1}{2}}[\mathbf{v}_t + \varepsilon\mathbf{v}\cdot\nabla\mathbf{v}] + 2F\hat{\mathbf{k}}\times\mathbf{v} = -\nabla p + 2T\hat{\mathbf{k}} + R\nabla^2\mathbf{v}, \tag{2.7}$$

$$\sigma^{\frac{1}{2}}[T_t + \varepsilon\mathbf{v}\cdot\nabla T] + 2w = R\nabla^2 T, \tag{2.8}$$

where the following parameters appear:

Prandtl number, $\sigma = \dfrac{\nu}{\varkappa}$,

Rayleigh number, $R_a = \dfrac{1}{R^2} = \dfrac{g\alpha\Delta T L^3}{\varkappa\nu}$,

Measure of non-linearity, $\varepsilon = \Delta T_c/\Delta T$,

Ratio of rotation and

Brunt-Väisälä frequencies, $F = \left[\dfrac{L\Omega^2}{g\alpha\Delta T\sigma}\right]^{\frac{1}{2}}$. (2.9)

Our second choice of non-dimensionalizing is based on the assumption that the quantity ΔT_c above is the temperature change resulting from the flow forced by applied velocity difference of magnitude V at the boundaries. Furthermore, if geostrophy is to be a natural consequence and the Ekman number is to appear as a significant parameter, we are led to the choice

$$\partial_t = \frac{\Omega}{\sqrt{\sigma}}\partial_{t'}, \quad \mathbf{v} = V\mathbf{v}', \quad T = 2\Delta T_c\sqrt{\sigma}FT',$$

$$p = \varrho L V\Omega p', \quad \nabla = \frac{1}{L}\nabla' \tag{2.10}$$

so that V as described above is related to ΔT_c by $V = g\alpha\Delta T_c\sqrt{\sigma}/\Omega$. Thus we derive

System Ω_S

$$\sigma^{-\frac{1}{2}}[\mathbf{v}_t + \varepsilon\mathbf{v}\cdot\nabla\mathbf{v}] + 2\hat{\mathbf{k}}\times\mathbf{v} = -\nabla p + \frac{2T}{F}\hat{\mathbf{k}} + E\nabla^2\mathbf{v}, \tag{2.11}$$

$$\sigma^{\frac{1}{2}}[T_t + \varepsilon\mathbf{v}\cdot\nabla T] + \frac{2w}{F} = E\nabla^2 T \tag{2.12}$$

instead of (2.7) and (2.8), with the parameters

$$\left.\begin{array}{l}\text{Rossby number, } \varepsilon = \nu\sqrt{\sigma}/\Omega L \\ \text{Ekman number, } E = \nu/\Omega L^2.\end{array}\right\} \tag{2.13}$$

Systems Ω_S and S_Ω have been chosen thus so as to preserve a certain symmetry. Depending on the specific conditions of the problem, particularly the type of forcing which is involved, we shall choose one or the other of the two systems.

Note that when $\partial_t = 0$ and $\varepsilon \to 0$, we are led to the equations

System S_Ω

$$2F\hat{\mathbf{k}}\times\mathbf{v} = -\nabla p + 2T\hat{\mathbf{k}} + R\nabla^2\mathbf{v}, \tag{2.14}$$

$$2w = R\nabla^2 T; \tag{2.15}$$

System Ω_S

$$2\hat{\mathbf{k}}\times\mathbf{v} = -\nabla p + \frac{2T}{F}\hat{\mathbf{k}} + E\nabla^2\mathbf{v} \tag{2.16}$$

$$\frac{2w}{F} = E\nabla^2 T. \tag{2.17}$$

These are to be supplemented by equation (2.2).

The particular forms of the above for two-dimensional flow show the same duality as for the pure rotating or stratified systems (see I). Thus if all variations with respect to y vanish, we have for (2.14)

$$-2Fv = -p_x + R\nabla^2 u, \tag{2.18a}$$

$$2Fu = R\nabla^2 v, \tag{2.18b}$$

$$0 = -p_z + 2T + R\nabla^2 w \tag{2.18c}$$

and for (2.16)

$$-2v = -p_x + E\nabla^2 u, \tag{2.19a}$$

$$2u = E\nabla^2 v, \tag{2.19b}$$

$$0 = -p_z + \frac{2T}{F} + E\nabla^2 w. \qquad (2.19\,\text{c})$$

The complete analogy between the two sets of equations is now evident if we make the following identifications:

Variables			Equations	
System S_Ω		System Ω_S	System S_Ω	System Ω_S
u	$\sim$	w	(2.18 a)	(2.19 c)
w	$\sim$	u	(2.18 b)	(2.17)
x	$\sim$	z	(2.18 c)	(2.19 a)
z	$\sim$	x	(2.15)	(2.19 b)
T	$\sim$	v		
R	$\sim$	E		
F	$\sim$	$\frac{1}{F}$		(2.20)

3. Boundary layers for the linear systems

We shall treat linear, steady flows in which R, or alternatively E, is very small. These conditions are typical of situations where boundary layer theory is applicable, and the present system is no exception. To investigate the form of the boundary layers it is instructive to inspect the two-dimensional equations for either system S_Ω or system Ω_S. If all but one of the variables are eliminated from equations (2.2), (2.15) and (2.18) the resulting equation contains the following operator

$$R^2\nabla^6 + 4\partial_{xx}^2 + 4F^2\partial_{zz}^2. \qquad (3.1)$$

Boundary-layer stretching in x (for $F = O(1)$) balances the sixth x-derivative of the first term with the second term and requires

$$\partial_x = \pm R^{-\frac{1}{3}}\partial_\xi, \quad \text{where} \quad \partial_\xi = O(1) \qquad (3.2)$$

Boundary-layer stretching in z balances the sixth z-derivative of the first term with the last term and requires

$$\partial_z = \pm\sqrt{\frac{F}{R}}\,\partial_\zeta, \quad \text{where} \quad \partial_\zeta = 0(1). \qquad (3.3)$$

In terms of E the above are

$$\partial_x = \pm\sqrt{\frac{F}{E}}\,\partial_\xi, \quad \partial_z = \pm E^{-\frac{1}{2}}\partial_\zeta. \qquad (3.4)$$

Now the x stretching corresponds to the buoyancy layer which is basic to problems in stratified systems and which was discussed in I. The equations governing the buoyancy layer can be derived from equations (2.15) and (2.18c) by stretching x as in (3.2) and neglecting the pressure term. Then denoting the dependent variables in the buoyancy layer by an overbar, we have the two equations

$$\bar{T}_{\xi\xi} - 2\bar{w} = 0, \quad \bar{w}_{\xi\xi} + 2\bar{T} = 0. \qquad (3.5)$$

The solutions which decay as $\xi \to \infty$ are

$$\bar{w} = (\bar{w}_0 \cos\xi + \bar{T}_0 \sin\xi)e^{-\xi}, \qquad (3.6\,\text{a})$$

$$\bar{T} = (\bar{T}_0 \cos\xi - \bar{w}_0 \sin\xi)e^{-\xi}, \qquad (3.6\,\text{b})$$

where subscript zero corresponds to values at $\xi = 0$. The two-dimensional continuity equation is

$$u_x + w_z = 0 \qquad (3.7)$$

or in the boundary layer

$$\bar{u}_\xi = \pm R^{-\frac{1}{3}}\bar{w}_z \qquad (3.8)$$

Integrating this equation from $\xi = 0$ to $\xi = \infty$, we derive

$$\bar{u}\big|_{\xi=0} = \pm\frac{R^{\frac{1}{3}}}{2}[\bar{w}_{0z} + \bar{T}_{0z}]. \qquad (3.9)$$

For a three dimensional system the results are essentially the same but ξ must be interpreted as the stretched coordinate normal to the boundary.

