An approximation to boundary layer wind profiles

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ABSTRACT

A procedure for computing approximate solutions of the boundary layer equations for a neutrally stratified atmosphere is described. The approximation equations, which are derived using perturbation expansion theory, are valid for cases when the cross-isobaric flow is small. The first approximation equations are integrated analytically and then applied to cases for which the gradient flow is either straight or curved. Comparison of the approximations with numerical solutions of the exact equations indicates that the approximations are generally useful under normal conditions of flow near the surface of the earth.

Introduction

A useful concept in the analysis of the atmospheric boundary layer is Prandtl's mixing length hypothesis. The hypothesis, when used in conjunction with the assumptions concerning the constancy of the turbulent shear stress and the linearity of the mixing length with height, leads to the familiar logarithmic wind profile. The logarithmic variation is generally observed in the atmosphere to be accurate under conditions of neutral thermal stratification and in the layers close to the ground. Figure 1 shows an example of wind observations by Thuillier & Lappe (1964) at the Cedar Hill tower located near Dallas, Texas. Note the decreasing accuracy of the logarithmic representation with height; this inaccuracy appears to be associated mainly with the decreasing validity of the constant stress assumption with height; the inertial and coriolis forces become increasingly important relative to the frictional forces at high elevations.

The logarithmic wind distributions which are obtained with the aid of the mixing length hypothesis may be considered as solutions of the full boundary layer equations for the highly restricted case for which the pressure gradient, inertial, and coriolis terms are negligible. When these terms are no longer negligible, the equations become highly nonlinear and solutions are obtainable only by numerical methods. Examples of numerical solutions for steady, straight, and uniformly horizontal flow are those obtained by Blackadar (1962) and Ohmstede & Appleby (1964). Solutions for the general case of curved, horizontally-nonuniform flow are much more difficult to obtain due to the greatly increased nonlinearity of the equations.

The difficulty of obtaining solutions to the basic equations indicates the desirability of analyzing the equations to find out if useful approximate solutions can be obtained by perturbation expansion methods (Bellman, 1964). For certain practical problems it would be desirable to find solutions which are valid when the coriolis and inertial effects are small but yet not negligible. The analysis has been attempted. It has been found that asymptotic solutions under conditions of small cross-isobaric flow may be calculated conveniently by using a stepwise procedure. In the succeeding sections we will describe the analysis and present example of calculated wind distributions.

Theoretical analysis

We will analyze the effects of curvature and horizontal non-homogeneity in the large scale flow by considering a flow pattern which may have these characteristics. A simple example of such a pattern is axially symmetric flow with variations along the vertical (z) and radial (r) directions. The appropriate equations in a cylindrical coordinate system are

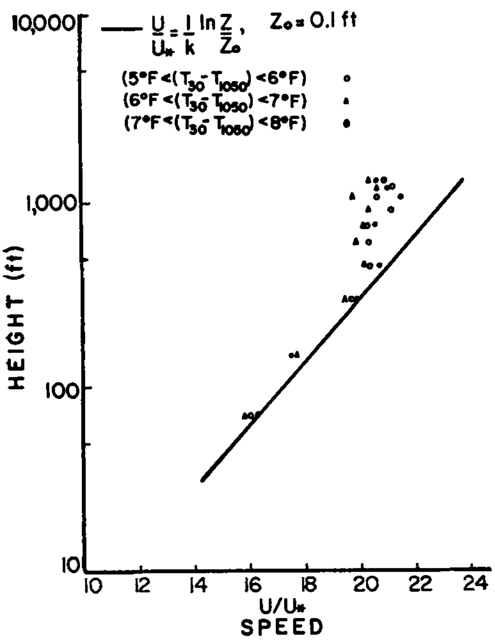

Fig. 1. Average wind profiles plotted as logarithmic function of height. (*After Thuillier & Lappe, 1964.*)

$$\frac{\partial u}{\partial t} + v\left(\frac{\partial u}{\partial r} + \frac{u}{r} + f\right) + w\frac{\partial u}{\partial z} - \frac{\partial}{\partial z}\left(K\frac{\partial u}{\partial z}\right) = 0, \tag{1}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} - \frac{u^2}{r} - fu - \frac{\partial}{\partial z}\left(K\frac{\partial v}{\partial z}\right)$$

