Wind-driven currents in a long rotating channel

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ABSTRACT

Boundary-layer type solutions are obtained for currents generated in a homogeneous fluid in a long rotating channel by a steady wind stress acting along or across the channel. Included are viscous wall boundary layers which arise necessarily from secondary circulations transverse to the axis of the channel, and from no slip conditions at the wall.

Introduction

The study of wind-driven currents in the presence of lateral boundaries has been primarily restricted to two-dimensional models in which the vertical coordinate has been parameterized in some way; see, for example, Munk (1950), Charney (1955), or Pedlosky (1965). The present paper examines a dynamical model of wind-generated currents in which some details of the vertical circulation are retained. A model of viscous-driven upwelling and sinking motions is obtained. The purpose of the paper is to explore those features of the circulation in a long rotating channel produced by a steady wind stress at the free surface, preliminary to the discussion of the closed channel or rectangular basin.

The model

In order to satisfactorily incorporate the vertical structure a very simple model is used. Since the wind stress is assumed independent of the coordinate along the channel, the motions are also assumed two dimensional, with the one exception that a constant pressure gradient along the channel is permitted as a special case. The horizontal scale is such that the rotation of the channel may be taken as a constant; the fluid is homogeneous. If A is the viscosity coefficient either molecular or eddy, Ω the angular velocity, $(u, v, w) = \mathbf{v}$ the velocity, (τ_1, τ_2) the horizontal wind stress, and the x-axis is placed across the channel, and y along it, the momentum and continuity equations are

$$A\nabla^2\mathbf{v} - 2\Omega\mathbf{k}\times\mathbf{v} = \left(\frac{u\partial}{\partial x} + w\frac{\partial}{\partial z}\right)\mathbf{v} + \frac{1}{\varrho}\nabla p + g\mathbf{k},$$

$$\nabla\cdot\mathbf{v} = 0,$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial z^2$. If the channel width is L, depth D, the boundary conditions are

$$z = h:\quad A\frac{\partial u}{\partial z} = \frac{\tau_1}{\varrho},\quad A\frac{\partial v}{\partial z} = \frac{\tau_2}{\varrho},\quad w = \frac{dh}{dt},\quad p = 0;$$

$$z = -D:\quad \mathbf{v} = 0;\quad x = 0,\quad L: \mathbf{v} = 0,$$

where h is the height of the free surface above $z = 0$. If the magnitude of the stress is τ_0, the starred nondimensional variables are defined:

$$(x, z) = D(x^*, z^*),\quad L = DL^*,$$

$$(u, v, w) = \tau_0/\varrho\Omega L(u^*, v^*, w^*),$$

$$p = \tau_0\, gD/\Omega^2 L^2,$$

$$(\tau_1, \tau_2) = (\tau_0/\varrho)(\tau_1^*, \tau_2^*).$$

The equations and boundary conditions written in these variables after dropping asterisks are

$$R^{-1}\nabla^2\mathbf{v} - 2\mathbf{k}\times\mathbf{v} = R_0\left(u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}\right)\mathbf{v} + \gamma\nabla p + \frac{\varrho g L}{\tau_0}\mathbf{k},$$

$$\nabla\cdot\mathbf{v} = 0, \tag{2.1}$$

$$z = h:\quad R^{-1}\frac{\partial u}{\partial z} = \tau_1,\quad R^{-1}\frac{\partial v}{\partial z} = \tau_2,\quad w = \frac{dh}{dt},\quad p = 0;$$

$$z = -1:\quad \mathbf{v} = 0;\quad x = 0,\quad L: \mathbf{v} = 0,$$

where $R \equiv L^2\Omega/A$, $R_0 \equiv \tau_0/\varrho L^2\Omega^2$ and $\gamma \equiv gD/\Omega^2 D^2$. It is assumed that not only is $R \gg 1$, and