In subsequent sections we shall make use of the buoyancy layer near the boundaries $x = -1$ and $x = 1$. In the former the substitution is $x + 1 = R^{\frac{1}{3}}\xi$, in the latter $1 - x = R^{\frac{1}{3}}\xi$. Hence near $x = 1$ we must choose a negative sign in (3.9) so that

$$\bar{u}\big|_{x=\mp 1} = \pm\frac{R^{\frac{1}{3}}}{2}[\bar{w}_{0z} + \bar{T}_{0z}]. \qquad (3.10)$$

The z-stretching corresponds to the Ekman layer which is fundamental to rotating fluid problems. It is convenient to use system Ω_S here and we derive from equations (2.18a, b) using a tilde for variables in the Ekman layer,

$$\tilde{u}_{\zeta\zeta} + 2\tilde{v} = 0, \quad \tilde{v}_{\zeta\zeta} - 2\tilde{u} = 0, \qquad (3.11)$$

with boundary layer solutions

$$\tilde{u} = (\tilde{u}_0 \cos \zeta + \tilde{v}_0 \sin \zeta) e^{-\zeta}, \qquad (3.12\,\mathrm{a})$$

$$\tilde{v} = (\tilde{v}_0 \cos \zeta - \tilde{u}_0 \sin \zeta) e^{-\zeta}, \qquad (3.12\,\mathrm{b})$$

where subscript zero corresponds to values at $\zeta = 0$. The continuity equation (2.2) takes the form

$$\tilde{w}_\zeta = \mp E^{\frac{1}{2}} [\tilde{u}_x + \tilde{v}_y] \qquad (3.13)$$

and upon integration from $\zeta = 0$ to $\zeta = \infty$ gives

$$\tilde{w}|_{\zeta=0} = \pm \frac{E^{\frac{1}{2}}}{2} \{\tilde{u}_{0x} + \tilde{v}_{0y} + \tilde{v}_{0x} - \tilde{u}_{0y}\}. \qquad (3.14)$$

Analogous to the buoyancy layer the above result with the positive sign holds near the lower boundary which we shall take as $z = -1$. Near $z = 1$ the substitution is $\zeta = (1 - z) E^{-\frac{1}{2}}$, so

$$\tilde{w}|_{z=\mp 1} = \pm \frac{E^{\frac{1}{2}}}{2} \{\tilde{u}_{0x} + \tilde{v}_{0y} + \tilde{v}_{0x} - \tilde{u}_{0y}\}. \qquad (3.15)$$

We have summarized the properties of the buoyancy and Ekman boundary layers for later use. These are the only two boundary layers which occur in the present system when $F = O(1)$. The Stewartson (1957) boundary layers of $O(E^{\frac{1}{3}})$ and $O(E^{\frac{1}{4}})$ and parallel to the direction z for rotating fluids (normal to z for stratified fluids) do not appear in the combined system. The point is that the rotational constraint generates Ekman layers in the vicinity of all horizontal surfaces and the stratification generates buoyancy layers near all vertical surfaces. The thicker 1/3 and 1/4 layers are thus preempted by the thinner $\frac{1}{2}$ layers.

As can be seen from the operator in equation (3.1) it is possible[1] for Stewartson boundary layers to exist in system S_Ω as long as $F \leqslant R^{\frac{1}{2}}$ and in system Ω_S as long as $F^{-1} \leqslant E^{\frac{1}{2}}$. However, we shall restrict our attention in this paper (as we did in I) to the similarities and differences between flows driven by heating (system S_Ω) along lateral boundaries and by imposed velocities (system Ω_S) along top and bottom boundaries. For these purposes it suffices to consider driving forces which vary harmonically along the respective boundaries.

[1] The results regarding the existence of these boundary layers have also been derived by Barcilon & Pedlosky (private communication).

4. Two-dimensional flows driven symmetrically and anti-symmetrically

We first investigate some simple flows in two dimensions and compare the results with those derived in I for the rotating homogeneous (Ω) and non-rotating, stratified (S) systems. It was shown in I that two-dimensional steady flow in system S has an exact analogy in system Ω, provided that the boundary conditions correspond. The stratified flows which we studied were driven by imposed temperatures at the side boundaries and the rotating flows by imposed tangential velocities at the top and bottom boundaries. We now add rotation to the problem in system S and stratification to the problem in system Ω and determine what effects these added features have on the flows previously derived. It should be kept in mind that we are looking at the flow in a stratified, rotating fluid and our designations Ω_S and S_Ω are to some extent artificial. However, if one keeps in mind the results which we derived for systems Ω and S, it is very instructive to see what changes are brought about by the respective additions of stratification and rotation.

As in I we shall derive detailed results for the stratified system (S_Ω) and point out the analogy in the rotating system (Ω_S).

Problem

Given a rotating, stratified fluid enclosed between two walls at $x = \pm 1$, where the velocity vanishes and the temperature is given by

$$T = -T_0 \sin z \quad \text{at} \quad x = -1,$$
$$T = \pm T_0 \sin z \quad \text{at} \quad x = 1 \qquad (4.1)$$

What are the resulting temperature and velocity distributions?

The flow is independent of y and the pertinent equations are (2.2), (2.15) and (2.18) with $R < 1$. We shall follow the procedure given in I (to which the reader may refer for a detailed description and justification) of writing the variables as the sum of two contributions: an interior part (subscript I) which is valid throughout the fluid and boundary layer parts (denoted by an overbar) which are valid only near $x = \pm 1$ and which decay as the distance from the boundary increases. The interior and boundary layer contributions are each expanded in powers of $R^{\frac{1}{2}}$ (vid. I) as follows:

$$\left.\begin{aligned} f &= f_I + \bar{f}, \\ f_I &= f^{(0)} + R^{\frac{1}{2}} f^{(1)} + R f^{(2)} + R^{\frac{3}{2}} f^{(3)} + \dots, \\ \bar{f} &= \bar{f}^{(0)} + R^{\frac{1}{2}} \bar{f}^{(1)} + R \bar{f}^{(2)} + R^{\frac{3}{2}} \bar{f}^{(3)} + \dots \end{aligned}\right\} \tag{4.2}$$

We then order the equations according to the different orders of $R^{\frac{1}{2}}$. The results depend on the magnitude of F in the equations and we study the cases where $F = O(1)$ and $F = O(R^{\frac{1}{2}})$.