$$= -\frac{1}{\varrho_s}\frac{\partial p}{\partial r} = -fu_g \tag{2}$$

$$\frac{\partial}{\partial z}(\varrho_s rw) + \frac{\partial}{\partial r}(\varrho_s rv) = 0, \tag{3}$$

where u, v, and w are the wind components parallel to the tangential, radial, and vertical directions, respectively; ϱ_s is a standard air density (function of z only), K is the eddy viscosity, f is the coriolis parameter, u_g is the geostrophic wind speed, p is pressure, and t is time. If the geostrophic wind and the eddy viscosity are given, Eqs. (1) to (3) form a set of simultaneous equations with u, v, and w as the unknowns and r, z, and t as the independent variables.

The variables will be rendered nondimensional such that all are of the order unity. For this purpose, let us introduce the characteristic scales (L, D, R), characteristic speeds (G, V), and characteristic eddy viscosity (K_m) and then define the dimensionless variables as

$$\hat{u} \equiv \frac{u}{G}, \quad \hat{v} \equiv \frac{v}{V}, \quad \hat{w} \equiv \frac{wL}{VD}, \quad \hat{t} \equiv \frac{Vt}{L},$$

$$\hat{r} \equiv \frac{r}{R}, \quad \hat{z} \equiv \frac{z}{D}, \quad \hat{K} \equiv \frac{K}{K_m}, \quad \frac{\partial}{\partial \hat{r}} \equiv L\frac{\partial}{\partial r}.$$

The characteristic tangential velocity component, G, will be assumed to be the gradient wind. Typical orders of magnitude of the characteristic quantities are

$L \sim R \sim 10^8$ cm, $D \sim 10^4$ cm, $K_m \sim 10^5$ cm^2 sec^{-1},

$G \sim 10^3$ cm sec^{-1}, $V \sim 10^2$ cm sec^{-1}.

Substituting the definitions into Eqs. (1) to (3), we obtain

$$\frac{D^2 V}{LK_m}\left[\frac{\partial \hat{u}}{\partial \hat{t}} + \hat{v}\left(\frac{\partial \hat{u}}{\partial \hat{r}} + \frac{L}{R}\frac{\hat{u}}{\hat{r}}\right) + \hat{w}\frac{\partial \hat{u}}{\partial \hat{z}}\right] + \frac{D^2 Vf}{GK_m}\hat{v}$$
$$= \frac{\partial \hat{K}}{\partial \hat{z}}\frac{\partial \hat{u}}{\partial \hat{z}} + \hat{K}\frac{\partial^2 \hat{u}}{\partial \hat{z}^2}, \tag{4}$$

$$\frac{D^2 V}{LK_m}\left(\frac{\partial \hat{v}}{\partial \hat{t}} + \hat{v}\frac{\partial \hat{v}}{\partial \hat{r}} + \hat{w}\frac{\partial \hat{v}}{\partial \hat{z}}\right) - \frac{GD}{K_m}\left(\frac{Df}{V}\right)(\hat{u} - \hat{u}_g)$$
$$- \frac{GD}{K_m}\left(\frac{DG}{VR}\right)\frac{\hat{u}^2}{\hat{r}} = \frac{\partial}{\partial \hat{z}}\left(\hat{K}\frac{\partial \hat{v}}{\partial \hat{z}}\right), \tag{5}$$

$$\frac{\partial}{\partial \hat{z}}(\varrho_s \hat{r}\hat{w}) + \frac{\partial}{\partial \hat{r}}(\varrho_s \hat{r}\hat{v}) = 0. \tag{6}$$

Since the characteristic quantities have been chosen such that the nondimensional variables are of the order of unity, the relative order of magnitude of the terms in the equations may be estimated by examining the values of the dimensionless number, D^2V/LK_m, D^2Vf/GK_m, GD/K_m, Df/V, and DG/VR.

