Energy partition in a barotropic atmosphere

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ABSTRACT

The response of an unbounded rotating barotropic atmosphere to a randomly distributed initial energy input is investigated. The initial condition is specified in terms of an energy density function and the solutions (kinetic and potential energies) of a one-dimensional and time-dependent linearized system are determined. The results show that the approach to a predominantly steady-state geostrophic energy distribution is rapid if the initial energy density spectrum is broad and/or the wave number of maximum initial energy density occurs at a relatively large wave-number, $k \geqslant 4$. If the initial energy density is concentrated in a narrow band in the low wave-number part of the spectrum a significant amount of energy goes into ageostrophic inertia-motion and the approach to the steady-state occurs slowly.

1. Introduction

Rossby (1938) first formulated the problem of the "adjustment" of an initially unbalanced current to the time-asymptotic state of geostrophic balance. Other authors (e.g. Cahn, 1945; Obukhov, 1949) have investigated the linearized initial-value problem for a barotropic fluid in which particular initial conditions are prescribed in a finite region R of an infinite rotating flat plane. Veronis (1956) has extended the adjustment problem to a simple two-layer oceanic model and has also investigated some of the energetics of the motion as a function of the input energy, due to a wind stress, which is applied for a finite duration. These investigations have been concerned with the approach to steady-state geostrophic balance in the region R due to horizontal dispersion of energy by fast moving gravity-inertia waves, which make the motion in the initial quiescent region outside R increasingly ageostrophic.

The results of these investigations do provide an insight into the process of "geostrophic adjustment" in a limited region; however the more general problem of adjustment, when motions in many regions of the atmosphere are not in geostrophic balance initially, is not considered.

In the present investigation we shall assume that atmospheric motions are in a state of imbalance at a particular instant ($t=0$), and that these motions are a random function of position over the whole infinite rotating flat plane. These initial conditions are more general than those of the above named authors, but by introducing random initial conditions only the statistics (mean kinetic and potential energies) of the transient and steady-state motion may be determined.

We shall further assume that the initial fields are statistically homogeneous in space, implying that the statistics are independent of position. This latter assumption simplifies the mathematical development while retaining the basic physics of the problem. To summarize, the purpose of this investigation is to investigate the response of a barotropic atmosphere to ageostrophic initial conditions which are randomly distributed in space.

2. Basic equations

A basic system of equations, which describe the field of vertically averaged motion, has been derived by Obukhov (1949). We shall use the linearized nondimensional equations of this system in the form

$$u_t - v = -\pi_x, \tag{2.1}$$

$$v_t + u = -\pi_y, \tag{2.2}$$

$$\pi_t + u_x + v_y = 0, \tag{2.3}$$

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where the constant Coriolis parameter $f = 2\omega \sin \phi$ and the radius of deformation $\lambda = (gH_0)^{\frac{1}{2}} f^{-1}$ (g is the acceleration of gravity and H_0 is the depth of a homogeneous atmosphere corresponding to standard values of atmospheric pressure and density) have been used to nondimensionalize the time and length scales respectively. Characteristic values of these parameters in middle latitudes are $f = 10^{-4}\text{sec}^{-1}$ and $\lambda = 2700$ km. The (x, y) components (u, v), which are density weighted vertically averaged velocities, have been nondimensionalized by a characteristic velocity U, chosen to make the time and space derivatives of the velocity field of order unity. The variable $\pi = (p(x, y, 0, t) - p_0)p_0^{-1}$, where p_0 denotes standard sea level pressure and p denotes the deviation from it. These equations also apply to motions in a homogeneous ocean of mean depth H_0 if π is the percentage variation in the depth.

If we multiply (2.1), (2.2) and (2.3) by u, v and π respectively and add them together we obtain the energy equation

$$\frac{\partial}{\partial t}\,\tfrac{1}{2}(u^2 + v^2 + \pi^2) = -\nabla \cdot (\mathbf{v}\pi), \tag{2.4}$$

where ∇ is the two-dimensional gradient operator and $\mathbf{V}$ is the two-dimensional velocity. The probability or ensemble average of a spatially homogeneous field (Yaglom, 1962) will be denoted by a bar. If the field of motion is spatially homogeneous the mean value of any quantity is independent of position. Then if we average (2.4) and integrate we obtain

$$\tfrac{1}{2}(\overline{u^2} + \overline{v^2} + \overline{\pi^2}) = \bar{E}_0, \tag{2.5}$$

where E_0 denotes the energy of the initial state. The expression in (2.5) is also the space average of the energy, since the field of motion is spatially homogeneous.