Case $F = O(1)$

Interior

$$O(R^0): \quad 2Fv^{(0)} = p_x^{(0)}, \; 2Fu^{(0)} = 0, \quad p_z^{(0)} = 2T^{(0)}, \; 2w^{(0)} = 0; \tag{4.3 a}$$

$$O(R^{\frac{1}{2}}): \quad 2Fv^{(1)} = p_x^{(1)}, \; 2Fu^{(1)} = 0, \quad p_z^{(1)} = 2T^{(1)}, \; 2w^{(1)} = 0; \tag{4.3 b}$$

$$O(R): \quad 2Fv^{(2)} = p_x^{(2)}, \; 2Fu^{(2)} = \nabla^2 v^{(0)}, \; p_z^{(2)} = 2T^{(2)}, \quad 2w^{(2)} = \nabla^2 T^{(0)}, \; u_x^{(2)} + w_z^{(2)} = 0. \tag{4.3 c}$$

The sets (4.3a) and (4.3b) are identical in form. We see that up to $O(R)$ stratification inhibits vertical flow in the interior, just as in system S. However, the presence of rotation also inhibits flow normal to the axis of rotation just as in system Ω. (Since the condition $F = O(1)$ means that the constraints of rotation and stratification are equally important, it is not surprising that the results reflect the presence of both processes.) The remaining variables are characterized by geostrophy and hydrostatic balance, but to $O(R^{\frac{1}{2}})$ the flow is otherwise undetermined. We note that in the set of equations at $O(R)$ the variables with superscript (2) can be eliminated to yield a single equation for $T^{(0)}$ or $v^{(0)}$. Thus

$$\nabla^2 [T_{xx}^{(0)} + F^2 T_{zz}^{(0)}] = 0 \tag{4.4 a}$$

or

$$\nabla^2 [F^2 v_{xx}^{(0)} + v_{zz}^{(0)}] = 0. \tag{4.4 b}$$

Boundary layer

The boundary layer equations and their solutions provide boundary conditions for the interior variables. Note that we stretch the coordinate x as given by equation (3.2).

$$O(R^{-\frac{1}{2}}): \quad \bar{p}_\xi^{(0)} = 0, \; \bar{u}_\xi^{(0)} = 0 \Rightarrow \bar{p}^{(0)} = 0, \; \bar{u}^{(0)} = 0; \tag{4.5 a}$$

$$O(R^0): \quad \bar{w}_{\xi\xi}^{(0)} + 2\bar{T}^{(0)} = 0, \quad \bar{T}_{\xi\xi}^{(0)} - 2\bar{w}^{(0)} = 0, \quad \bar{v}_{\xi\xi}^{(0)} = 0, \quad 2F\bar{v}^{(0)} = \pm \bar{p}_\xi^{(1)}, \quad \bar{u}_\xi^{(1)} \pm \bar{w}_z^{(0)} = 0. \tag{4.5 b}$$

Since $\bar{v}_{\xi\xi}^{(0)} = 0$, it is necessary that $\bar{v}^{(0)} = 0$ and also that $\bar{p}^{(1)} = 0$. Hence to $O(R^0)$ we derive the buoyancy layer equations (3.5) and use the continuity equation to solve for $\bar{u}^{(1)}$.

The solution for the buoyancy layer has the form (3.6). Now the velocity vanishes on the boundaries so $w = 0$ on $x = \pm 1$ and since $w^{(0)} = 0$ everywhere by equation (4.3a), we must have $\bar{w}^{(0)} = 0$ on $x = \pm 1$. Thus the solution in the buoyancy layer is

$$\bar{w}^{(0)} = \bar{T}_0^{(0)} \sin \xi e^{-\xi}, \quad \bar{T}^{(0)} = \bar{T}_0^{(0)} \cos \xi e^{-\xi}, \tag{4.6}$$

where $\bar{T}_0^{(0)} \equiv \bar{T}^{(0)}|_{\xi=0}$.

It is now possible to derive boundary conditions in terms of $T^{(0)}$ for the problem posed by equation (4.4a). We have specified that $v = 0$ on $x = \pm 1$ and since $\bar{v}^{(0)} \equiv 0$ we deduce $v^{(0)} = 0$ on $x = \pm 1$ or, making use of (4.3a)

$$T_x^{(0)} = 0 \quad \text{on} \quad x = \pm 1. \tag{4.7}$$

Furthermore the outflow condition (3.9) for the buoyancy layer can be written

$$\bar{u}^{(0)}|_{\xi=0} = \pm \tfrac{1}{2} \bar{T}_{0z}^{(0)}. \tag{4.8}$$

Since $u = 0$ at $x = \pm 1$ and since $u^{(0)} = 0$ everywhere, $\bar{u}^{(0)}|_{\xi=0} = 0$ so $\bar{T}_{0z}^{(0)} = 0$. The boundary layer temperature can be expressed in terms of the interior and total temperatures by use of (4.2) so we deduce

$$T_{0z}^{(0)} = T_z|_{\xi=0}. \tag{4.9}$$

The problem for $T^{(0)}$ is now completely specified by (4.1), (4.4a), (4.7) and (4.9). The solution with *symmetric* heating, i.e. $T = -T_0 \sin z$ on $x = \pm 1$ is

$$T^{(0)} = (A \cosh x + B \cosh Fx) \sin z, \tag{4.10a}$$

where

$$A = \frac{-FT_0 \sinh F}{F \cosh 1 \sinh F - \sinh 1 \cosh F},$$

$$B = -\frac{\sinh 1}{F \sinh F} A. \tag{4.10 b}$$

From the zero-order geostrophic and hydrostatic balances we conclude

$$v^{(0)} = -(A \sinh x + FB \sinh Fx) \cos z. \quad (4.11)$$

This solution is very close to that which was found in I for system S, where we deduced that the fluid was simply restratified by symmetric heating with uniform temperature across each level. Here there is a slight departure from a uniform temperature when $F = 1$. The point is that diffusion controls the temperature distribution and gives rise to an almost uniform temperature across the fluid. There is no cross velocity, u, and only enough zonal velocity, v, to compensate geostrophically for the slight deviation of the temperature from a horizontally uniform value.

When the heating is *antisymmetric*, i.e., $T = \pm T_0 \sin z$ on $x = \pm 1$, the solution is similar to (4.10) and (4.11) but with cosh and sinh interchanged. The result shows a temperature distribution which departs not too far from a linear dependence on x when $F = 1$. This result is *very* different from the one for a non-rotating, stratified fluid where we found that the temperature deviation vanished in the interior and that boundary layer contributions brought about the change from the boundary to the interior values. Also, in system S the fluid at each level flowed uniformly across the interior from one boundary to the other.

Now, the outflow, u, from the buoyancy layer is inhibited by the constraint of rotation and thus the buoyancy layer cannot be maintained. The temperature variation is dominated by diffusion as in the case of symmetric heating. A relatively strong zonal velocity, v, is necessary to balance the horizontal temperature gradient locally.

The analogous picture for system Ω_S can be derived by noting the substitutions listed at the end of section 2. In this case if we suppose that the flow is driven by an imposed zonal velocity

$$v = -v_0 \sin x \text{ on } z = -1, \; v = \pm v_0 \sin x \text{ on } z = 1, \quad (4.12)$$

we deduce exactly analogous results. Thus the solution with *symmetric* driving conditions is

$$v^{(0)} = (A \cosh z + B \cosh Fz) \sin x, \quad (4.13\text{a})$$

where

$$A = \frac{-v_0 F \sinh F}{F \cosh 1 \sinh F - \sinh 1 \cosh F},$$

$$B = -\frac{\sinh 1}{F \sinh F} A \quad (4.13\text{b})$$

and

$$T^{(0)} = -(A \sinh z + FB \sinh Fz) \cos x. \quad (4.14)$$

The solutions (4.13) and (4.14) can be translated from (4.10) and (4.11) with the use of the equivalences of section 2. Thus when $F = 1$ the zonal velocity is nearly uniform in z and there is only a slight temperature variation which accompanies the non-uniformity of v.