$R_0 < 1$, but also that $R_0 R < 1$. It is convenient to introduce a stream function and also to scale v by a factor of $R^{\frac{1}{2}}$, that is, let

$$\mathbf{v} = -\mathbf{j} \times \nabla\psi(x, z) + R^{\frac{1}{2}} v \mathbf{j}. \tag{2.2}$$

A vorticity equation is obtained by eliminating the pressure from the x and z momentum equations. The system then becomes:

$$\left.\begin{aligned}
&-R^{-\frac{3}{2}}\nabla^4\psi + 2\frac{\partial v}{\partial z} = 0,\\
&R^{-\frac{1}{2}}\nabla^2 v + 2\frac{\partial \psi}{\partial z} = \gamma\frac{\partial p}{\partial y} \equiv P,\\
z = 0:\quad &R^{-\frac{1}{2}}\frac{\partial v}{\partial z} - \tau_2,\quad R^{-1}\frac{\partial^2\psi}{\partial z^2} = -\tau_1\\
&w = p = 0,\\
z = -1:\quad &\frac{\partial \psi}{\partial z} = \psi = v = 0,\\
x = 0, L:\quad &\psi = \frac{\partial \psi}{\partial x} = v = 0,
\end{aligned}\right\} \tag{2.3}$$

where $P \sim O(1)$ is assumed a constant, and will remain arbitrary. The equations, now linear, may be solved separately for the stress component along the channel and across the channel.

Considering the stress to be along the channel, if the Ekman depth, here defined as $D^* R^{-\frac{1}{2}}$, is much less than D^* there is a frictional boundary layer at the free surface, and possibly at the bottom of the channel. In the surface layer in addition to possible vertical flux out of the layer, there is the Ekman transport across the channel. Because of the lateral boundaries there must be a compensating return flow across the channel; this may be accomplished by either a sinking motion to the bottom and return in a bottom boundary layer, or by a return flow across the channel in the interior, or by a combination of the two motions.

In any case it is evident that boundary layers are needed along the walls, if for no other reason than to reduce the interior velocities to zero on the lateral boundary; the structure of these boundary layers involve two scales of motion and resembles those investigated in detail by Greenspan & Howard (1963).

Solution for wind stress along the channel

The method of solution of equations (2.3) is based upon classical boundary layer theory for viscous flow. The method used is that of Carrier (1953), or the systematic theory of Lagerstrom & Cole (1955). For convenience the various boundary regions have been labeled in Fig. 1. If (ψ, v) are the solutions of (2.3), a uniformly valid approximation, as a sum of asymptotic expansions in R, is constructed of the form

$$\psi(x, z) \sim \sum_j \psi_j, \quad v(x, z) \sim \sum_j v_j, \tag{3.1}$$

where (ψ_j, v_j) is the *component* solution in region j of Fig. 1. In particular (ψ_1, v_1) represents an asymptotic expansion of (ψ, v) to N terms say, as $R \to \infty$, with x, z fixed. The interior component solution looks like, for example:

$$\psi_1 = \psi_{11}(x, z) + R^{-1/2}\psi_{12}(x, z) + \dots,$$
$$v_1 = v_{11}(x, z) + R^{-1/2}v_{12}(x, z) + \dots.$$

that is, $(\psi_1, v_1) = O(1)$. Differential equations for the lowest order term in (ψ_1, v_1) are obtained by substituting (3.1) into (2.3) and taking the limit as $R \to \infty$, x, z fixed; as described below all other component solutions tend to zero away from their respective boundaries; hence the resulting equations are equivalent to those obtained by dropping the viscous terms in (2.3). Since these are of lower order than (2.3), the solution cannot satisfy all boundary conditions. By adding to the interior solution a component solution which decays exponentially away from the boundary, the *composite* solution is made to satisfy the boundary conditions to a certain approximation in R (the interior solution is "matched" to the boundary layer solution). The stretchings of the variables, given in Table 1, are consistent with those in a number of similar problems, for example as in Carrier (1966). The differential equations and boundary conditions for the lowest order component solutions are found by substituting (3.1) into (2.3) after writing (2.3) in the appropriate stretched variables, and applying lim $R \to \infty$, holding the local stretched variables fixed.

