Analogous behavior of homogeneous, rotating fluids and stratified, non-rotating fluids

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ABSTRACT

It has been pointed out in the past that analyses of homogeneous, rotating fluids (system Ω) and stratified, non-rotating fluids (system S) lead to the same mathematical problem in certain circumstances. The analogy is explored here for the case of steady, linear two-dimensional flows where the temperature variations imposed along side boundaries drive the stably stratified flow and swirl or zonal velocity imposed at the top and bottom boundaries drive the rotating flow. In the fluid S buoyancy boundary layers exist near the side boundaries and these serve the same function in adjusting the temperature between boundary and interior values that the Ekman layer serves to adjust the zonal velocity in the rotating case. The mathematical properties of the two layers are identical. In system S any changes which take place in the interior and tend to alter the stratification are resisted just as changes in the interior which tend to alter the vertical vorticity are resisted in system Ω. Processes which give rise to changes in the interior are concentrated in the boundary layers. Simple problems are worked out which exhibit the analogous behavior for two-dimensional systems driven by symmetric and by anti-symmetric conditions imposed at the boundaries. The temperature in the interior of the stratified fluid is the average of the imposed boudary values. This means that with symmetric driving temperatures the fluid is simply restratified by the imposed conditions and with anti-symmetric boundary values the temperature of the fluid interior is unchanged. Buoyancy boundary layers are necessary to adjust interior to boundary temperatures in the latter case. The rotating system has an analogous behavior. In three dimensions the analogy breaks down. The reason is that the third dimension in system S does not involve the constraint and therefore is literally an added degree of freedom whereas in system Ω the third dimension also involves the constraint. The difference between the two systems is exhibited by an analysis of the three-dimensional stratified problem with symmetric and anti-symmetric boundary conditions. The symmetric problem in system S differs qualitatively from both the two-dimensional case and the rotating fluid problem. The three-dimensional problem in system Ω is qualitatively similar to the two-dimensional one.

1. Introduction and summary

Often in the course of attempting to understand a particular physical phenomenon we try to construct an analogy in which the problem is expressed in terms of more familiar concepts or which exhibits the principal aspects of the original phenomenon in a simpler form. When the analogy can be made exact, it has the advantage of providing a different set of ideas, intuitive guesses, mathematical experiences, etc., which can be brought to bear on the original problem.

In this paper we shall explore the analogy between a homogeneous rotating fluid (system Ω) and a stably stratified, non-rotating fluid (system S). This analogy has been recognized for a long time. For example, the stability of two-dimensional Bénard convection can be posed mathematically as the same stability problem as Couette flow between two rotating cylinders when the inner cylinder is rotating slightly faster than the outer but when the differential rotation is small compared to the mean and the gap between the cylinders is small compared to the mean radius (Chandrasekhar (1961)). Aspects of internal waves in a stably stratified fluid bear an exact analogy to waves generated in a rotating fluid (Yih (1965), Chapter V).

We shall be concerned with slow, steady flow in the two systems and shall develop an approach which enables us to treat problems when the effects of the constraints are more important than the diffusive processes everywhere except in certain boundary layers. Formally the latter situations can be described by the statements

$$E = \frac{\nu}{\Omega L^2} \ll 1, \quad R = \sqrt{\frac{\varkappa \nu}{g\alpha \Delta T L^3}} = \frac{1}{R_a^{\frac{1}{2}}} \ll 1, \tag{1.1}$$

where E is the Ekman number (or the reciprocal of the square root of the Taylor number) and R_a is the Rayleigh number. The remaining symbols are described in section 2.

The equations are non-dimensionalized in section 2 and we arrive at two final sets of non-dimensional equations, one pertinent to system Ω, the other to system S. In each case we assume that the constraint dominates in the equations and the final form is such that the principal balance is free of parameters. The small parameters, E and R, multiply the remaining terms. It is then shown that for two-dimensional flow (where in each case one of the directions must be parallel and the other direction normal to the direction of the constraining mechanism) the two systems are identical in form. Hence, a complete mathematical analogy exists provided one makes the identifications indicated by equation (2.19). Effectively, this means that in going from one system to the other one exchanges horizontal and vertical directions (i.e., equates the directions parallel to the constraining mechanism in each case) and identifies the swirl or zonal velocity, v in system Ω with the temperature in system S. The analogy does not follow through in a general three-dimensional problem as can be seen by the presence of the constraint in two equations for system Ω and in only one in system S. For unsteady and for non-linear problems in two dimensions the analogy is not identical because of the presence of the Prandtl number, σ, in the stratified system, although it does hold exactly even for these expanded systems if $\sigma = 1$.

Even though we do not discuss problems where boundary layers exist parallel to the constraining direction, it is nevertheless obvious from the analogy that $R^{\frac{1}{3}}$ and $R^{\frac{1}{4}}$ layers can exist exactly as $E^{\frac{1}{3}}$ and $E^{\frac{1}{4}}$ layers exist in rotating fluids (Stewartson, 1957, and Greenspan & Howard, 1963). R. Dickinson (unpublished manuscript) has solved a problem in system S, where $R^{\frac{1}{3}}$ and $R^{\frac{1}{4}}$ layers were necessary.[1]

[1] The latter work is reported in the 1966 GFD lectures of the Wood Hole Oceanographic Institution.