In the antisymmetric case where $v = \pm v_0 \sin x$ on $z = \pm 1$, the zonal velocity varies nearly linearly with z when $F = 1$ and is accompanied by a strong temperature variation across the fluid. Because of the constraint of the stratification there is no vertical velocity anywhere in the fluid, hence no Ekman layer suction and no Ekman layer. Again the system is dominated by diffusive processes.

Case $F = O(R^{\frac{1}{2}})$

Let $F = fR^{\frac{1}{2}}$ where f is a constant of $O(1)$. Then proceeding as before we derive an expanded set of equations from (2.2), (2.15) and (2.18) for the interior and the boundary layer regions.

Interior

$$O(R^0): \quad p_x^{(0)} = 0, \; u_x^{(0)} = 0, \; w^{(0)} = 0, \; p_z^{(0)} = 2T^{(0)}; \quad (4.15\text{a})$$

$$O(R^{\frac{1}{2}}): \quad p_x^{(1)} = 2fv^{(0)}, \; u^{(0)} = 0, \; w^{(1)} = 0, \; u_x^{(1)} = 0, \; p_z^{(1)} = 2T^{(1)}; \quad (4.15\text{b})$$

$$O(R): \quad p_x^{(2)} = 2fv^{(1)}, \; \nabla^2 v^{(0)} = 2fu^{(1)}, \; 2w^{(2)} = \nabla^2 T^{(0)}, \; p_z^{(2)} = 2T^{(2)}, \; u_x^{(2)} + w_z^{(2)} = 0; \quad (4.15\text{c})$$

$$O(R^{\frac{3}{2}}): \quad p_x^{(3)} = 2fv^{(2)} + \nabla^2 u^{(1)}, \; \nabla^2 v^{(1)} = 2fu^{(2)}, \; 2w^{(3)} = \nabla^2 T^{(1)}, \; p_z^{(3)} = 2T^{(3)}, \; u_x^{(3)} + w_z^{(3)} = 0. \quad (4.15\text{d})$$

Boundary layer

$$O(R^{-\frac{1}{2}}): \quad \bar{p}_\xi^{(0)} = 0, \; \bar{u}_\xi^{(0)} = 0 \;\Rightarrow\; \bar{p}^{(0)} = 0, \; \bar{u}^{(0)} = 0; \quad (4.16\text{a})$$

$$O(R^0): \quad \bar{w}_{\xi\xi}^{(0)} + 2\bar{T}^{(0)} = 0, \; \bar{T}_{\xi\xi}^{(0)} - 2w^{(0)} = 0, \; \bar{v}_{\xi\xi}^{(0)} = 0, \; \bar{p}_\xi^{(1)} = 2f\bar{v}^{(0)}, \; \bar{u}_\xi^{(1)} + \bar{w}_z^{(0)} = 0. \quad (4.16\text{b})$$

Hence, $\bar{v}^{(0)} = 0,\ \bar{p}^{(1)} = 0;$

$$O(R^{\frac{1}{2}}):\ \bar{p}^{(2)}_{\xi} = \bar{u}^{(1)}_{\xi\xi},\ \bar{v}^{(1)}_{\xi\xi} = 0,\ \bar{w}^{(1)}_{\xi\xi} + 2\bar{T}^{(1)} = 0,$$
$$\bar{T}^{(1)}_{\xi\xi} - 2\bar{w}^{(1)} = 0,\ \bar{u}^{(2)}_{\xi} + \bar{w}^{(1)}_{z} = 0. \qquad (4.16\text{c})$$

Again we shall look at the flows determined by the conditions (4.1) together with $\mathbf{v} = 0$ on $x = \pm 1$.

The reader can either refer to I or verify for himself that the set of equations leading to a solution for $u^{(0)}$, $p^{(0)}$, $w^{(0)}$, $T^{(0)}$, $u^{(1)}$ is exactly the same as that for the *non-rotating* system ($F \equiv 0$). Specifically, we find that the problem with *symmetric heating*, i.e. $T = -T_0 \sin z$ on $x = \pm 1$, leads to

$$u^{(0)} = 0,\ w^{(0)} = 0,\ T^{(0)} = -T_0,\ u^{(1)} = 0.$$

Hence, using the second of equations (4.15c) with $u^{(1)} = 0$ and the derived result $\bar{v}^{(0)} = 0$, we deduce

$$v^{(0)} = 0. \qquad (4.18)$$

The solution with symmetric heating is thus identical to the one derived when $F = 0$. no changes are brought about by higher-order corrections.

For *antisymmetric heating* ($T = \pm T_0 \sin z$ at $x = \pm 1$) the zero-order solution in the interior is again identical to that for the non-rotating case. Thus,

$$u^{(0)} = 0,\ w^{(0)} = 0,\ T^{(0)} = 0,\ u^{(1)} = \frac{T_0}{2} \cos z. \qquad (4.19)$$

Letting $v^{(0)} = \hat{v}(x) \cos z$, we see that the second of equations (4.15c) leads to

$$(\partial^2_{xx} - 1)\hat{v} = fT_0. \qquad (4.20)$$

Since $v = 0$ on $x = \pm 1$ and $\bar{v}^{(0)} \equiv 0$ by (4.16b) the boundary condition on $\hat{v}$ is

$$\hat{v} = 0 \text{ on } x = \pm 1,$$

and the solution for $v^{(0)}$ is

$$v^{(0)} = fT_0 \left(\frac{\cosh x}{\cosh 1} - 1\right) \cos z. \qquad (4.21)$$

Thus, a weak rotation (with $F = O(R^{\frac{1}{2}})$) gives rise to a zero-order zonal velocity in the interior. The Coriolis force associated with this value of the zonal velocity is, of course, $O(R^{\frac{1}{2}})$ because v is multiplied by F in the original equation.

Through the geostrophic and hydrostatic relations in (4.15b) we find that an $O(R^{\frac{1}{2}})$ temperature is associated with $v^{(0)}$. The equation to be solved is

$$T^{(1)}_x = fT_0 \left[1 - \frac{\cosh x}{\cosh 1}\right]$$

and we find that

$$T^{(1)} = fT_0 \left[x - \frac{\sinh x}{\cosh 1}\right], \qquad (4.22)$$

where we have made use of the boundary layer equations (4.16c) to show that $T^{(1)}$ is anti-symmetric, i.e. that the integration constant is zero. (The anti-symmetry of $T^{(1)}$ is obvious intuitively.)

In the anti-symmetric situation it is important to note that the stratification requires a buoyancy layer to adjust the boundary values. When rotation is equally strong ($F = 1$), the buoyancy layer vanishes because the outflow required for the existence of a buoyancy layer is suppressed by the rotation. A weak rotation ($F = O(R^{\frac{1}{2}})$), on the other hand, is not sufficiently strong to inhibit the outflow and a flow similar to that of the non-rotating fluid is established. Diffusion in the interior of the fluid balances the weak effect of rotation to give rise to a zonal velocity. The latter balances a horizontal variation of temperature through the thermal wind relation. It should be kept in mind that the roles of both $v^{(0)}$ and $T^{(1)}$ are rather passive, the principal dynamical balance being that of the non-rotating system.