For regions 1, 2, 4, $(\psi_j, v_j) \sim O(1)$; the low order equations are, in the interior:

$$\left.\begin{aligned}
2\frac{\partial v_{11}}{\partial z} &= 0,\\
2\frac{\partial \psi_{11}}{\partial z} &= P.
\end{aligned}\right\} \tag{3.2}$$

For region 2, the equations (2.3) in variables (x, ζ_2) are

$$\left.\begin{aligned} -\frac{\partial^4\psi}{\partial\zeta_2^4}+2\frac{\partial v}{\partial\zeta_2} &= -2R^{-1}\frac{\partial^4\psi}{\partial x^2\partial\zeta_2^2}+\dots,\\ \frac{\partial^2 v}{\partial\zeta_2^2}+2\frac{\partial\psi}{\partial\zeta_2} &= -R^{-1}\frac{\partial^2 v}{\partial x^2},\end{aligned}\right\} \quad (3.3)$$

which, on substituting in (3.1) and applying lim $R\to\infty$, (x, ζ_2) fixed, are identical to (3.3) if $(\psi_{21}, v_{21})\to(\psi, v)$, and the right side set to zero. The boundary conditions at $z=\zeta_2=0$ are

$$\psi_{11}(x,0)+\psi_{21}(\psi,0)=\frac{\partial^2\psi_{21}}{\partial\zeta_2^2}=\tau_2(x)-\frac{\partial v_{21}}{\partial\zeta_2}=0.$$

The equations in the bottom layer are identical with (3.3) if $(\psi_4, v_4)\to(\psi_2, v_2)$ and $\zeta_4\to\zeta_2$. Boundary conditions at $z+1=\zeta_4=0$ are

$$\begin{aligned}\psi_{11}(x_1,-1)+\psi_{41}(x,0)&=\frac{\partial\psi_{41}}{\partial\zeta_4}\\ &=v_{11}(x)+v_{41}(x,0)=0. \quad (3.4)\end{aligned}$$

This is the Ekman problem for a horizontally unbounded ocean. The component solutions are

$$\left.\begin{aligned} &\psi_{11}(x,z)=\tfrac{1}{2}[\tau_2(x)+zP],\\ &v_{11}(x)=\tau_2(x)-P,\\ &\psi_{21}(x,\zeta_2)=-\tfrac{1}{2}\tau_2(x)\,e^{\zeta_2}\cos\zeta_2,\\ &v_{21}(x,\zeta_2)=\tfrac{1}{2}\tau_2(x)\,e^{\zeta_2}(\cos\zeta_2+\sin\zeta_2),\\ &\psi_{41}(x,\zeta_4)=\tfrac{1}{2}[P-\tau_2(x)]\,\bar{e}^{\zeta_4}(\cos\zeta_4+\sin\zeta_4),\\ &v_{41}(x,\zeta_4)=[P-\tau_2(x)]\,\bar{e}^{\zeta_4}\cos\zeta_4.\end{aligned}\right\} \quad (3.5)$$

In addition to the surface Ekman transport $\frac{1}{2}\tau_2(x)$ across the channel, there are gradient currents v_{11} along the channel and $u_{11}=-\frac{1}{2}P$ across the channel. There is the Ekman divergence of mass $\frac{1}{2}(\partial\tau_2(x)/\partial x)$ out of the boundary layer which, since w_{11} is independent of z, flows to the bottom layer and is returned to the surface via the walls.