The field of motion may be expressed as

$$u = u_g + u', \quad v = v_g + v', \quad \pi = \pi_g + \pi'. \tag{2.6}$$

The subscript g denotes steady geostrophic motion, which has the property $\partial/\partial t \equiv 0$ and, as Obukhov has shown, it may be determined from the equation of potential vorticity

$$\nabla^2 \pi_g - \pi_g = \Omega(x, y). \tag{2.7}$$

$\Omega(x, y)$ is determined from the initial conditions and is defined as

$$\Omega(x, y) \equiv \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} - \pi_0, \tag{2.8}$$

∇^2 is the two-dimensional Laplacian and

$$u_g = -\partial \pi_g/\partial y, \quad v_g = \partial \pi_g/\partial x. \tag{2.9}$$

Further, the ageostrophic motion has the property that its potential vorticity is

$$\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} - \pi' = 0 \tag{2.10}$$

and it follows from (2.10) that u', v' or π' satisfy

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + 1\right)(u', v', \pi') = 0. \tag{2.11}$$

If we substitute (2.6) into (2.5), then the mean energy equation becomes

$$\tfrac{1}{2}(\overline{u'^2} + \overline{v'^2} + \overline{\pi'^2} + \overline{u_g^2} + \overline{v_g^2} + \overline{\pi_g^2}) = \bar{E}_0 \tag{2.12}$$

since

$$\begin{aligned}\overline{u'u_g} + \overline{v'v_g} + \overline{\pi'\pi_g} &= -\overline{u'\partial\pi_g/\partial y + v'\partial\pi_g/\partial x} + \overline{\pi'\pi_g} \\ &= -\overline{\pi_g(\partial v'/\partial x - \partial u'/\partial y - \pi')} = 0,\end{aligned}$$

using (2.9) and (2.10).

3. Solution

In the following development we shall permit no variation in the x-direction ($\partial/\partial x \equiv 0$). Cahn (1945) has solved this simplified system of equations for a particular initial condition. Here we shall investigate the response of this system to a random initial condition, characterized by a given energy density function. At the initial time, $t = 0$,

$$\left.\begin{aligned} &v = v' = 0, && \partial v'/\partial t = -u_0(y), \\ &u = u_0(y), && \partial u/\partial t = 0, \\ &\pi = 0, && \partial \pi/\partial t = 0, \end{aligned}\right\} \tag{3.1}$$

where $u_0(y)$ is a homogeneous random velocity field, which may be represented by a Fourier-Stieltjes integral (Yaglom, 1962)

$$u_0(y) = \int_{-\infty}^{\infty} dU_0 e^{iky}. \tag{3.2}$$

Since the velocity field is statistically homogeneous,

$$\tfrac{1}{2}\overline{u_0(y)\, u_0^*(y+r)} = \tfrac{1}{2}\int_{-\infty}^{\infty} \Psi(k)\, e^{ikr} dk, \tag{3.3}$$

where $\Psi(k)/2$ is the initial energy density spectrum defined as:

$$\tfrac{1}{2}\Psi(k) = \tfrac{1}{2}\,\frac{\overline{dU_0(k)\, dU_0^*(k)}}{dk}, \tag{3.4}$$

the asterisk denoting a complex conjugate. The conditions on $\Psi(k)$ are:

$$\left.\begin{aligned} &\Psi(k) > 0, \\ \text{and}\quad &\Psi(k) = \Psi(-k), \\ &\int_{-\infty}^{\infty} \Psi(k)\, dk < \infty. \end{aligned}\right\} \tag{3.5}$$

Similarly Cahn's solution of (2.11) with initial conditions (3.1) may be expressed in the form

$$v'(y,t) = -\int_{-\infty}^{\infty} dU_0 \frac{\sin \omega t}{\omega} e^{iky}, \tag{3.6}$$

where $\omega^2 \equiv 1 + k^2$. From (2.1), (2.3) and (3.1), with $\partial/\partial x \equiv 0$, we obtain

$$u(y,t) = \int_{-\infty}^{\infty} dU_0 \left[1 + \frac{1}{\omega^2}(\cos \omega t - 1)\right] e^{iky} \tag{3.7}$$

and

$$\pi(y,t) = i\int_{-\infty}^{\infty} dU_0 \frac{k}{\omega^2}(1 - \cos \omega t)\, e^{iky} \tag{3.8}$$

respectively. With the aid of (2.7), (2.8) and (2.9), we obtain

$$\pi_g = i\int_{-\infty}^{\infty} dU_0 \frac{k}{\omega^2} e^{iky} \tag{3.9}$$

and

$$u_g = \int_{-\infty}^{\infty} dU_0 \frac{k^2}{\omega^2} e^{iky}. \tag{3.10}$$

The ageostrophic components u' and π' are

$$u' = u - u_g = \int_{-\infty}^{\infty} dU_0 \frac{\cos \omega t}{\omega^2} e^{iky} \tag{3.11}$$

and

$$\pi' = \pi - \pi_g = -i\int_{-\infty}^{\infty} dU_0 \frac{k}{\omega^2} \cos \omega t e^{iky}. \tag{3.12}$$