The system Ω_S has exactly analogous solutions for the case $1/F = O(E^{\frac{1}{2}})$. Thus, with symmetric boundary conditions (recall that in system Ω_S we impose zonal velocities given by aquation (4.12) at the top and bottom boundaries) the zonal velocity achieves a value uniform in the vertical and the system behaves exactly as if there were no stratification. With antisymmetric boundary conditions an $O(E^0)$ temperature is generated and is balanced by a zonal velocity of $O(E^{\frac{1}{2}})$ by means of the thermal wind relation.

5. Extension of results to three-dimensional problems

One of the results which we derived in I is that the rotating system in three dimensions is essentially similar to the two-dimensional one, the main difference being the quantitative one of evaluating the vertical vorticity as $v_x - u_y$ instead as only v_x. The same conclusions hold with respect to system Ω_S, i.e. the addition of the third dimension brings no new qualitative results. It is not a difficult matter for the reader to verify that result, so we shall leave the system Ω_S and concentrate on S_Ω.

We found in I that the addition of the third dimension in the stratified system brought about qualitative changes because motion in the added dimension is unconstrained by the stratification. We shall proceed with the formal analysis of the system S_Ω pointing out the changes brought about by the addition of the third dimension and then shall summarize the results in the next section.

Problem

Given a rotating stratified fluid enclosed between two walls at $x = \pm 1$ where the velocity vanishes and the temperature is given by

$$T = -T_0 \sin y \sin z \quad \text{at } x = -1,$$
$$T = \pm T_0 \sin y \quad \sin z \quad \text{at } x = 1. \tag{5.1}$$

What are the resulting temperature and velocity distributions?

We shall proceed as in section 4, writing the variables in the expanded form (4.2).

CASE $F = O(1)$

Interior

$$O(R^0):\quad 2Fv^{(0)} = p_x^{(0)},\ 2Fu^{(0)} = -p_y^{(0)},\ p_z^{(0)} = 2T^{(0)},$$
$$w^{(0)} = 0,\ u_x^{(0)} + v_y^{(0)} = 0; \tag{5.2 a}$$

$$O(R^{\frac{1}{2}}):\quad 2Fv^{(1)} = p_x^{(1)},\ 2Fu^{(1)} = -p_y^{(1)},\ p_z^{(1)} = 2T^{(1)}$$
$$w^{(1)} = 0,\ u_x^{(1)} + v_y^{(1)} = 0; \tag{5.2 b}$$

$$O(R):\quad 2Fv^{(2)} = p_x^{(2)} - \nabla^2 u^{(0)},\ 2Fu^{(2)} = -p_y^{(2)} + \nabla^2 v^{(0)},$$
$$p_z^{(2)} = 2T^{(2)},\ 2w^{(2)} = \nabla^2 T^{(0)},$$
$$u_x^{(2)} + v_y^{(2)} + w_z^{(2)} = 0; \tag{5.2 c}$$

$O(R^{\frac{3}{2}})$: Same as $O(R)$ but with superscripts (1) and (3). (5.2 d)

Therefore, the equations at $O(R^0)$ and $O(R^{\frac{1}{2}})$ are geostrophic, hydrostatic, and involve only horizontal motions. At $O(R)$ we can eliminate the superscript (2) variables to derive the equation

$$\nabla^2[\nabla_1^2 T^{(0)} + F^2 T_{zz}^{(0)}] = 0. \tag{5.3}$$

Similarly, from the equations at $O(R^{\frac{3}{2}})$ we deduce

$$\nabla^2[\nabla_1^2 T^{(1)} + F^2 T_{zz}^{(1)}] = 0. \tag{5.4}$$

Boundary layers

$$O(R^{-\frac{1}{2}});\quad \bar{p}_\xi^{(0)} = 0,\ \bar{u}_\xi^{(0)} = 0 \ \Rightarrow\ \bar{p}^{(0)} = 0,\ \bar{u}^{(0)} = 0; \tag{5.5 a}$$

$$O(R^0):\quad \bar{v}_{\xi\xi}^{(0)} = 0,\ 2F\bar{v}^{(0)} = \bar{p}_\xi^{(1)},\ \overline{w}_{\xi\xi}^{(0)} + 2\overline{T}^{(0)} = 0,$$
$$\overline{T}_{\xi\xi}^{(0)} - 2\overline{w}^{(0)} = 0,\ \bar{u}_\xi^{(1)} = \mp \overline{w}_z^{(0)}. \tag{5.5 b}$$

The first of set (5.5b) gives $\bar{v}^{(0)} = 0$ and the second $\bar{p}^{(1)} = 0$. $\bar{v}^{(0)} = 0$ has been used in the last of (5.5b).

$$O(R^{\frac{1}{2}}):\quad 2F\bar{v}^{(1)} = \bar{p}_\xi^{(2)} - \bar{u}_{\xi\xi}^{(1)},\ 2F\bar{u}^{(1)} = \bar{v}_{\xi\xi}^{(1)},$$
$$\overline{w}_{\xi\xi}^{(1)} + 2\overline{T}^{(1)} = 0,\ \overline{T}_{\xi\xi}^{(1)} - 2\overline{w}^{(1)} = 0,$$
$$\bar{u}_\xi^{(2)} + \bar{v}_y^{(1)} + \overline{w}_z^{(1)} = 0. \tag{5.5 c}$$

We shall use the boundary layer equations and their solutions to derive boundary conditions on $T^{(0)}$ and $T^{(1)}$ for equations (5.3) and (5.4).

From (5.5a) and (5.5b) we have $\bar{u}^{(0)} = 0$, $\bar{v}^{(0)} = 0$. Now since $u = 0$ and $v = 0$ on $x = \pm 1$ by assumption, we have $u^{(0)} = v^{(0)} = 0$ on $x = \pm 1$. Thus from the first three of equations (5.2a) we deduce

$$\overline{T}^{(0)} = 0 \quad \text{and} \quad \overline{T}_x^{(0)} = 0 \quad \text{on} \quad x = \pm 1. \tag{5.6}$$

The solution to the problem posed by equations (5.3) and (5.6) is easily seen to be

$$T^{(0)} = 0. \tag{5.7}$$

To get the boundary conditions on $T^{(1)}$ we proceed as follows: Since $w^{(0)} = 0$ everywhere and $w = 0$ on $x = \pm 1$, it follows that $\overline{w}^{(0)} = 0$ on $x = \pm 1$. Then the buoyancy layer solution from equations (3.6) is the same as equation (4.6). Integration of equation (3.8) yields

$$\bar{u}^{(1)} = \pm \frac{\overline{T}_{0z}^{(0)}}{\sqrt{2}} \sin\ (\xi + \pi/4) e^{-\xi}, \tag{5.8}$$

where $\overline{T}_0^{(0)} = \overline{T}^{(0)}|_{\xi=0}$. Since $T^{(0)} \equiv 0$ from (5.7) and $\overline{T}_0^{(0)} = = T|_{\xi=0} - T^{(0)}$ we get

$$\tilde{u}^{(1)} = \pm\frac{1}{\sqrt{2}} T_z\big|_{\xi=0} \sin(\xi+\pi/4)e^{-\xi} \text{ near } x = \mp 1. \tag{5.9}$$