In section 3 we introduce an expansion procedure which makes a certain class of problems tractable. In justifying the expansion procedure we introduce the buoyancy boundary layer for system S and in passing show the analogy between it and the Ekman layer in the rotating system. The boundary layers are developed assuming that variations along the boundaries are $O(1)$. Through an argument about the order of magnitude of the flux into and out of the buoyancy (or Ekman) layer we justify an expansion in powers of $R^{\frac{1}{2}}$ (or $E^{\frac{1}{2}}$). The foregoing procedure then allows us to solve problems by means of a stepwise procedure where we derive an ordered set of equations at each stage of the expansion for the interior and boundary layer regions. The boundary layer equations must be combined with the simpler interior set to satisfy boundary conditions. The idea of Ekman (or buoyancy) layer suction (Greenspan & Howard, 1963) is basic to the present procedure.

With the structure developed in section 3, two simple two-dimensional problems are solved in section 4. The flow in system S is generated by heating a stratified fluid either symmetrically or anti-symmetrically at the sides and it is shown that symmetric heating essentially restratifies the fluid whereas anti-symmetric heating requires boundary layer and interior flows to maintain the resulting thermal field. In both cases the temperature of the interior fluid is the average of the imposed boundary temperatures at the same level. The corresponding rotating problem has an analogous solution when vertical vorticity is imposed at the horizontal boundaries. Here, the vorticity of the fluid achieves a value equal to the average of the imposed boundary vorticities at a given horizontal location.

The three-dimensional rotating problem is qualitatively similar to the two-dimensional problem, the main difference being that vertical vorticity is now expressed by two terms instead of one. The analysis for this case is not given.

We conclude the paper in section 5 with the treatment of the three-dimensional thermal problem. The qualitative difference between the rotating and stratified systems in three-dimensions is evident from the totally different solution for the thermal problem when the fluid is heated symmetrically at the lateral boundaries. In two-dimensions the fluid in the interior simply achieves the mean value of the boundary temperature at the two sides in direct analogy to the rotating problem (in two *or* three dimensions). In three dimensions the $O(1)$ temperature of the fluid interior is unchanged by symmetric heating. A boundary layer is formed at each boundary and the fluid particles execute a closed circuit (see Fig. 2), never passing the plane of symmetry which separates the two boundaries in this case. The fluid motion is such as to maintain the interior temperature at the same value that it had before the heating was applied. The point is that the third dimension allows the fluid to flow unconstrained in a new direction whereas in the rotating problem the third dimension is one in which the constraint still operates.

The difference between systems Ω and S is the following: The constraint in system Ω is the imposed vorticity of the basic rotation and any change in the system which requires a change *in the vorticity is resisted strongly*. The only possible means to bring about a change in the local vorticity in the linear steady system is to do so through the boundary layer where diffusive processes can balance the strong rotational constraint. It does not matter whether the flow is two- or three-dimensional, the essential point being that generation of local vorticity is resisted.

In system S the constraint is the stratification and any process which tends to alter the stratification is resisted. Hence, vertical flows in the interior are inhibited but can occur near boundaries where diffusive processes can balance the constraints. The restriction to two-dimensions is now severe because any flow in the horizontal must be accompanied by a vertical flow and the latter can take place only near boundaries. The addition of the third dimension allows the fluid to flow in horizontal planes and this does not require that the fluid try to change the stratification. Furthermore, variations in the lateral boundary conditions along the horizontal direction can generate horizontal flows which are unconstrained.

2. Equations and non-dimensionalization

System Ω

For the rotating system we begin with the Navier-Stokes equations for an incompressible fluid

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} + 2\mathbf{\Omega}\times\mathbf{v} = -\frac{1}{\varrho}\nabla p + \nu\nabla^2\mathbf{v}, \quad (2.1)$$

$$\nabla\cdot\mathbf{v} = 0, \quad (2.2)$$

where $\mathbf{v}$ is the velocity vector with components (u,v,w) in the directions (x,y,z) respectively; Ω is the angular velocity of the system about the z axis; ϱ is (constant) density; ν is the coefficient of kinematic viscosity; p includes the pressure, the potential of the external force field and the centrifugal force (in this treatment of contained rotating fluids neither external forces nor centrifugal accelerations contribute to the motions of interest). Equation (2.2) expresses conservation of mass for an incompressible fluid.

We are interested in fluids which are nearly uniformly rotating, i.e., where the system is near geostrophic balance. Hence, we non-dimensionalize the above system so that the pressure and Coriolis terms are not multiplied by parameters. Thus, the substitutions

$$\partial_t = \Omega\partial_{t'}, \quad \mathbf{v} = V\mathbf{v}', \quad p = L\Omega V\varrho p', \quad \nabla = \frac{1}{L}\nabla',$$

yield the non-dimensional equations (dropping primes)

$$\mathbf{v}_t + \varepsilon\mathbf{v}\cdot\nabla\mathbf{v} + 2\hat{\mathbf{k}}\times\mathbf{v} = -\nabla p + E\nabla^2\mathbf{v}. \quad (2.3)$$

Equation (2.2) is unchanged in form. In equation (2.3) $\hat{\mathbf{k}}$ is the unit vector in the z direction and the parameters

$$\varepsilon = \frac{V}{\Omega L}, \quad E = \frac{\nu}{\Omega L^2} \quad (2.4)$$

are the Rossby and Ekman numbers respectively and represent measures of non-linearity and friction. In the limit of steady, small amplitude motions ($\partial_t = 0$, $\varepsilon\to 0$) equations (2.3) reduce to

$$2\hat{\mathbf{k}}\times\mathbf{v} = -\nabla p + E\nabla^2\mathbf{v} \quad (2.5)$$

or in component form

$$-2v = -p_x + E\nabla^2 u, \quad (2.6\text{a})$$

$$2u = -p_y + E\nabla^2 v, \quad (2.6\text{b})$$

$$o = -p_z + E\nabla^2 w. \quad (2.6\text{c})$$

When all motions are independent of the coordinate y the Laplacean operator is $\nabla^2 = \partial^2_{xx} + \partial^2_{zz}$ and equation (2.6 b) becomes

$$2u = E\nabla^2 v \tag{2.7}$$

System S

The equations for the stratified system are written in terms of the deviation of the system from a basic state of hydrostatic balance with a positive constant temperature gradient in the z direction, i.e. the system is stably stratified.[1] The equations in the Boussinesq approximation have the form