For the flow along the wall, that is, the return flow or upwelling, a boundary layer of thickness $O(R^{-1/4})$ is required, denoted as region a. To match the interior geostrophic flow $(\psi_a, v_a)\sim O(1)$ is required. The equations for this region are

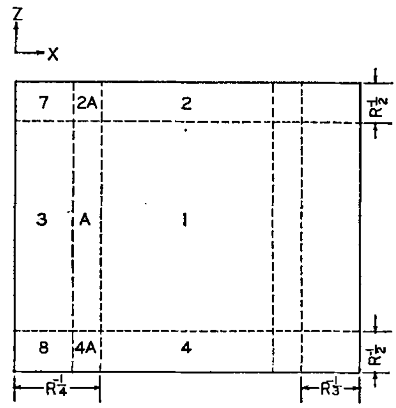

Fig. 1. Cross-section of the channel showing the width of the various boundary-layer regions and the identification number for each region used in the text.

$$\left.\begin{aligned} 2\frac{\partial v}{\partial z}&=O(R^{-\frac{1}{4}}),\\ \frac{\partial^2 v}{\partial\eta^2}+2\frac{\partial\psi}{\partial z}&=O(R^{-\frac{1}{4}}).\end{aligned}\right\} \quad (3.6)$$

where $x=R^{-1/4}\eta$. With substitution of (3.1) into (3.6) and applying the lim $R\to\infty$, η, z fixed we have

$$\left.\begin{aligned} 2\frac{\partial v_{a1}}{\partial z}&=0,\\ \frac{\partial^2 v_{a1}}{\partial\eta^2}+2\frac{\partial\psi_{a1}}{\partial z}&=0.\end{aligned}\right\} \quad (3.7)$$

As in the interior, the order in z of the equations (3.7) is reduced requiring Ekman layers at the top and bottom of the region a. Although both coordinates are stretched, the equations in regions $2a$ and $4a$ are the Ekman equations to

Table 1. *Coordinates for the regions of the channel cross-section.*

Region	Variable $X=$	$Z=$
1	x	z
2	x	$R^{-1/2}\zeta_2$
3	$R^{-1/3}\xi$	z
4	x	$R^{-1/2}\zeta_4-1$
7	$R^{-1/3}\xi$	$R^{-1/2}\zeta_2$
8	$R^{-1/3}\xi$	$R^{-1/2}\zeta_4-1$
a	$R^{-1/4}\eta$	z
$2a$	$R^{-1/4}\eta$	$R^{-1/2}\zeta_2$
$4a$	$R^{-1/4}\eta$	$R^{-1/2}\zeta_4-1$

the lowest order, that is the same form as (3.3); again for matching with region a, (ψ_{2a}, v_{2a}) and $(\psi_{4a}, v_{4a}) \sim O(1)$.

The boundary conditions for (ψ_{2a}, v_{2a}), (ψ_{4a}, v_{4a}) are found by substituting (3.1) into the z conditions in (2.3) and applying lim $R \to \infty$, η, $\zeta_{2,4}$ held fixed. At $z = 0$,

$$\psi_{2a1} + \psi_{a1} = \frac{\partial^2 \psi_{2a1}}{\partial \zeta_2^2} = \frac{\partial v_{2a1}}{\partial \zeta_2} = 0$$

and at $z = -1$:

$$\psi_{a1} + \psi_{4a1} = \frac{\partial \psi_{4a1}}{\partial \zeta_4} = v_{a1} + v_{4a1} = 0.$$

Solving together equations (3.7), and the Ekman equations (3.3) written for (ψ_{2a}, v_{2a}) and (ψ_{4a}, v_{4a}) it is found that

$$\left.\begin{aligned} \psi_{a1} &= -\tfrac{1}{2} z e^{-\eta} V_0, \\ v_{a1} &= V_0 e^{-\eta}, \\ \psi_{2a1} &= v_{2a1} = 0, \\ \psi_{4a1} &= -\tfrac{1}{2} V_0 e^{-\eta-\zeta_4} (\cos \zeta_4 + \sin \zeta_4), \\ v_{4a1} &= -V_0 e^{-\eta-\zeta_4} \cos \zeta_4, \end{aligned}\right\} \quad (3.8)$$

where V_0 is a constant to be determined.