Integrating the second of equations (5.5c) with $\tilde{v}^{(1)} \to 0$ as $\xi \to \infty$ yields

$$\tilde{v}^{(1)} = \mp\frac{F}{\sqrt{2}} T_z\big|_{\xi=0} \cos(\xi+\pi/4)e^{-\xi} \text{ near } x = \pm 1. \tag{5.10}$$

Now $u = 0$ and $v = 0$ so $\tilde{u}^{(1)} = -u^{(1)}$, $\tilde{v}^{(1)} = -v^{(1)}$ on $x = \pm 1$. Thus we have

$$u^{(1)} = \pm\tfrac{1}{2} T_z\big|_{\xi=0},\ v^{(1)} = \mp\frac{F}{2} T_z \text{ on } x = \mp 1. \tag{5.11}$$

Now we use the geostrophic relations from equations (5.2b) to write the conditions (5.11) in terms of $T^{(1)}$. We have for *symmetric heating*, i.e. $T = -T_0 \sin y \sin z$ on $x = \pm 1$, the conditions

$$\left.\begin{aligned} T_x^{(1)}\big|_{x=-1} &= -\frac{F^2 T_0}{2}\sin y \sin z, \\ T_x^{(1)}\big|_{x=1} &= \frac{F^2 T_0}{2}\sin y \sin z. \\ T^{(1)}\big|_{x=-1} &= -\frac{F T_0}{2}\cos y \sin z, \\ T^{(1)}\big|_{x=1} &= \frac{F T_0}{2}\cos y \sin z \end{aligned}\right\} \tag{5.12}$$

and for *asymmetric heating* the same but with a negative sign for each term at $x = +1$.

The solution to equation (5.4) with the conditions (5.12) is

$$T^{(1)} = (A(x)\sin y + B(x)\cos y)\sin z, \tag{5.13}$$

where for *symmetric heating*

$$A(x) = \frac{F^2 T_0}{2} \times \left[\frac{\sinh\sqrt{1+F^2}\,\sinh(\sqrt{2}x) - \sinh\sqrt{2}\,\sinh(\sqrt{1+F^2}\,x)}{\sqrt{2}\cosh\sqrt{2}\,\sinh\sqrt{1+F^2} - \sqrt{1+F^2}\cosh\sqrt{1+F^2}\,\sinh\sqrt{2}}\right], \tag{5.14a}$$

$$B(x) = \frac{F T_0}{2} \times \left[\frac{\sqrt{1+F^2}\cosh\sqrt{1+F^2}\,\sinh(\sqrt{2}x) - \sqrt{2}\cosh\sqrt{2}\,\sinh(\sqrt{1+F^2}\,x)}{\sqrt{1+F^2}\cosh\sqrt{1+F^2}\,\sinh\sqrt{2} - \sqrt{2}\cosh\sqrt{2}\,\sinh\sqrt{1+F^2}}\right]. \tag{5.14b}$$

With *anti-symmetric heating* the solution is the same as above with T_0 replaced by $-T_0$, and with sinh and cosh interchanged.

Because the interior flow to this order is two-dimensional a stream function can be defined by

$$v^{(1)} = \psi_x, \quad u^{(1)} = -\psi_y$$

and we have

$$\psi = -\frac{1}{F}(A \sin y + B\cos y)\cos z. \tag{5.15}$$

Hence, the results for this problem show that $T^{(0)}$ vanishes irrespective of the symmetry of the heating and the first correction to interior temperature is at $O(R^{\frac{1}{2}})$. The result differs very much from the two-dimensional system in which the interior temperature varied roughly linearly from the value at one boundary to that at the other. It also differs from the three-dimensional picture for the system S (vid. I) in which we concluded that the interior temperatures, $T^{(0)}$ and $T^{(1)}$, vanished identically. In terms of order-of-magnitude the present result lies between the two dimensional-system S_Ω and the three-dimensional system S.

Note that at any level the interior streamlines coincide (either parallel or anti-parallel) with the isotherms. We show the horizontal plan form of these contours in figures 1 and 2. For symmetric heating a fluid particle in a region of heating moves upward in the boundary layer, enters the interior in the region where $\partial T_0/\partial z < 0$, executes the path partially shown in Fig. 1 and returns to the boundary at a value of y where cooling takes place, sinks as it cools, leaves the boundary layer when it has achieved the temperature of the ambient fluid, follows the same path as before but with opposite direction, enters the boundary layer again at the original value of y and rises to its original position and temperature. The particle path differs from that of the non-rotating case in that the fluid enters

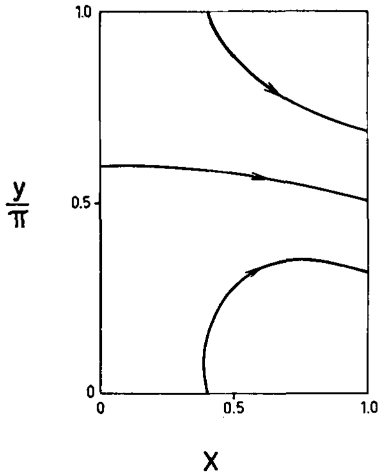

Fig. 1. The pattern of streamlines for interior flow of $O(R^{\frac{1}{2}})$ is exhibited in the region $0 \leqslant x \leqslant 1$ for the three-dimensional case with symmetric heating. The fluid is heated at $x = 1$ with $\partial T0/\partial z > 0$ in the region where the flow leaves the boundary and is cooled at $x = 1$ where the flow enters the boundary. The effect of rotation is evident from the skewness of the trajectories; in the non-rotating case the streamlines enter and leave the boundaries symmetrically. The boundary layer near $x = 1$ where the fluid rises before it leaves the boundary region and sinks after it enters is not shown. The pattern at a lower level is anti-parallel to the one shown. The flow pattern is repeated in y with period 2π. In the region $\pi \leqslant y \leqslant 2\pi$ the pattern is identical except for the sign, i.e., $\psi(y + \pi) = -\psi(y)$.

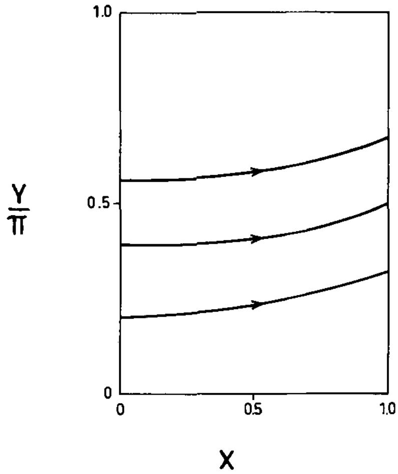

Fig. 2. The pattern of streamlines for interior flow of $O(R^{\frac{1}{2}})$ is exhibited in the region $0 \leqslant x \leqslant 1$ for the three-dimensional case with anti-symmetric heating. The fluid is heated with $\partial To/\partial z > 0$ at $x = -1$ and cooled at $x = 1$. Particles sink in the boundary layer (not shown) near $x = 1$. At a lower level the flow is anti-parallel to the one in the figure. Rotation causes the streamlines to leave and enter the boundary at an angle and to bend further to the right than they do in the non-rotating case. Also, in the latter case the streamline pattern is symmetric about the planes $y = (2m - 1)\pi/2$ (m an integer). In the present flow the pattern is anti-symmetric about $y = m\pi$.

and leaves the boundary layer at an angle to the normal and the streamlines in the interior reflect the presence of rotation through the skewness of the pattern. In the non-rotating case the streamlines are symmetric about the plane $y = \pi/2$ and anti-symmetric about the plane $y = 0$.