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\varrho}\nabla p + g\alpha T\hat{\mathbf{k}} + \nu\nabla^2\mathbf{v}, \tag{2.8}$$

$$T_t + \mathbf{v}\cdot\nabla T + \frac{4\Delta T}{L} w = \varkappa\nabla^2 T, \tag{2.9}$$

$$\nabla\cdot\mathbf{v} = 0 \tag{2.10}$$

where the equation of state (with ϱ^1 = perturbation density),

$$\varrho^1 = -\varrho\alpha T \tag{2.11}$$

has been used in the gravitational force term and where the basic temperature gradient (the constant 4 is included for convenience),

$$\frac{\partial T}{\partial z} = \frac{4\Delta T}{L} \tag{2.12}$$

is written explicitly in the third term of the heat equation (2.9).

In this system we suppose that the motions are caused or accompanied by a deviation of temperature, $2\Delta T_c$, where $\Delta T_c < \Delta T$. The non-dimensionalization is less straightforward than that of the Ω system but can be determined by the following considerations: The basic balance is such that the pressure gradient is offset by the constraint, hence we non-dimensionalize the pressure so as to balance the gravitational term with no parameters involved. We also expect that the nonlinear terms will come in at the same order in the equations of motion and the heat equation. Both of these considerations plus a desire for a certain symmetry between systems Ω and S are satisfied by the substitutions

$$p = g\alpha L\varrho\Delta T_c\sqrt{\sigma}p', \quad T = 2\Delta T_c\sqrt{\sigma}T', \quad \nabla = \frac{1}{L}\nabla',$$

$$\mathbf{v} = \sqrt{\frac{g\alpha L}{\Delta T}}\Delta T_c\mathbf{v}', \quad \partial_t = \sqrt{\frac{L}{g\alpha\Delta T}}\partial_{t'}.$$

Then we have (dropping primes)

$$\sigma^{-\frac{1}{2}}[\mathbf{v}_t + \varepsilon\mathbf{v}\cdot\nabla\mathbf{v}] = -\nabla p + 2T\hat{\mathbf{k}} + R\nabla^2\mathbf{v}, \tag{2.13}$$

$$\sigma^{\frac{1}{2}}[T_t + \varepsilon\mathbf{v}\cdot\nabla T] + 2w = R\nabla^2 T, \tag{2.14}$$

where

$$\sigma = \frac{\nu}{\varkappa}, \quad \varepsilon = \frac{\Delta T_c}{\Delta T}, \quad R = \sqrt{\frac{\nu\varkappa}{g\alpha\Delta T L^3}} \tag{2.15}$$

are respectively the Prandtl number, the measure of non-linearity, and the reciprocal of the square root of the Rayleigh number. In the limit of steady, small-amplitude motions ($\partial_t \equiv 0$, $\varepsilon \to 0$) the equations reduce to

$$0 = -\nabla p + 2T\hat{\mathbf{k}} + R\nabla^2\mathbf{v}, \tag{2.16}$$

$$2w = R\nabla^2 T \tag{2.17}$$

or in component form for (2.16)

$$0 = -p_x + R\nabla^2 u, \tag{2.18a}$$

$$0 = -p_y + R\nabla^2 v, \tag{2.18b}$$

$$-2T = -p_z + R\nabla^2 w. \tag{2.18c}$$

When all variations with respect to y vanish, equation (2.18b) is not pertinent. It is then evident that there is a one-to-one analogy between systems Ω and S if we make the identifications:

System Ω		System S
u	$\sim$	w
w	$\sim$	u
x	$\sim$	z
z	$\sim$	x
v	$\sim$	T
P	$\sim$	P
E	$\sim$	R

(2.19)

Of course, for any particular situation it is also necessary that the boundary conditions in terms of the corresponding variables be equivalent for the analogy to hold.

[1] If the fluid is contained between metal boundaries at the sides, the linear basic temperature profile would be established by fixing the top and bottom edges of the side boundaries at different temperatures. If the metal has a constant diffusion coefficient, it attains a linear temperature profile which is transmitted to the fluid by diffusion.

We note also that for three-dimensional flows the equations are no longer equivalent because the constraint of rotation appears in two equations whereas the constraint of stratification appears in only one. Non-linearities and/or time dependence introduce the Prandtl number, σ, into the stratified system so that for flows which involve either of these terms the analogy cannot hold unless $\sigma = 1$ and the motions are two-dimensional.

It is now clear that any problem which is formulated in terms of one of the two systems can be translated in terms of the other provided the restrictions cited above are satisfied. In particular, this means that the Ekman boundary layer along the top and bottom boundaries in system Ω has a counterpart in a buoyancy layer along the side boundaries in system S. The Stewartson (1957) boundary layers which can exist along the direction of Ω in the rotating system and have thicknesses of orders $E^{\frac{1}{3}}$ and $E^{\frac{1}{4}}$ exist also in the stratified system along the horizontal direction and have thicknesses of orders $R^{\frac{1}{3}}$ and $R^{\frac{1}{4}}$.

3. Expanded forms of the equations for two-dimensional flow

We shall not attempt to develop the analogy in detail for each of the different phenomena in the two systems. Instead we shall formulate a simple problem which exhibits the essential physical processes in each of the two analogous systems and then show how the analogy breaks down when we go from two to three dimensions.