The flow along the channel in the interior is matched by the geostrophic current $V_0 e^{-\eta}$ in this boundary layer. Although V_0 can be chosen to bring the interior flow to zero at the wall, the stream function cannot be brought to zero there, and neither can $\partial\psi/\partial x$ be nulled at the wall; these conditions require a $R^{-1/3}$ layer adjacent to the wall.

In this region the equations (2.2) become

$$2\frac{\partial v}{\partial z} - R^{-1/6} \frac{\partial^4 \psi}{\partial \xi^4} = O(R^{-5/6}), \quad (3.9)$$

$$\frac{\partial^2 v}{\partial \xi^2} + 2R^{-1/6} \frac{\partial \psi}{\partial z} = O(R^{-2/3}), \quad (3.10)$$

where $x = R^{-1/3}\xi$.

From (3.9) it is seen that if $\psi_3 \sim 1$ for matching with the interior, then $v_3 \sim R^{-1/6}$. On substituting (3.1) in (3.9) and (3.10) and taking the limit $R \to \infty$, ξ, z fixed we have

$$\begin{aligned} 2\frac{\partial v_{31}}{\partial z} - \frac{\partial^4 \psi_{31}}{\partial \xi^4} &= 0, \\ \frac{\partial^2 v_{31}}{\partial \xi^2} + 2\frac{\partial \psi_{31}}{\partial z} &= 0 \end{aligned} \quad \text{or} \quad \frac{\partial^6 \psi_{31}}{\partial \xi^6} + 4\frac{\partial^2 \psi_{31}}{\partial z^2} = 0. \quad (3.11)$$

It would be suspected that again component Ekman solutions would be required at the top and bottom of this layer. It is found, however, that if $\psi_{31}(\xi, z)$ can be made to vanish at top and bottom (which it can), then the composite solution in the region where $\xi \sim 1$ and $\zeta_2 \sim 1$ or $\zeta_4 \sim 1$, does satisfy the boundary conditions to the lowest order. Substitution of (3.1) into the boundary conditions at $x = 0$, and taking the lim $R \to \infty$, ξ, z fixed, demands

$$\left.\begin{aligned} \psi_{31}(0, z) + \psi_{a1}(0, z) + \psi_{11}(0, z) &= 0, \\ \frac{\partial \psi_{31}}{\partial \xi} = v_{11}(0) + v_{a1}(0) = v_{31}(0, z) &= 0. \end{aligned}\right\} \quad (3.12)$$

Solution of (3.11) subject to these conditions. yields

$$\left.\begin{aligned} \psi_{31}(\xi, z) &= -\tfrac{1}{2}\tau_2(0)\, G(\xi, z), \\ v_{31}(\xi, z) &= -\tfrac{1}{2}\tau_2(0)\, H(\xi, z), \end{aligned}\right\} \quad (3.13)$$

where

$$G(\xi, z) = -2 \sum_{n=1}^{\infty} \alpha_n^{-1} \sin \alpha_n z \times \left\{\cos \frac{\sqrt{3}}{2} \beta_n^{1/3} \xi + \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} \beta_n^{1/3} \xi\right\} e^{-\beta_n^{1/3}\xi/2},$$

$$H(\xi, z) = -\frac{2^{7/3}}{\sqrt{3}} \sum_{n=1}^{\infty} \alpha_n^{-2/3} \cos \alpha_n z e^{-\beta_n^{1/3}\xi/2} \sin \frac{\sqrt{3}}{2} \beta_n^{1/3} \xi,$$

and $\beta_n = 2\alpha_n = 2n\pi$.