In tropical latitudes where $f \sim 10^{-5}$ sec^{-1} the magnitudes of these numbers which correspond to the typical values of the nondimensionalyzing quantities listed above are:

$$\frac{D^2 V}{LK_m} \sim 10^{-3}, \quad \frac{D^2 Vf}{GK_m} \sim 10^{-3}, \quad \frac{GD}{K_m} \sim 10^2,$$

$$\frac{Df}{V} \sim 10^{-3}, \quad \frac{DG}{VR} \sim 10^{-3}.$$

It may be seen that, in general, the terms involving D^2V/LK_m and D^2Vf/GK_m are relatively small. We will now restrict our analysis to the class of motions where these two numbers (both denoted by ε) are small compared to unity. It follows that, to the order of ε, the boundary layer equations may be approximated by

$$\frac{\partial}{\partial z}\left(K\frac{\partial u_0}{\partial z}\right) = 0, \tag{7}$$

$$\frac{\partial}{\partial z}\left(K\frac{\partial v_0}{\partial z}\right) = -\frac{GD}{K_m}\left(\frac{Df}{V}\right)(u_0 - u_g) - \frac{GD}{K_m}\left(\frac{DG}{VR}\right)\frac{u_0^2}{r}, \tag{8}$$

$$\frac{\partial}{\partial z}(\varrho_s r w_0) = -\frac{\partial}{\partial r}(\varrho_s r v_0). \tag{9}$$

Here the sign "$\wedge$" has been omitted for simplicity; henceforth the variables refer to the dimensionless ones unless otherwise specified. The subscript in u_0, v_0, and w_0 denote that these velocity components are the lowest order or first approximations. The equations for the higher order approximations may be obtained conveniently by an ordering procedure which is based on a power series representation in ε of the dependent variables. Thus

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots.. \varepsilon^n u_n,$$

$$v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots. \varepsilon^n v_n,$$

$$w = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \ldots. \varepsilon^n w_n.$$

Substituting these in Eqs. (4) to (6) and equating like powers in ε, one obtains the equations for the higher order approximations. In particular, the equations governing the first order effects of ε are

$$\frac{\partial}{\partial z}\left(K\frac{\partial u_1}{\partial z}\right) = \frac{\partial u_0}{\partial t} + v_0\left(\frac{\partial u_0}{\partial r} + \frac{L}{R}\frac{u_0}{r} + \frac{fL}{G}\right) + w_0\frac{\partial u_0}{\partial z}. \tag{10}$$

$$\frac{\partial}{\partial z}\left(K\frac{\partial v_1}{\partial z}\right) = \frac{\partial v_0}{\partial t} + v_0\frac{\partial v_0}{\partial r} + w_0\frac{\partial v_0}{\partial z} - \frac{GD}{K_m}\left(\frac{Df}{V}\right)u_1 - 2\left(\frac{DG}{K_m}\right)\left(\frac{DG}{VR}\right)\frac{u_1 u_0}{r}, \tag{11}$$

$$\frac{\partial}{\partial z}(\varrho_s r w_1) = -\frac{\partial}{\partial r}(\varrho_s r v_1). \tag{12}$$

The nth order perturbations are governed by equations which have exactly the same form as Eqs. (10) to (12); they can be written formally by replacing the subscripts "1" and "0" by "n" and "n-1", respectively. The euqations indicate that the various approximations could be calculated successively, starting with the lowest order approximation (in the order u_0, v_0, w_0, u_1, v_1, w_1, etc., using Eqs. (7), (8), (9), (10), (11), (12), etc. Thus the set of nonlinear equations, Eqs. (1) to (3), is transformed into a simpler set. The first equation of this set, Eq. (7), is homogeneous while the rest are inhomogeneous. The inhomogeneous terms, which act as forcing functions, depend on previously calculated solutions. This procedure therefore represents a considerable simplification relative to a straightforward numerical integration of the basic equations, (1) to (3).

A first approximation solution

Before the equations for the various approximations could be integrated, one has to specify the functional form of the eddy viscosity. A rather realistic distribution has been suggested by Blackadar (1962); the expression is

$$K = \left(\frac{kz}{1 + kz\lambda^{-1}}\right)^2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right]^{\frac{1}{2}}, \tag{13}$$

where k is the Karman constant and λ is an empirical parameter representing the upper limit which the mixing length approaches asymptotically at upper levels. The expression for λ used by Blackadar is

$$\lambda = \frac{27 \times 10^{-5} G}{f},$$

where G is effectively the gradient wind.