We shall outline an expansion procedure which is useful for deducing the flows when R and E are small.

Consider a stratified fluid between two walls at $x = \pm 1$ where boundary conditions have an $O(1)$ variation in z and are independent of y. Elimination of all variables except T from (2.2), (2.17), (2.18a) and (2.18c) yields the equation

$$R^2\nabla^6 T + 4\,T_{zz} = 0. \tag{3.1}$$

With the identifications made in (2.19) we see that this equation is the stratified analogy of the sixth-order equation for rotating fluids

$$E^2\nabla^6 v + 4\,v_{zz} = 0. \tag{3.2}$$

Just as the latter has *Ekman boundary layers* of thickness $E^{\frac{1}{2}}$ near $z = \pm 1$, the former has *buoyancy boundary layers* of thickness $R^{\frac{1}{2}}$ near $x = \pm 1$. The properties of the buoyancy boundary layer can be derived by analogy from those of the Ekman boundary layers. We shall make use of the properties of the latter as the need arises.

By analogy then, derivatives in the buoyancy layers near $x = \pm 1$ are $O(R^{-\frac{1}{2}})$. Formally, we shall stretch the x variable by the transformations

$$\partial_x = \pm R^{-\frac{1}{2}}\partial_\xi,\quad \partial_\xi = 0(1),\quad \xi = (1 \pm x)\,R^{-\frac{1}{2}} \tag{3.3}$$

near $x = \mp 1$.

To simplify later analyses we introduce an expansion of the variables in powers of R. Experience with rotating problems tells us that variations of $O(1)$ variables in the Ekman layer induce an $O(E^{\frac{1}{2}})$ flow into or out of the interior of the fluid. Hence, an expansion in powers of $E^{\frac{1}{2}}$ is called for in system Ω. In system S this means expansion in powers of $R^{\frac{1}{2}}$. Furthermore, we shall divide each variable into two parts: a) a boundary layer part (denoted by an overbar) which decays as distance from the boundary increases and b) an interior part (subscript I) which has $O(1)$ derivatives with respect to x and z. Thus

$$f = f_I + \bar{f}, \tag{3.4}$$

where

$$f_I = f^{(0)} + R^{\frac{1}{2}}f^{(1)} + Rf^{(2)} + R^{\frac{3}{2}}f^{(3)} + \ldots, \tag{3.5a}$$

$$\bar{f} = \bar{f}^{(0)} + R^{\frac{1}{2}}\bar{f}^{(1)} + R\bar{f}^{(2)} + R^{\frac{3}{2}}\bar{f}^{(3)} + \ldots \tag{3.5b}$$

and f stands for any of the variables.

In the interior regions of the fluid we have the following ordered sets of equations from (2.2), (2.17), (2.18a) and (2.18c):

$$O(R^0):\quad w^{(0)} = 0,\ u_x^{(0)} = 0,\ p_x^{(0)} = 0,\quad p_z^{(0)} = 2T^{(0)} \ \Rightarrow\ T_x^{(0)} = 0; \tag{3.6a}$$

$$O(R^{\frac{1}{2}}):\quad w^{(1)} = 0,\ u_x^{(1)} = 0,\ p_x^{(1)} = 0,\quad p_z^{(1)} = 2T^{(1)} \ \Rightarrow\ T_x^{(1)} = 0; \tag{3.6b}$$

$$O(R):\quad w^{(2)} = \nabla^2 T^{(0)},\ u_x^{(2)} + w_z^{(2)} = 0,\quad p_x^{(2)} = \nabla^2 u^{(0)},\ p_z^{(2)} = 2T^{(2)}. \tag{3.6c}$$

The lowest order equations specify that there be no vertical velocity and that all variables be independent of x. The system is hydrostatically balanced. Equations of $O(R^{\frac{1}{2}})$ are identical in form to the set at $O(R^0)$. At $O(R)$ diffusion contributes to the balance but since $w^{(0)} = 0$, hydrostatic balance is still valid.

The corresponding boundary layer equations with the coordinate stretching (3.3) yield:

$$O(R^{-\frac{1}{2}}): \quad \bar{p}_\xi^{(0)} = 0,\ \bar{u}_\xi^{(0)} = 0 \ \Rightarrow\ \bar{p}^{(0)} = 0,\ \bar{u}^{(0)} = 0: \tag{3.7a}$$

$$O(R^0): \quad \bar{T}_{\xi\xi}^{(0)} = 2\bar{w}^{(0)},\ \bar{w}_{\xi\xi}^{(0)} = 2\bar{T}^{(0)},$$
$$\bar{p}_\xi^{(1)} = 0,\ \bar{u}_\xi^{(1)} \pm \bar{w}_z^{(0)} = 0; \tag{3.7b}$$

$$O(R^{\frac{1}{2}}): \quad \bar{T}_{\xi\xi}^{(1)} = 2\bar{w}^{(1)},\ \bar{w}_{\xi\xi}^{(1)} = -2\bar{T}^{(1)},$$
$$\pm\bar{p}_\xi^{(2)} = \bar{u}_{\xi\xi}^{(1)},\ \bar{u}_\xi^{(2)} \pm \bar{w}_z^{(1)} = 0, \tag{3.7c}$$

where $\pm$ holds near $x = \mp 1$.