Anti-symmetric heating gives rise to the interior streamline (and isotherm) pattern shown in figure 2. Again the rotation causes the streamlines to be inclined to the normal at the boundaries and the entire pattern between $y = 0$ and $y = \pi$ shows a general curvature in the same direction. In the non-rotating case the pattern is symmetric about the plane $y = \pi/2$.

Case $F = O(R^{\frac{1}{2}})$. ($F = fR^{\frac{1}{2}}$, $f = O(1)$)

We shall not treat this case in the same detail as the previous ones because we can show relatively easily that the results differ from those of system S (i.e. the non-rotating case) only in some of the details of the flow in higher order.

Interior

$$O(R): \quad p_x^{(0)} = 0,\ p_y^{(0)} = 0,\ p_z^{(0)} = 2T^{(0)},$$
$$w^{(0)} = 0,\ u_x^{(0)} + v_y^{(0)} = 0, \tag{5.16 o}$$

hence $\quad T_{0x} = 0,\ T_{0y} = 0;$

$$O(R^{\frac{1}{2}}): \quad p_x^{(1)} = 2fv^{(0)},\ p_y^{(1)} = -2fu^{(0)},\ p_z^{(1)} = 2T^{(1)},$$
$$w^{(1)} = 0,\ u_x^{(1)} + v_y^{(1)} = 0; \tag{5.16 b}$$

$$O(R): \quad p_x^{(2)} = 2fv^{(1)} + \nabla^2 u^{(0)},\ p_y^{(2)} = -2fu^{(1)} + \nabla^2 v^{(0)},$$
$$p_z^{(2)} = 2T^{(2)},\ 2w^{(2)} = \nabla^2 T^{(0)},\ u_x^{(2)} + v_y^{(2)} + w_z^{(2)} = 0. \tag{5.16 c}$$

Boundary layer

The pertinent boundary layer equations are (4.5) and the conclusions are effectively the same for the boundary layer variables. Thus

$$\bar{p}^{(0)} = 0,\ \bar{u}^{(0)} = 0,\ \bar{v}^{(0)} = 0,\ \bar{p}^{(1)} = 0,\ \bar{u}^{(1)}\big|_{x=\mp 1}$$
$$= \pm \tfrac{1}{2} T_z\big|_{x=\mp 1}. \tag{5.17}$$

We note immediately that $T^{(0)} = 0$ throughout because of the imposed harmonic y dependence. That $T^{(1)}$ likewise vanishes can be shown by the following argument: For the first two of equations (5.16c) we note that

$$\nabla^2 (u_y^{(0)} - v_x^{(0)}) = 0. \tag{5.18}$$

Since $\bar{u}^{(0)} = 0$ and $\bar{v}^{(0)} = 0$, the boundary conditions $u = 0$, $v = 0$ on $x = \pm 1$, require that $u^{(0)} = 0$, $v^{(0)} = 0$ on $x = \pm 1$ and that means that equation (5.18) together with the last of (5.16a) requires that

$$u^{(0)} = 0, \; v^{(0)} = 0 \quad \text{throughout}. \tag{5.19}$$

Hence, from the set (5.16b) we conclude that

$$T_x^{(1)} = 0, \; T_y^{(1)} = 0. \tag{5.20}$$

We can now go on to solve for $u^{(1)}$ and $v^{(1)}$ and hence $T^{(2)}$, but we note that we are simply solving for higher-order corrections to the solutions for the non-rotating system just as we might expect would be introduced by a higher order F. The point is that no lower-order changes of the type that we found in the two-dimensional systems are present in the three-dimensional S_Ω system when $F = O(R^{\frac{1}{2}})$.

6. Summary and discussion

We first summarize the results of the simple systems S and Ω and then discuss the results of the combined systems S_Ω and Ω_S.

For a homogeneous, rotating fluid, slow steady flow is determined by the processes which take place in Ekman boundary layers. When boundary conditions on vertical vorticity are applied to the upper and lower boundaries, the resultant flow in the interior of the fluid is such that the relative vorticity is the average of the vorticities at the two boundaries. When the applied conditions are symmetric, the interior vorticity equals the boundary vorticity and no boundary layers exist in the steady state (although boundary layers are necessary in the transient evolution to the steady state in an initial value problem).

When the applied conditions are anti-symmetric, there is no relative vorticity in the interior. In the latter case Ekman boundary layers are necessary to adjust the boundary vorticity to the interior. If the boundary vorticity is uniform, the Ekman layer acts solely to bring about this adjustment. If the applied vorticity varies along the boundary, an inflow or outflow from the boundary layer is created and the Ekman layer provides the necessary flow out of or into the interior. These remarks are pertinent for both two- and three-dimensional flow.

A stratified, non-rotating fluid has an analogous behavior when the flow is steady, linear and two-dimensional (uniform in one horizontal dimension). In this case, heating (or cooling) at the lateral boundaries is transmitted to a thin vertical slab of fluid by conduction and causes the fluid to rise (or sink). If the applied heating is symmetric, the rise of fluid near the two boundaries necessitates a sinking of fluid in the interior during a transient period and eventually the interior of the fluid is heated because the forced sinking brings fluid of higher temperature to lower levels. In the steady state there is no relative motion once the temperature of the interior fluid corresponds to the applied temperature and the system is thus effectively restratified.

When the applied heating is anti-symmetric, warmed fluid rises and cooled fluid sinks near the appropriate boundaries and the interior fluid maintains its original temperature. In this case buoyancy layers are necessary to adjust the temperature of the fluid from boundary to interior values. When the applied temperatures vary along the boundaries, this causes more rapid vertical velocities in regions of more intense heating and a vertical divergence results. Thus, a horizontal flux of fluid from the interior is necessary for mass conservation. This flux of fluid to or from the interior through the action of the buoyancy layer is the thermal counterpart of Ekman suction in rotating fluids.

When the applied temperatures can vary also in the horizontal direction along the boundaries, the fluid can adjust to the variable boundary temperatures by flowing in the horizontal direction parallel to the boundaries. Flow in this direction is unimpeded by the stratification. This added degree of freedom brings about distinct changes in the behavior of the fluid. The case with symmetric heating now involves horizontal flows in the interior with no change in the interior temperature at either zero or first order, i.e. the fluid is not restratified. In fact the thermal structure of the flow is the same in the cases with symmetric and anti-symmetric heating, the only difference between the two cases being in the detailed pattern of the interior streamlines. The interior flow is dominated in both cases by diffusive processes; the velocities are first-order and join to buoyancy boundary layers, which are necessary to adjust the zero interior temperatures to boundary values.

The combined problem, i.e., adding stratification to the rotating system or adding rotation to the stratified case, shows quite different behavior from the simple systems. In the two-dimensional flows the systems again behave analogously.