It is now possible to compute the various terms in the energy equation (2.12), which becomes

$$\tfrac{1}{2}(\overline{u'^2} + \overline{v'^2} + \overline{\pi'^2} + \overline{u_g^2} + \overline{\pi_g^2}) = \frac{\overline{u_0^2}}{2}. \tag{3.13}$$

With $r = 0$, (3.3) reduces to

$$\frac{\overline{u_0^2}}{2} = \frac{1}{2}\int_{-\infty}^{\infty} \Psi(k)\, dk. \tag{3.14}$$

Similarly,

$$\frac{\overline{u'^2}}{2} = \frac{1}{4}\int_{-\infty}^{\infty} \frac{(1 + \cos 2\omega t)}{\omega^4} \Psi(k)\, dk, \tag{3.15}$$

$$\frac{\overline{v'^2}}{2} = \frac{1}{4}\int_{-\infty}^{\infty} \frac{(1 - \cos 2\omega t)}{\omega^2} \Psi(k)\, dk, \tag{3.16}$$

$$\frac{\overline{\pi'^2}}{2} = \frac{1}{4}\int_{-\infty}^{\infty} k^2 \frac{(1 + \cos 2\omega t)}{\omega^4} \Psi(k)\, dk, \tag{3.17}$$

$$\frac{\overline{u_g^2}}{2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{k^4}{\omega^4} \Psi(k)\, dk, \tag{3.18}$$

and

$$\frac{\overline{\pi_g^2}}{2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{k^2}{\omega^4} \Psi(k)\, dk, \tag{3.19}$$

where $\Psi(k)$ is defined by (3.4). It should be noted that the time-dependent terms in the energy equation cancel, thus (3.13) is satisfied by the steady-state expressions in (3.15) through (3.19).

In order to evaluate the various terms in (3.13), $\Psi(k)$ must be specified. The expression for $\Psi(k)$ may be somewhat arbitrary as long as the conditions in (3.5) are satisfied.[1] Physically, one might expect that the initial energy density in wave-number space has one or more maxima with the bulk of energy stored in the relatively

[1] However $\Psi(k)$ may be equal to a delta function, indicating a perfect correlation of the initial velocity field (3.3).

low wave numbers. We shall simulate this situation by letting

$$\frac{\overline{u_0^2}}{2}=\frac{1}{2}\int_{-\infty}^{\infty}\Psi(k)\,dk$$
$$=\frac{1}{2}\int_{-\infty}^{\infty}\frac{b}{\pi}\left[\frac{1}{(k-a)^2+b^2}+\frac{1}{(k+a)^2+b^2}\right]dk=1. \tag{3.20}$$

If $b<\sqrt{3}a$, then the wave number of maximum energy density (for $k>0$) occurs at

$$k=\sqrt{a^2+b^2}\sqrt{\sqrt{2\left(1+\frac{a^2-b^2}{a^2+b^2}\right)}-1}. \tag{3.21}$$

If $b\ll\sqrt{3}a$ then $k_{\max}\approx a$; if $b\geqslant\sqrt{3}a$ the maximum occurs at $k=0$ and $\Psi(k)$ decreases monotonically for $k>0$. $\Psi(k)/2$ is displayed in Fig. 5 for various a and b.

4. Time-dependent energy

Consider the expression for the ageostrophic component $\overline{v'^2}/2$, given by (3.16),

$$\tfrac{1}{2}\overline{v'^2}=\tfrac{1}{2}\int_{-\infty}^{\infty}\frac{(1-\cos 2\omega t)}{\omega^2}\Psi(k)\,dk. \tag{4.1}$$

If the substitution $\omega\equiv\sqrt{1+k^2}=\eta$ is made, in order to apply the Riemann-Lebesgue lemma (Lighthill, 1959) to the time-dependent part of the integral in (4.1), we find that the absolute integrability of $\Psi(k)$ (3.5) is more than sufficient to ensure that

$$\tfrac{1}{2}\overline{v'^2}\to\tfrac{1}{2}\int_{-\infty}^{\infty}\frac{\Psi(k)}{\omega^2}\,dk=C$$