In sets (3.7b) and (3.7c) the equations which couple $\bar{w}$ and $\bar{T}$ are identical in form to the equations which couple u and v in Ekman boundary layers. Here the solution gives the buoyancy spiral

$$\bar{w}^{(0)} = (\bar{w}_{\mp 1}^{(0)} \cos \xi + \bar{T}_{\mp 1}^{(0)} \sin \xi) e^{-\xi}, \tag{3.8a}$$

$$\bar{T}^{(0)} = (\bar{T}_{\mp}^{(0)} \cos \xi - \bar{w}_{\mp 1}^{(0)} \sin \xi) e^{-\xi}, \tag{3.8b}$$

where $(\bar{w}_{\mp 1}^{(0)}, \bar{T}_{\mp 1}^{(0)}) = (\bar{w}^{(0)}, \bar{T}^{(0)})$ at $x = \mp 1$. Since the buoyancy spiral relates w and T rather than u and v it cannot be seen physically in a flow as can the Ekman spiral.

We can expand the variables in system Ω in powers of $E^{\frac{1}{2}}$ and derive the analogous sets to (3.5), (3.6), (3.7) and (3.8) or alternatively we can use (2.19) to write down the corresponding sets for the rotating system.

The principal results which we need for the analyses in the following sections is the buoyancy layer outflow corresponding to the Ekman layer outflow. This can be derived from the continuity equation (the last of 3.7b) and the deduced value of $\bar{w}^{(0)}$ in (3.8a). Thus using

$$\bar{u}_\xi^{(1)} = \mp \bar{w}_z^{(0)} = \mp[\partial_z \bar{w}_{\mp 1}^{(0)} \cos \xi + \partial_z \bar{T}_{\mp 1}^{(0)} \sin \xi] e^{-\xi}, \tag{3.9}$$

we integrate from $\xi = \infty$ where $\bar{u}^{(1)} = 0$ to $\xi = \xi$ to get

$$\bar{u}^{(1)} = \mp \frac{1}{\sqrt{2}} [\partial_z \bar{w}_{\mp 1}^{(0)} \sin (\xi - \pi/4) - \partial_z \bar{T}_{\mp 1}^{(0)} \sin (\xi + \pi/4)] e^{-\xi}. \tag{3.10}$$

We shall generally take the lateral boundaries as non-slip and the flow will be driven by temperatures imposed at the sides. Then with $w = 0$ at $x = \pm 1$ and, since $w^{(0)} = 0$ everywhere by (3.6a), it follows that

$$\bar{w}_{\mp 1}^{(0)} = 0. \tag{3.11}$$

Furthermore, since, $u = 0$ at $x = \pm 1$,

$$u^{(1)}\big|_{x=\mp 1} = -\bar{u}^{(1)}\big|_{x=\mp 1}. \tag{3.12}$$

Then with (3.10), (3.11) and (3.12) we deduce

$$u^{(1)}\big|_{x=\mp 1} = \mp \tfrac{1}{2}\partial_z \bar{T}_{\mp 1}^{(0)} = \pm \tfrac{1}{2}\partial_z (T_{\mp 1}^{(0)} - T_{\mp 1}), \tag{3.13}$$

where the last expression is derived by substituting the total minus the interior temperature for the boundary layer temperature. Here $T_{\mp 1} = T$ at $x = \mp 1$.

Expression (3.13) is the stratified analogy of the interior flow induced by the Ekman layer in a rotating fluid. Physically we have the following picture: Suppose that the fluid is heated along lateral boundaries and that the imposed boundary temperature increases with height. Then in the buoyancy layer conduction transmits the increased temperature to the fluid and vertical convection ensues. The resultant vertical velocity will be stronger where the heating is stronger and this creates a vertical divergence. Hence there must be a compensating flux of fluid into the boundary layer from the interior. If at a given level there is a net upward flux of fluid in the boundary layer because of net heating, there must be a net downward flux of fluid in the interior and since the fluid is stably stratified this means that warmer fluid is convected downward thereby warming the interior.

4. Simple examples of two-dimensional flow in systems Ω and S

Suppose that in system S the boundary conditions are

$$T = T_0,\ \mathbf{v} = 0 \quad \text{at} \quad x = -1,$$
$$T = \pm T_0,\ \mathbf{v} = 0 \quad \text{at} \quad x = 1, \tag{4.1}$$

where the positive and negative signs at $x = 1$ specify two problems, one driven symmetrically the other anti-symmetrically. It is a straightforward matter to solve the two problems using the foregoing expansions.

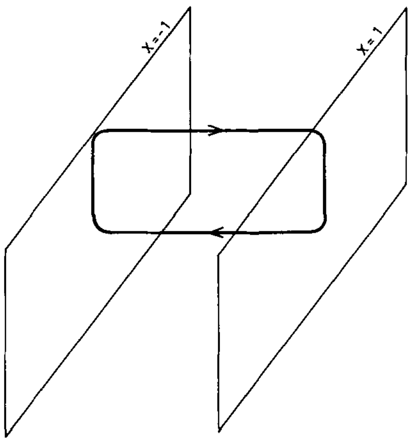

Fig. 1. A typical path of a fluid particle is shown for the case where the fluid is being heated at $x=1$ and cooled at $x=+1$, i.e., anti-symmetric heating, as described in the text. For the three-dimensional problem with a harmonic y dependence the path would be reversed in regions where the fluid is being cooled at $x=-1$ and warmed at $x=1$.