The use of Eq. (13) in our analysis, although theoretically possible, introduces a considerable amount of complexity in the solution of the equations. However, if the total wind shear is approximated by the shear of the first approximation for u, $\partial u_0/\partial z$, one may obtain analytic solutions. We have, therefore, approximated the expression for K in Eq. (13), by K_0

$$K_0 \equiv \left[\frac{k(z+z_0)}{1 + k(z+z_0)\lambda^{-1}}\right]^2\left|\frac{\partial u_0}{\partial z}\right|,$$

where z_0 is the roughness parameter. The replacement of z by $z+z_0$ shifts the origin of the vertical coordinate so that the no-slip boundary condition could be applied at $z=0$. This expression for the exchange coefficient, together with the dimensional forms of Eqs. (7), (8), and (9),

$$\frac{\partial}{\partial z}\left(K_0\frac{\partial u_0}{\partial z}\right)=0, \tag{14}$$

$$\frac{\partial}{\partial z}\left(K_0\frac{\partial v_0}{\partial z}\right)=-\frac{u_0^2}{r}-f(u_0-u_g), \tag{15}$$

$$\frac{\partial}{\partial z}(\varrho_s r w_0)=-\frac{\partial}{\partial r}(\varrho_s r v_0), \tag{16}$$

provides a set of equations for determining the first approximation to the motion field.

The integrals of (14) and (15) are

$$u_0=\frac{u_*}{k}\ln\frac{z+z_0}{z_0}+\frac{u_* z}{\lambda},$$

$$v_0=\left[\frac{fz_0}{u_*}\frac{F(z)}{k}+\frac{z_0 E(z)}{\lambda}\right],$$

where $F(z)$

$$\begin{aligned}
&\equiv\left(\frac{z+z_0}{z_0}\right)\left\{u_g-\frac{u_*}{k}\left(\ln\frac{z+z_0}{z_0}-2\right)+\frac{u_* z_0}{\lambda}\right.\\
&\quad-\frac{1}{fr}\left(\frac{u_*}{k}\right)^2\left[\left(\ln\frac{z+z_0}{z_0}\right)^2-4\ln\frac{z+z_0}{z_0}+6\right]\\
&\quad\left.+\frac{2}{fr}\left(\frac{u_*}{k}\right)\left(\frac{u_* z_0}{\lambda}\right)\left(\ln\frac{z+z_0}{z_0}-2\right)-\frac{1}{fr}\left(\frac{u_* z_0}{\lambda}\right)^2\right\}\\
&\quad+\left(\frac{z+z_0}{z_0}\right)^2\left[\frac{1}{2fr}\left(\frac{u_* z_0}{\lambda}\right)^2-\frac{u_* z_0}{4\lambda}-\frac{u_*}{2frk}\left(\frac{u_* z_0}{\lambda}\right)\right.\\
&\quad\left.\times\left(\ln\frac{z+z_0}{z_0}-1\right)\right]-\frac{1}{9fr}\left(\frac{u_* z_0}{\lambda}\right)^2\left(\frac{z+z_0}{z_0}\right)^3\\
&\quad+c_1\ln\frac{z+z_0}{z_0}
\end{aligned}$$

and $E(z)$

$$\begin{aligned}
&\equiv\frac{\lambda c_2}{z_0}+c_1\left(\frac{z+z_0}{z_0}\right)+\left(\frac{z+z_0}{z_0}\right)^2\left\{\frac{u_g}{2}-\frac{u_*}{2k}\right.\\
&\quad\times\left(\ln\frac{z+z_0}{z_0}-\frac{3}{2}\right)+\frac{u_* z_0}{2\lambda}-\frac{1}{2fr}\left(\frac{u_*}{k}\right)^2\\
&\quad\times\left[\left(\ln\frac{z+z_0}{z_0}\right)^2-3\ln\frac{z+z_0}{z_0}+\frac{7}{2}\right]+\frac{u_*}{frk}\left(\frac{u_* z_0}{\lambda}\right)\\
&\quad\left.\times\left(\ln\frac{z+z_0}{z_0}-\frac{3}{2}\right)-\frac{1}{2fr}\left(\frac{u_* z_0}{\lambda}\right)^2\right\}+\left(\frac{z+z_0}{z_0}\right)^3\\
&\quad\times\left[\frac{1}{3fr}\left(\frac{u_* z_0}{\lambda}\right)^2-\frac{u_* z_0}{6\lambda}-\frac{u_*}{3frk}\left(\frac{u_* z_0}{\lambda}\right)\right.\\
&\quad\left.\times\left(\ln\frac{z+z_0}{z_0}-\frac{5}{6}\right)\right]-\left(\frac{z+z_0}{z_0}\right)^4\left(\frac{1}{12fr}\right)\left(\frac{u_* z_0}{\lambda}\right)^2.
\end{aligned}$$