(C = constant) as $t\to\infty$. In particular, the spectrum given by (3.20) satisfies the conditions under which the time-dependent part of (4.1) will approach zero as $t\to\infty$. When $t\ll1$, $\overline{v'^2}/2\sim t^2$; while for $t\gg1$, application of the method of stationary phase (Erdélyi, 1956) shows that

$$\tfrac{1}{2}\overline{v'^2}\sim C+\frac{b}{a^2+b^2}\,\frac{\cos 2t}{t^{\frac{1}{2}}}. \tag{4.2}$$

After the initial energy increase the ageostrophic energy component $\overline{v'^2}/2$ oscillates about its steady-state value with a period equal to one half the inertial period and its amplitude approaches zero like $t^{-\frac{1}{2}}$. Further, the greatest amplitude of this oscillation occurs when the maximum initial energy density appears in the longest wave ($a=0$) and the energy density is concentrated in the long waves (b small).

The numerical solution of $\overline{v'^2}/2$ was accomplished by expanding $\overline{v'^2}/2$ in a Taylor's series, $\overline{v'^2}/2\,|_{t+\tau}\approx\overline{v'^2}/2\,|_t+\partial(\overline{v'^2}/2)/\partial t\,|_t\tau$ and retaining only the first two terms. The time derivative, found from (4.1), is

$$\frac{\partial}{\partial t}\frac{\overline{v'^2}}{2}=\frac{1}{2}\int_{-\infty}^{\infty}\Psi(k)\frac{\sin 2\omega t}{\omega}\,dk$$
$$=\frac{1}{4}\int_{-\infty}^{\infty}\Psi(k)\,e^{-ik\beta}dk\int_{-2t}^{2t}J_0(\sqrt{(2t)^2-\beta^2})\,d\beta, \tag{4.3}$$

(see Magnus & Oberhettinger, 1949, p. 118) where J_0 is the zero-order Bessel function. Since $\Psi(k)=\Psi(-k)$, (4.3) may be written

$$\frac{\partial}{\partial t}\frac{\overline{v'^2}}{2}=\frac{1}{2}\int_{0}^{2t}J_0(\sqrt{(2t)^2-\beta^2})\,B(\beta)\,d\beta, \tag{4.4}$$

where

$$\tfrac{1}{2}B(\beta)=\tfrac{1}{2}\int_{-\infty}^{\infty}\Psi(k)\,e^{ik\beta}dk=e^{-b\beta}\cos a\beta \tag{4.5}$$

if $\Psi(k)$ is given by (3.20).

Some numerical computations of $\overline{v'^2}/2$ are presented in Fig. 1. The solid horizontal line represents the steady-state value $\overline{v_s'^2}/2$ for a given a and b. As noted above the energy initially increases like t^2; it then grows linearly and finally it settles into the asymptotic state, given by (4.2). Further, the wave-amplitude as a function of a and/or b and/or t is given by (4.2) for $t\gg1$. When $a=4$, $b=1$, two maxima appear in the range $0<t<1.5$. This phenomenon also appears in other computations (not shown) when $a>b$ ($a\geqslant4$) and the two parameters are sufficiently separated. The exact criterion for the appearance of the two maxima has not been determined since it is always associated with a very small-amplitude oscillation (4.2) and is therefore relatively unimportant.

In the special case of a perfect correlation in the initial field ($a=b=0$), (4.4) yields