The second of equations (3.6b) states

$$u_x^{(1)}=0 \quad \text{or} \quad u^{(1)}|_{x=-1}=u^{(1)}|_{x=1}. \tag{4.2}$$

Hence, from (3.13)

$$\partial_z(T^{(0)}-T)_{x=-1}=\partial_z(T-T^{(0)})_{x=1}. \tag{4.3}$$

Since $T^{(0)}$ is independent of x from (3.6a),

$$T^{(0)}|_{x=-1}=T^{(0)}|_{x=1}=T^{(0)}$$

and (4.3) becomes

$$2T_z^{(0)}=\partial_z(T_1+T_{-1}). \tag{4.4}$$

We can integrate this last equation with respect to z and write

$$2T^{(0)}=T_1+T_{-1}. \tag{4.5}$$

For the *symmetric situation* where $T_1=T_{-1}=T_0$, we get

$$T^{(0)}=T_0, \tag{4.6}$$

i.e., the interior temperature is everywhere the same as the temperature of the boundary. In the boundary layer (4.6) states that

$$\overline{T}^{(0)}|_{x=-1}= {}^{(0)}|_{x=1}=0. \tag{4.7}$$

Thus the fluid temperature adjusts to the imposed boundary values at each level and there is no internal flow, consequently, no boundary layer. The conclusion here is that to 0(1) the fluid is simply restratified by the imposed boundary temperatures. There is a small $O(R)$ correction due to diffusion in the interior which we do not calculate explicitly.

For the anti-symmetric situation, where $T_1=-T_{-1}$,

$$T^{(0)}=0 \tag{4.8}$$

hence

$$\overline{T}_{-1}^{(0)}=T_0 \quad \text{and} \quad \overline{T}_1^{(0)}=-T_0. \tag{4.9}$$

Here the temperature of the fluid interior vanishes at each level and a buoyancy layer exists at each boundary. A fluid particle in one boundaryl ayer receives heat, say, from the boundary by diffusion, rises to a region of vertically converging flow and where it leaves the boundary layer with the temperature of the ambient fluid, crosses to the other boundary layer, where it enters in a region of vertically diverging flow and is cooled, and sinks until it leaves that layer with the temperature of the ambient fluid, crosses back to its original boundary layer and rises to its original level and temperature. The path is exhibited in figure 1 where $\partial T_0/\partial z>0$ in regions of flow into the boundary layers and $\partial T_0/\partial z<0$ in regions of outflow.

The corresponding problems in the rotating system involve zero values of u and w on the boundaries and

$$v=v_0 \quad \text{at} \quad z=-1, \quad v=\pm v_0 \quad \text{at} \quad z=1. \tag{4.10}$$

The analogous flow can be derived in exactly the same manner as the above. Symmetric conditions yield values of v uniform in the vertical whereas the anti-symmetric system has zero zonal velocity, v, plus boundary layers.

In system S the problem analogous to the spin-up problem of Greenspan and Howard is the heat-up problem where at time $t=0$ temperatures varying in z are imposed symmetrically at $x=\pm 1$. This problem has been treated in an unpublished manuscript by R. Dickinson. We mention it in passing only to point out that there is an analogy also for this transient problem.

We finally note that if the S system is bounded in z or if there are sharp gradients in z ($\partial_z \gg O(1)$), thermal boundary layers of $O(R^{\frac{1}{3}})$ and $O(R^{\frac{1}{4}})$ can exist just as they do in the rotating problem as analyzed by Stewartson (1957). The exact analogy is evident from the form of the equations.

5. Three-dimensional flows

The three-dimensional rotating problem is qualitatively identical to the two-dimensional rotating problem. This fact can easily be verified by going through exactly the same procedure as for the two-dimensional flows in sections 3 and 4 and we shall not give the analysis here since nothing new is added.

Three-dimensional flow in the stratified system, however, is qualitatively different from two-dimensional flow. We proceed to demonstrate this fact by investigating the equations (2.2), (2.17) and (2.18) using the expansions (3.4) and (3.5).

Interior flow

$O(R^0)$: $p_x^{(0)} = 0,\ p_y^{(0)} = 0,\ p_z^{(0)} = 2T^{(0)}$

$$w^{(0)} = 0,\ u_x^{(0)} + v_y^{(0)} = 0. \tag{5.1}$$

Our conclusions from equations (5.1) are that there are no vertical velocity and no horizontal gradients of pressure but there can be horizontally non-divergent flow. From the first three equations we deduce

$$T_x^{(0)} = 0,\ T_y^{(0)} = 0. \tag{5.2}$$

$O(R^{\frac{1}{2}})$: The equations are identical to the set $O(R^0)$ with $f^{(0)}$ replaced by $f^{(1)}$. Our conclusions are the same and again we have

$$T_x^{(1)} = 0,\ T_y^{(1)} = 0. \tag{5.3}$$

$O(R)$: $p_x^{(2)} = \nabla^2 u^{(0)},\ p_y^{(2)} = \nabla^2 v^{(0)},\ p_z^{(2)} = 2T^{(2)},$

$$2w^{(2)} = \nabla^2 T^{(0)},\ u_x^{(2)} + v_y^{(2)} + w_z^{(2)} = 0. \tag{5.4}$$

Diffusion enters at this order and balances the terms which in previous orders vanished. From the first two equations (5.4) we derive

$$\nabla^2 [v_x^{(0)} - u_y^{(0)}] = 0 \tag{5.5}$$

and this together with the last of equations (5.1) yields

$$\nabla^2 [u_{xx}^{(0)} + u_{yy}^{(0)}] = 0. \tag{5.6}$$

Hence, the zero-order interior velocity is not only horizontally non-divergent but the (three-dimensional) diffusion of vertical vorticity vanishes locally.