Component solutions have been determined except for the small corner regions where $\xi < 1$, $\zeta \sim 1$. Since the equations (2.3) are symmetrical with respect to x, the component solutions on the right wall may be found by replacing x by $L - x$ in the above solutions. Collecting the above, the *composite* solution for each region can be written:

$$\psi_1(x, z) \sim \tfrac{1}{2}[\tau_2(x) + zP] + \dots,$$

$$v_1(x, z) \sim \tau_2(x) - P + \dots,$$

$$\psi_2(x, \zeta_2) \sim \tfrac{1}{2}\tau_2(x)\,[1 - e^{\zeta_2} \cos \zeta_2] + \tfrac{1}{2}zP + \dots,$$

$$v_2(x, \zeta_2) \sim \tau_2(x)\,[1 + \tfrac{1}{2}e^{\zeta_2}(\cos \zeta_2 + \sin \zeta_2)] - P + \dots,$$

$$\psi_4(x, \zeta_4) \sim \tfrac{1}{2}\tau_2(x)\,[1 - e^{-\zeta_4}(\cos \zeta_4 + \sin \zeta_4)] + \tfrac{1}{2}P[e^{-\zeta_4}(\cos \zeta_4 + \sin \zeta_4) + z] + \dots,$$

$$v_4(x, \zeta_4) \sim \tau_2(x)\,[1 - e^{-\zeta_4} \cos \zeta_4] + P[e^{-\zeta_4} \cos \zeta_4 - 1] + \dots,$$

$$\psi_a(x,z) \sim \tfrac{1}{2}\tau_2(x) + \tfrac{1}{2}ze^{-\eta}\tau_2(0) + \tfrac{1}{2}zP(1-e^{-\eta}) + \dots,$$

$$v_a(x,z) \sim \tau_2(x) - \tau_2(0)\,e^{-\eta} + P(e^{-\eta}-1) + \dots,$$

$$\psi_{2a}(x,z) \sim \psi_a + \psi_2; \quad v_{2a} \sim v_a + v_2,$$

$$\begin{aligned}\psi_{4a}(x,z) \sim{}& \tfrac{1}{2}\tau_2(x)\,[1 - e^{-\zeta_4}(\cos\zeta_4 + \sin\zeta_4)] \\ &+ \tfrac{1}{2}\tau_2(0)\,e^{-\eta}[z + e^{-\zeta_4}(\cos\zeta_4 + \sin\zeta_4)] \\ &+ \tfrac{1}{2}P[z + e^{-\zeta_4}(\cos\zeta_4 + \sin\zeta_4)] \\ &\times (1 - e^{-\eta}) + \dots, \qquad (3.14)\end{aligned}$$

$$\begin{aligned}v_{4a}(x,z) \sim{}& [\tau_2(x) - \tau_2(0)\,e^{-\eta}]\,(1 - e^{-\zeta_4}\cos\zeta_4) \\ &+ P[-1 + e^{\zeta_4}\cos\zeta_4]\,(1 - e^{-\eta}) + \dots,\end{aligned}$$

$$\begin{aligned}\psi_3(x,z) \sim{}& \tfrac{1}{2}\tau_2(x) - \tfrac{1}{2}\tau_2(0)\,[e^{-\eta}z + G(\xi,z)] \\ &+ \tfrac{1}{2}zP(1 - e^{-\eta}) + \dots,\end{aligned}$$

$$\begin{aligned}v_3(x,z) \sim{}& \tau_2(x) - \tau_2(0)\,e^{-\eta} - \frac{\tau_2(0)}{2}R^{-1/6}H(\xi,z) \\ &+ P(e^{-\eta} - 1) + \dots,\end{aligned}$$

$$\psi_7(x,z) \sim \psi_2 + \psi_3; \quad v_7 \sim v_2 + v_3,$$

$$\begin{aligned}\psi_8(x,z) \sim{}& \tfrac{1}{2}\tau_2(x)\,[1 - e^{-\zeta_4}(\cos\zeta_4 + \sin\zeta_4)] \\ &+ \tfrac{1}{2}\tau_2(0)\,\{e^{-\eta}[z + e^{-\xi_4}(\cos\zeta_4 + \sin\zeta_4)] \\ &- G(\xi,z)\} + \tfrac{1}{2}P[z + e^{-\zeta_4}(\cos\zeta_4 + \sin\zeta_4)] \\ &\times (1 - e^{-\eta}) + \dots,\end{aligned}$$