One constant of integration has been eliminated by applying the no-slip boundary condition for u_0 at $z=0$. The three remaining constants of integration (u_*, c_1, c_2) may be evaluated by applying the no-slip condition for v_0 at $z=0$ and by prescribing boundary values for u_0 and v_0 at upper levels.

The expression for u_0 indicates that, to the first approximation, the wind component along the isobars is essentially logarithmic for small values of z; it becomes increasingly linear as the altitude increases.

Using the expression for v_0, one may integrate Eq. (16) to obtain the corresponding expression for w_0.

The accuracy of the first approximations may be tested by comparing them with solutions of the exact equations, Eqs. (1), (2), (3), and (13). In the case of steady, horizontally uniform, straight flow, Eqs. (1), (2), and (3) reduce to

$$\frac{\partial}{\partial z}\left(K\frac{\partial u}{\partial z}\right)=fv,$$

$$\frac{\partial}{\partial z}\left(K\frac{\partial v}{\partial z}\right)=-f(u-u_g),$$

$$w=0.$$

The corresponding first approximation equations are

$$\frac{\partial}{\partial z}\left(K_0\frac{\partial u_0}{\partial z}\right)=0,$$

$$\frac{\partial}{\partial z}\left(K_0\frac{\partial v_0}{\partial z}\right)=-f(u_0-u_g),$$

$$w_0=0.$$

These two sets of equations have been integrated for the case where

$$f=5\times10^{-5}\ \text{sec}^{-1},\quad U_g=1000\ \text{cm sec}^{-1},\quad z_0=0.1\ \text{cm}.$$

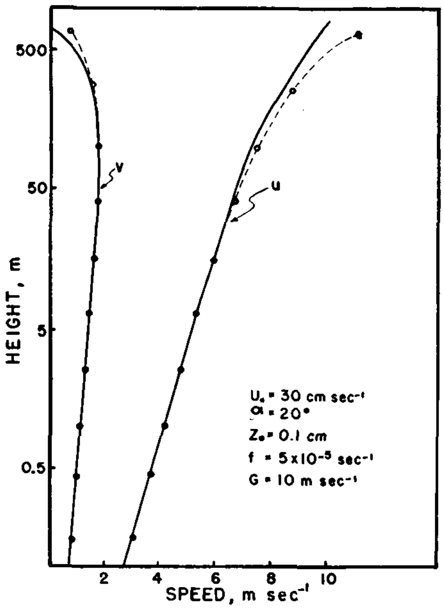

Fig. 2. Comparison between the exact solution (solid line) and the first approximation (dashed line) for straight flow. U_* and α are the friction velocity and the cross-isobar inflow angle, respectively.

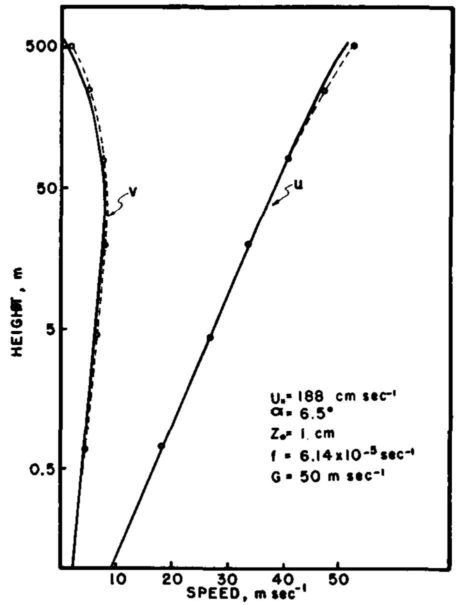

Fig. 3. Comparison between the exact solution (solid line) and the first approximation (dashed line) for curved flow. U_* and α are the friction velocity and the cross-isobar inflow angle, respectively.