$O(R^{\frac{3}{2}})$: The questions are identical to the set at $O(R)$ and we write for future use:

$$\nabla^2 (v_x^{(1)} + u_y^{(1)}) = 0, \tag{5.7 a}$$

$$u_x^{(1)} + v_y^{(1)} = 0, \tag{5.7 b}$$

or combining these two

$$\nabla^2 (u_{xx}^{(1)} + u_{yy}^{(1)}) = 0. \tag{5.8}$$

Boundary layer flow

$O(R^{-\frac{1}{2}})$: $\bar{p}^{(0)} = 0,\ \bar{u}^{(0)} = 0. \quad (5.9)$

$O(R^0)$: $\bar{v}^{(0)} = 0,\ \bar{p}^{(1)} = 0,\ \bar{w}_{\xi\xi}^{(0)} + 2\bar{T}^{(0)} = 0,$

$$\bar{T}_{\xi\xi}^{(0)} - 2\bar{w}^{(0)} = 0,\ \bar{u}_\xi^{(1)} = \mp \bar{w}_z^{(0)}. \tag{5.10}$$

We have made use of the fact that $\bar{v}_{\xi\xi}^{(0)} = 0$ because $\bar{v}^{(0)}$ must vanish for large ξ.

Now we note that $u|_{x=\pm 1} = 0$ plus $\bar{u}^{(0)} = 0$ gives

$$u^{(0)} = 0 \quad \text{on} \quad x = \pm 1 \tag{5.11}$$

and $v|_{x=\pm 1} = 0$ plus $\bar{v}^{(0)} = 0$ gives

$$v^{(0)} = 0 \quad \text{on} \quad x = \pm 1$$

or

$$u_x^{(0)} = 0 \quad \text{on} \quad x = \pm 1. \tag{5.12}$$

The general problem posed by equations (5.6), (5.11) and (5.12) requires boundary conditions on y and z boundaries as well. However, for our purposes it suffices to investigate the cases where

$$T = 2T_0 \sin y \sin z \text{ on } x = -1,$$
$$T = \pm 2T_0 \sin y \sin z \text{ on } x = 1, \tag{5.13}$$

i.e. where we force the system harmonically in y and z and where we consider as before a symmetric and an anti-symmetric heating. Note that this assumed harmonic dependence plus conditions (5.2) and (5.3) yield

$$T^{(0)} = 0, \quad T^{(1)} = 0, \tag{5.14}$$

i.e. the interior temperatures of $O(R^0)$ and $O(R^{\frac{1}{2}})$ vanish throughout.

It is evident also that with this harmonic dependence the solution to equations (5.6), (5.11) and (5.12) yields[1]

$$u^{(0)} \equiv 0, \quad v^{(0)} \equiv 0. \tag{5.15}$$

[1] In fact, it can be shown that the general three dimensional problem with homogeneous boundary conditions has only the solution given by (5.15).

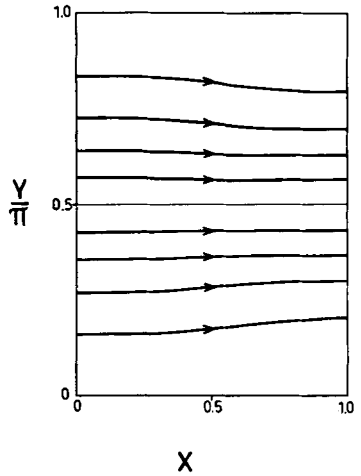

Fig. 2. Streamlines of interior velocity for the case of anti-symmetric heating and in the region $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant \pi$ is shown at a level where $\partial To/\partial z > 0$ at $x = -1$ and $\partial To/\partial z < 0$ at $x = 1$. The pattern is anti-symmetric about $y = 0$ and $y = \pi$, and is symmetric about $y = \pi/2$. A sketch of a particle trajectory is shown in Fig. 1.

The first non-vanishing contributions to the interior flow come in at $O(R^{\frac{1}{2}})$. With the substitutions

$$u^{(1)} = U(x)\sin y \cos z,\ v^{(1)} = V(x)\cos y \cos z \quad (5.16)$$

equation (5.8) becomes

$$(\partial^2_{xx} - 2)(\partial^2_{xx} - 1)\,U = 0. \quad (5.17)$$

We need boundary conditions on U. First note that from (5.11) $\bar{p}^{(1)} = 0$ and from the second equation of motion in the boundary layer we can conclude that $\bar{v}^{(1)}_{\xi\xi} = 0$ and therefore $\bar{v}^{(1)} = 0$. Since $v = 0$ at $x = \pm 1$ this means that

$$v^{(1)} = 0 \quad \text{at} \quad x = \pm 1$$

or from (5.7b) $u^{(1)}_x = 0$ and from (5.16)

$$U_x = 0 \quad \text{at} \quad x = \pm 1. \quad (5.18)$$

The remaining boundary conditions are derived from the buoyancy layer equations which are the same as those discussed in section 3. We substitute the harmonic dependences from (5.16) and (5.13) into (3.13) to derive for the *system heated symmetrically*

$$U = \pm T_0 \quad \text{on} \quad x = \pm 1. \quad (5.19)$$

The solution to (5.17), (5.18) and (5.19) is

$$U = A \sinh \sqrt{2}x + B \sinh x$$

$$A = \frac{T_0 \cosh 1}{\cosh 1 \sinh \sqrt{2} - \sqrt{2} \sinh 1 \cosh \sqrt{2}},$$

$$B = \frac{-\sqrt{2} \cosh \sqrt{2}\, A}{\cosh 1}. \quad (5.20)$$

The corresponding solution for $V(x)$ is

$$V = \sqrt{2} A \cosh \sqrt{2}x + B \cosh x. \quad (5.21)$$

With *anti-symmetric heating* we have

$$U = -T_0 \quad \text{on} \quad x = \pm 1 \quad (5.22)$$

and the solution is

$$U = C \cosh \sqrt{2}x + D \cosh x$$

$$C = -\frac{T_0 \sinh 1}{\sqrt{2} \sinh \sqrt{2} \cosh 1 - \sinh 1 \cosh \sqrt{2}},$$

$$D = \frac{-\sqrt{2} \sinh \sqrt{2}C}{\sinh 1} \quad (5.23)$$

with

$$V = \sqrt{2}C \sinh \sqrt{2}x + D \sinh x. \quad (5.24)$$

The above solutions show that in the three-dimensional stratified system the interior $O(R^0)$ and $O(R^{\frac{1}{2}})$ temperatures vanish whether the imposed boundary conditions are symmetric or anti-symmetric. This marks a significant change from the two-dimensional system where the interior temperature assumes a value which is the average of the imposed boundary values at each level, as well as from the three-dimensional rotating system where the behavior is similar to that of the two-dimensional case.