$$\begin{aligned}v_8(x,z) \sim{}& [\tau_2(x) - \tau_2(0)\,e^{-\eta}]\,(1 - e^{-\eta_4}\cos\zeta_4) \\ &+ P(1 - e^{-\eta})\,(-1 + e^{-\zeta_4}\cos\zeta_4) \\ &- \tfrac{1}{2}\tau_2(0)\,R^{-1/6}H(\xi,z) + \dots,\end{aligned}$$

where the notation ψ_j refers to the composite solution, whether on the left or right side of the equation; further, v, the velocity component along the channel, has not been rescaled, as in (2.2).

The wind stress across the channel

With the wind stress exerted across the channel, the Ekman transport, being along the channel, will no longer require upwelling and sinking motions at the walls for conservation of mass. There will be small corner regions wherein the velocity must be reduced to zero, but these apparently are not susceptible to the boundary layer approach. The equations and boundary conditions are the same as (2.3) except at $z=0$:

$$\frac{\partial v}{\partial z} = 0, \quad R^{-1}\frac{\partial^2\psi}{\partial z^2} = -\tau_1(x), \quad w = p = 0.$$

The equations are obtained for the regions 1, 2 and 4 in the same way as before. It is easily verified that although $(\psi_2, v_2) \sim 1$, it is necessary that $(\psi_1, v_1) \sim (\psi_4, v_4) \sim R^{-1}$, that is, the interior solution arises from the second order terms in the Ekman layer; these solutions have coefficients depending not on $\tau_1(x)$, but $\partial^2\tau_1/\partial x^2 \equiv \tau_1''(x)$, since the terms in (3.3) of order R^{-1}, will appear as nonhomogeneous forcing terms. Since these solutions are derived with little variation in method from those found above, the composite solution will merely be written down for these regions.

$$\psi_1(x,z) \sim -R^{-1}\tau_1''(x)/4 + \dots,$$

$$v_1(x,z) \sim -R^{-1}\tau_1''(x)/2 + \dots,$$

$$\begin{aligned}\psi_2(x,z) \sim{}& -\tfrac{1}{2}\tau_1(x)\,e^{\zeta_2}\sin\zeta_2 + R^{-1}\frac{\tau_1''(x)}{4} \\ &\times \left\{-1 + e^{\zeta_2}\cos\zeta_2 + \frac{3}{4}\zeta_2 e^{\zeta_2}(\sin\zeta_2 - \cos\zeta_2)\right\} + \dots,\end{aligned}$$

$$\begin{aligned}v_2(x,z) \sim{}& -\tau_1(x)\,e^{\zeta_2}(\cos\zeta_2 - \sin\zeta_2) + R^{-1}\frac{\tau_1''(x)}{2} \\ &\times \left[-1 + \frac{3}{4}(\zeta_2 e^{\zeta_2}\cos\zeta_2 - \tfrac{1}{2}e^{\zeta_2}(\cos\zeta_2 + \sin\zeta_2)\right] + \dots,\end{aligned}$$

$$\psi_4(x,z) \sim R^{-1}\frac{\tau''(x)}{4}[-1 + e^{-\zeta_4}(\cos\zeta_4 + \sin\zeta_4)] + \dots,$$

$$v_4(x,z) \sim R^{-1}\frac{\tau_1''(x)}{2}[-1 + e^{-\zeta_4}] + \dots.$$

The solutions in the interior and bottom are identical with those for the case of wind stress along the channel (3.14), if $-\tfrac{1}{2}\tau''(x)R^{-1}$ replaces $\tau_2(x)$, and P is set to zero. Hence the wall solutions in the case of stress across the channel may be found directly from (3.14). Instead of repeating the lengthy solution with the above replacement, it is merely noted that the presence of the lateral boundary current depends on the nonvanishing of $\partial^2\tau_1(0)/\partial x^2$.