The addition of stratification of $O(1)$ to system Ω inhibits vertical velocities. For the case of symmetric boundary conditions the stratification does not drastically alter the results of the simple rotating system which involved a uniform vorticity from one boundary to the other. The only change which is brought about by the stratification is to cause the process of diffusion to play a large role but diffusion does not have a large effect on variables which are already uniform in one direction and slowly varying in the other.

The flow in the anti-symmetric situation, on the other hand, is completely altered. Since the stratification inhibits vertical velocities, the required vertical velocity of the Ekman suction is no longer possible and in fact the Ekman layer disappears. The entire flow is now dominated by diffusive processes and the interior vorticity varies uniformly from the value at one boundary to the value of the other. Associated with this zero-order vertical shear in the zonal velocity is a zero-order temperature variation across the fluid. This is the familiar thermal wind balance.

Even though we have drawn a distinction between the cases with symmetric and anti-symmetric boundary conditions it should be noted that there is no qualitative difference in the physical processes which dominate when stratification is added to system Ω. In the symmetric case there was uniformity to begin with and the effect of diffusive processes in the interior does not change the situation very much. In the anti-symmetric situation stratification inhibits the vertical velocities which dominated the interior in the simple rotating system and diffusion now brings about a huge change in the behavior of the interior. But the difference between the symmetric and anti-symmetric situations is quantitative, not qualitative.

The two-dimensional system S_Ω is analogous to the system Ω_S. The addition of rotation of $O(1)$ to system S inhibits the outflow from the buoyancy layer so that diffusive processes dominate in the interior. Again the effect is not quantitatively noticeable in the case with symmetric heating and the system takes on the appearance of being restratified as it was for the non-rotating case. With anti-symmetric heating, however, the inhibition of the buoyancy layer outflow by rotation eliminates the buoyancy layer and diffusion dominates, causing a uniform change of temperature across the interior.

When the rotation is only $O(R^{\frac{1}{4}})$ in system S_Ω, the two dimensional symmetric problem is unchanged from the non-rotating one. Slow rotation introduces a small Coriolis term which is balanced by diffusion in the interior. However, since there was no buoyancy layer outflow in system S (the latter provides the zero-order solution) there is no Coriolis force to balance the diffusion of zonal velocity so there is no change introduced by slow rotation. For the case of anti-symmetric heating there is a buoyancy layer outflow at zero-order (the solution to system S) and even though the Coriolis force is small, a zero-order zonal velocity is generated, the diffusion of which balances the Coriolis force. These two effects are not very significant dynamically because $F = O(R^{\frac{1}{4}})$ but the zonal velocity is $O(1)$. The zero-order zonal velocity is associated with a first-order temperature change through the thermal wind relation.

Analogous results hold for the system Ω_S when the stratification is $O(E^{\frac{1}{4}})$.

When the boundary conditions vary in both directions parallel to the boundaries, the system Ω_S shows no qualitative change from the two-dimensional rotating case with stratification. The three-dimensional stratified system with rotation of $O(1)$ does change, however. The velocity is still of first order and is dominated by diffusion just as in the non-rotating case but the interior temperature has a non-zero value at $O(R^{\frac{1}{4}})$, i.e., an order lower then in the non-rotating case and active buoyancy layers exist. The effects of rotation can also be seen in the pattern of streamlines in the interior. The flow enters and leaves the boundary layer at an angle to the normal for both symmetric and anti-symmetric heating and the interior streamlines at any given level are shown in figures 1 and 2.

Rotation of $O(R^{\frac{1}{4}})$ has no significant effect on the three-dimensional stratified problem.

Finally, we note that the form of the equations in the stratified, rotating system (with both processes of $O(1)$) precludes the existence of the Stewartson (1957) boundary layers. However, these can exist in the cases above where we investigated the effects of small rotation in stratified flows and vice versa.

A concluding remark

The foregoing investigation and that given in paper I contain some results in the study of fluids which are rotating and/or stratified. It is interesting to see how the analogy works in these linear steady systems and these two papers may be of some help in extending the investigation to non-linear or non-steady problems. My own feeling about the present approach is that the restrictions to steady, linear flows is much too drastic a limitation for the combine drotatings,tratified system. The interior fluid is dominated by diffusive processes and one cannot help but feel uncomfortable about the applicability of a method which relies on the dominating influence of diffusion throughout the domain of interest. For the individual systems (Ω or S) the approach promises to be more useful although even in system S the three-dimensional case contains a flow in the interior which is governed by diffusion. However, this result is higher-order and one does deduce significant lower-order results independent of the diffusive processes.

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Note

After this paper was submitted for publication I had an opportunity to see the manuscript of a second paper by Barcilon & Pedlosky (A Unified Linear Theory of Homogeneous and Stratified Rotating Fluids, J. Fluid Mech. in press) in which they discuss the behaviour of fluids which are stratified and rotating with $E^{\frac{1}{3}} \leqslant F^{-1} \leqslant E^{\frac{1}{4}}$. They show that side wall boundary layers exhibit a triple structure with a buoyancy layer of thickness $R^{\frac{1}{4}}$ c "hydrostatic, baroclinic layer of thickness F^{-1} and a Stewartson layer of thickness $E^{\frac{1}{4}}$. As in their earlier paper their discussion is concerned with the changes which stratification brings to system Ω. It is clear from the foregoing work that exactly analogous results can be derived for system S_{Ω}, with a triple boundary layer structure when $R^{\frac{1}{3}} \leqslant F \leqslant R^{\frac{1}{4}}$.

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АНАЛОГИЧНОЕ ПОВЕДЕНИЕ ВРАЩАЮЩИХСЯ И СТРАТИФИЦИРОВАННЫХ ЖИДКОСТЕЙ

В настоящей статье продолжается исследование аналогии между вращающимися и стратифицированными жидкостями. Анализ расширен на включение вращения в стратифицированную систему и стратификации во вращающуюся систему. Хотя основные уравнения одни и теже, мы различаем эти две системы на основе сил, вызывающих движение. Система S_Ω соответствует добавлению вращения в стратифицированную систему S предыдущей статьи, где течение поддерживалось дифференциальным разогревом на боковых границах. В систему Ω_S добавляется стратификация к вращающейся жидкости Ω, движение в которой поддерживалось дифференциальной вертикальной завихренностью на верхней и нижней границах. Показано, что точная аналогия продолжает иметь место, когда течение двумерно и когда определенные переменные идентифицированы друг с другом в обеих системах. Комбинирование вращения и стратификации одного порядка подавляет отток от пограничных слоев, необходимость существования которого была найдена для простых систем, при этом также силы плавучести и экмановские пограничные слои не играют роли в двумерном течении. Вся жидкость целиком управляется процессами диффузии. В трехмерных течениях система S_Ω имеет существенные пограничные слои. Когда вводится дополнительный процесс (вращение в систему S_Ω или стратификация в систему Ω_S) в качестве поправки более высокого порядка, то он может вести к изменениям в нулевом порядке для двумерных течений, но вызывает лишь эффекты более высоких порядков в трехмерной системе S_Ω. Показано также, что вращающаяся стратифицированная жидкость не имеет пограничных слоев типа, найденных Стюартсоном (1957) для вращающихся однородных жидкостей.