The trajectory of a particle for the case with anti-symmetric heating is similar to that for the two-dimensional problem (see fig. 1) and is described essentially by the paragraph below equation (4.7). There is now a weak component of velocity in the y direction and the pattern of streamlines for the interior velocity in a region where the fluid flows to the right is shown in figure 2.

When the fluid is heated symmetrically so that the solution is given by equations (5.20) and (5.21), the flow in the interior is still of $O(R^{\frac{1}{4}})$ but the fluid does not cross the plane $x = 0$. Instead, a fluid particle which starts, say, in the boundary layer where it is being heated rises and leaves the boundary layer with the ambient temperature of the interior fluid, makes a half loop in the horizontal plane returning to the boundary layer in a region where the fluid is being cooled by the boundary, sinks and gives off its heat, leaves the boundary layer at a lower level where it executes a half loop travelling in the direction opposite to the one it had in the first loop, enters the boundary layer at a position below its original one and rises to its original position and temperature. This circuit is shown in fig. 3. The particle enters the boundary layer in a region where $\overline{w}_z^{(0)} > 0$ (or $T_z^{(0)} > 0$) and leaves where $\overline{w}_z^{(0)} < 0$ (or $T_z^{(0)} < 0$). The above picture is reflected in the plane $x = 0$.

The principal difference between this case and the rotating one is that in system S the fluid is constrained only in the z direction and has freedom to travel in both horizontal directions, whereas the rotating fluid is constrained in both x and y. When we add a third dimension to the original two-dimensional problem in system Ω, we add a direction along which the flow is constrained in the same manner as it had already been constrained. In system S on the other hand we add a degree of freedom and the fluid can flow in the added directions unconstrained.

The effect of the stratification in this problem leads to the relatively complicated loop which a particle executes. Were it not for the stratification a particle could move vertically in the interior as well and the resultant particle path would be much more direct.

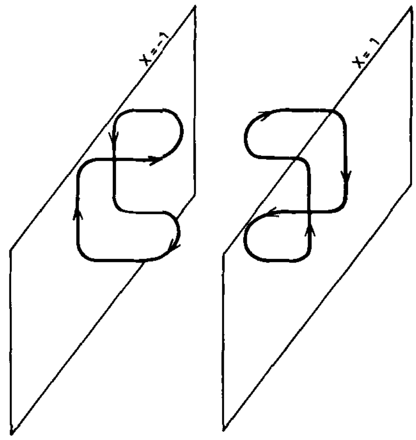

Fig. 3. Typical particle paths are shown for the three-dimensional case with symmetric heating and cooling along the lateral boundaries. Upward motion occurs where the fluid is being heated and downward motion where it is being cooled. The trajectories are anti-symmetric about the plane $x = 0$.

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АНАЛОГИЯ В ПОВЕДЕНИИ ОДНОРОДНЫХ ВРАЩАЮЩИХСЯ ЖИДКОСТЕЙ И СТРАТИФИЦИРОВАННЫХ НЕВРАЩАЮЩИХСЯ ЖИДКОСТЕЙ

Ранее отмечалось, что анализ однородных, вращающихся жидкостей (система Ω) и стратифицированных, невращающихся жидкостей (система S) в некоторых случаях приводит к одной и той же математической проблеме. Здесь исследуется аналогия для случая стационарного линейного двумерного потока, в котором температурные изменения вдоль боковых границ возбуждают устойчивое стратифицированное течение, а завихренность или зональная скорость на верхней и нижней границах возбуждают вращающееся течение. В жидкости S связанные с плавучестью пограничные слои существуют вблизи боковых границ и выполняют ту же функцию в приспособлении температуры границы и внутренней части среды, что и слой Экмана в приспособлении зональной скорости в случае вращения. Математические свойства обоих слоев идентичны. В системе S любые изменения, которые имеют место внутри и стремятся изменить стратификацию, встречают сопротивление так же, как и внутренние изменения, которые стремятся изменить вертикальную скорость в системе Ω. Процессы, которые возбуждают изменения внутри, концентрируются в пограничных слоях. Разбираются простые задачи, которые обраруживают аналогию в поведении двумерных систем, возбуждаемых симметричными и антисимметричными условиями на границах. Температура внутри стратифицированной жидкости является средней от значений на границах. Это означает, что при симметричных возбуждающих температурах жидкость просто меняет стратификацию, а при антисимметричных граничных условиях температура внутри жидкости остается неизменной. Плавучие пограничные слои необходимы в последнем случае для приспособления температуры внутри среды к граничным значениях.

Вращающаяся система имеет аналогичное поведение. В случае трех измерений аналогия нарушается. Причина в том, что третье измерение в системе S не содержит ограничения и поэтому фактически является дополнительной степенью свободы, тогда как в системе Ω третье измерение также содержит ограничение. Разница между двумя системами обнаруживается путем анализа трехмерной стратифицированной задачи с симметричным и антисимметричным граничными условиями. Симметричная задача в системе S качественно отличается как от двумерного случая, так и от задачи с вращающейся жидкостью. Трехмерная задача в системе Ω качественно подобна двумерной задаче.