The no-slip condition was prescribed at $z = 0$; in addition, the vertical derivatives of corresponding velocity components were made equal to one another at this level. The solutions—$u(z)$, $v(z)$, $u_0(z)$, and $v_0(z)$—are compared in Fig. 2. It may be seen that the first approximations are indeed very close to the exact solutions even at moderately high elevations.

An accurate test of the first approximation for the case of curved, horizontally nonuniform flow is not possible because there is neither a solution to the exact equations nor an observation for this case which could be used for comparison. Only a very crude test could be made. We have done this by comparing the first approximations for axially symmetric flow with solutions to a simple linearization of Eqs. (1) and (2) about gradient flow. The linearized equations are

$$\frac{\partial}{\partial z}\left(K\frac{\partial u}{\partial z}\right) = v\left(\frac{\partial G}{\partial r} + \frac{G}{r} + f\right), \qquad (17)$$

$$\frac{\partial}{\partial z}\left(K\frac{\partial v}{\partial z}\right) = -\frac{G(2u-G)}{r} - f(u-u_g). \qquad (18)$$

The corresponding first approximation equations are

$$\frac{\partial}{\partial z}\left(K_0\frac{\partial u_0}{\partial z}\right) = 0, \qquad (19)$$

$$\frac{\partial}{\partial z}\left(K_0\frac{\partial v_0}{\partial z}\right) = -\frac{u_0^2}{r} - f(u_0-u_g). \qquad (20)$$

The inaccuracy of the simple linearization equations arises from the fact that the assumption concerning the smallness of the velocity perturbations is not valid at levels near the ground while the inaccuracy of the first approximations stems from the fact that the cross-isobaric effect is entirely neglected in Eq. (19). On the other hand, the centrifugal force term appears to be described better by Eq. (20) than by Eq. (18).

Eqs. (17), (18), (19) and (20) were integrated for data obtained from Hurricane Donna (1960) where

$f = 6.14 \times 10^{-5}\ \text{sec}^{-1}$, $u_g = 594\ \text{m sec}^{-1}$, $G = 50\ \text{m sec}^{-1}$, $z_0 = 1$ cm, $r = 75$ km, $\frac{\partial G}{\partial r} + \frac{G}{r} + f = 3 \times 10^{-4}\ \text{sec}^{-1}$.

The boundary conditions used were similar to those used in the preceding integrations for straight flow.

Figure 3 shows a comparison between the first approximations and the solutions to the linearized equations. The agreement between the solutions is rather remarkable and entirely unexpected on the basis of the seemingly large differences between the governing equations.

It is possible that the accuracy of the first approximations may be improved by replacing $\partial u_0/\partial z$ in the expression for K_0 with

$$\left[\left(\frac{\partial u_0}{\partial z}\right)^2 + \left(\frac{\partial v_0}{\partial z}\right)^2\right]^{\frac{1}{2}},$$

and then using this improved exchange coefficient in solving for new values of u_0 and v_0 in Eqs. (14) and (15).

Concluding remarks

It has been demonstrated that a useful approximation to the basic boundary layer equations may be formulated for motions where the cross-isobaric flow is relatively small. In general, this condition is satisfied best at low latitudes where the coriolis effect is small and over relatively smooth ground and ocean surfaces. The approximation equations are considerably simpler in form than the basic boundary layer equations and are conveniently solvable using a stepwise procedure.

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АППРОКСИМАЦИЯ ПРОФИЛЕЙ ВЕТРА В ПОГРАНИЧНОМ СЛОЕ

Описывается процедура численного приближенного решения уравнений пограничного слоя для нейтрально стратифицированной атмосферы. Приближенные уравнения, которые выведены с использованием теории возмущений, выполняются для случая, когда поток, пересекающий изобару, мал. Уравнения первого приближения интегрируются аналитически и затем применяются в случаях, когда градиентное течение как прямолинейно, так и криволинейно. Сравнение приближенных решений с численными решениями точных уравнений показывает, что аппроксимация полезна при нормальных условиях в потоке около поверхности Земли.