Discussion of the solution and conclusions

The circulation transverse to the axis of the channel for the case of stress along the axis is the following. There is the rightward surface Ekman transport $\tau_2(x)/2$, the gradient current transport $u_{11} = -\frac{1}{2}P$, and the horizontal transport in the friction layer at the bottom $\frac{1}{2}(P-\tau_2(x))$, in addition to the vertical transport of mass out of the surface layer, $-\frac{1}{2}(\partial\tau_2/\partial x)$. The presence of the friction layer at the bottom, associated with the transport $\frac{1}{2}(P-\tau_2(x))$ requires the interior geostrophic current v_{11} along the channel.

At the left wall the vertical transport by the combined $R^{-1/4}$ and $R^{-1/3}$ scales of motion is the negative sum of the two component stream functions evaluated at the wall, obtained from (3.8), (3.9) and (3.13)

$$\begin{aligned} -\psi_{a1}(0,z)-\psi_{31}(0,z) &= -\tfrac{1}{2}z[\tau_2(0)-P]+\tfrac{1}{2}\tau_2(0)\,G(0,z) \\ &= -\tfrac{1}{2}z[\tau_2(0)-P]+\tfrac{1}{2}\tau_2(0)\,(1+z) \\ &= -\tfrac{1}{2}\tau_2(0)+\tfrac{1}{2}zP; \end{aligned}$$

similarly at the right wall the vertical transport is $-\frac{1}{2}\tau_2(L)-\frac{1}{2}zP$. Here $V_0 = P-\tau_2(0)$ previously obtained from (3.12), has been used. The sum of these two transports represents the balance to the net downward transport in the interior from the surface Ekman layer. The maximum vertical velocities occur in the $R^{-1/3}$ region; from (3.14) it is found that the ratio of w to the magnitude of the horizontal velocities occurring in the surface friction layer $|w/v| \sim R^{1/3-1/2} = R^{-1/6}$, where v has been rescaled consistent with (2.2).

The currents produced by the constant pressure gradient P along the channel represent a counterclockwise circulation ($P>0$), as seen from the solutions in regions 1 and 4 in (3.14).

A case of special interest is that of uniform stress along the channel. If in addition, $P=\tau_2=$ constant, the gradient current u_{11} across the channel exactly balances the surface Ekman transport. The friction layer at the bottom is no longer required to the lowest order; hence the gradient current v_{11} disappears and, since no vertical transport from the bottom is required, the $R^{-1/4}$ scale of motion at the walls vanishes. Maximum horizontal speeds occur in the Ekman layer. This case must be close to representing flow in a rectangular basin due to uniform wind stress along one axis; since to the lowest order only u differs from zero in the interior, insertion of walls across the channel, say at $y=0$, M, would require only a lateral friction layer to null the current at the walls.

Returning to the variable stress case, if L is a characteristic horizontal scale, an estimate of the width of the lateral boundary layer thickness is $\delta \sim R^{-1/4}L$. If $\Omega \sim 0.5\times10^{-4}$ sec^{-1} and $10^4 < A < 10^6$ cm^2 sec^{-1} and if L is in km, $4\,L^{1/2} < \delta < 1.2\,L^{1/2}$; if $L=100$ km, then $4<\delta<12$ km. Since δ depends on the fourth root of the viscosity coefficient, it is not very sensitive to the latter.

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ТЕЧЕНИЯ, ВОЗБУЖДАЕМЫЕ ВЕТРОМ В ДЛИННОМ ВРАЩАЮЩЕМСЯ КАНАЛЕ

Получены решения типа пограничного слоя для течений, возбуждаемых в однородной жидкости в длинном вращающемся канале при помощи напряжения скорости ветра, действующего вдоль или поперек канала. Эти решения, в частности, описывают вязкие пограничные слои у стенок, которые с необходимостью возникают из вторичных циркуляционных течений в плоскости, перпендикулярной оси канала.