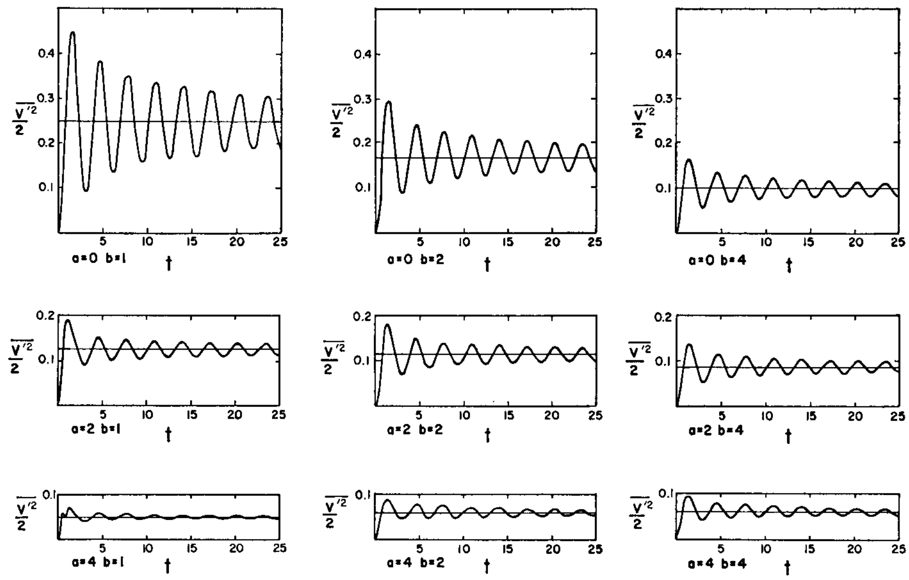

Fig. 1. The ageostrophic energy component $\overline{v'^2}/2$ as a function of time for various a and b. The solid horizontal line in each graph is the asymptotic steady-state value of $\overline{v'^2}/2$ for the corresponding a and b. If $f = 10^{-4}\ \mathrm{sec}^{-1}$, one nondimensional time unit corresponds to 2.78 hours.

$$\frac{\partial}{\partial t}\frac{\overline{v'^2}}{2} = \int_0^{2t} J_0\left(\sqrt{(2t)^2-\beta^2}\right) d\beta = \sin 2t \qquad (4.6)$$

(Magnus & Oberhettinger, 1949, p. 34). The amplitude of the oscillation does not damp with time. This case represents a pure inertia-oscillation with equipartition of energy between the ageostrophic velocity components and no potential energy present.

These computations agree qualitatively with the results of Washington (1964), who showed that in a barotropic fluid the steady-state is approached rapidly if the scale of the initial velocity field is much smaller than the radius of deformation λ; while the steady-state is approached more slowly as the scale of the initial field is increased. In the limit, as the scale of the initial field approaches an infinite length, a time-independent steady-state is never reached since these one-dimensional pure inertia-waves have zero group-velocity and wave dispersion cannot occur.

5. Steady-state energy

The time independent expressions for the energies in (3.15) through (3.19) may be integrated exactly when $\Psi(k)$ is expressed by (3.20). In the upper half of the complex k-plane each expression has simple poles at $k = \pm a + ib$ and a simple pole (or a pole of order two) at $k = i$. Therefore, each integration may be accomplished by summing the residues at the poles. These integrations were carried out for the steady-state energies $\overline{v_s'^2}/2$ and $\overline{\pi_s'^2}/2$, given by

$$\frac{\overline{v_s'^2}}{2} = \frac{1}{2}\left[\frac{b(a^2+b^2-1)}{(a^2+b^2-1)^2+4a^2} + \frac{a^2-b^2+1}{(a^2-b^2+1)^2+4a^2b^2}\right]$$

and (5.1)

$$\frac{\overline{\pi_s'^2}}{2} = \frac{1}{4}\left\{\frac{2(a^2-b^2)[(a^2-b^2+1)^2-4a^2b^2]+16a^2b^2(a^2-b^2+1)}{[(a^2-b^2+1)^2-4a^2b^2]^2+16a^2b^2(a^2-b^2+1)^2}\right.$$
$$\left. + \frac{b(a^2+b^2+1)[(a^2+b^2-1)^2-4a^2]}{[(a^2+b^2-1)^2-4a^2]^2+16a^2(a^2+b^2-1)^2}\right\}. \qquad (5.2)$$

The remaining energies were computed using the following expressions, which are clearly deducible from (3.13) and (3.15) through (3.20),

$$\frac{\overline{u_s'^2}}{2} = -\left(\frac{\overline{\pi_s'^2}}{2} - \frac{\overline{v_s'^2}}{2}\right), \qquad (5.3)$$

$$\frac{\overline{\pi_g^2}}{2} = 2\left(\frac{\overline{\pi_s'^2}}{2}\right) \qquad (5.4)$$

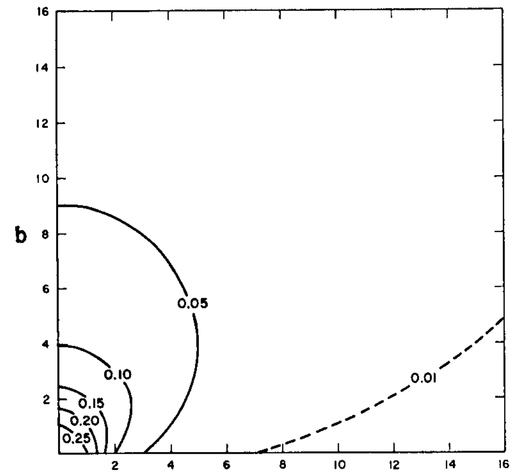

Fig. 2. The steady-state ageostrophic energy component $\overline{v_s'^2}/2$ as a function of a and b.

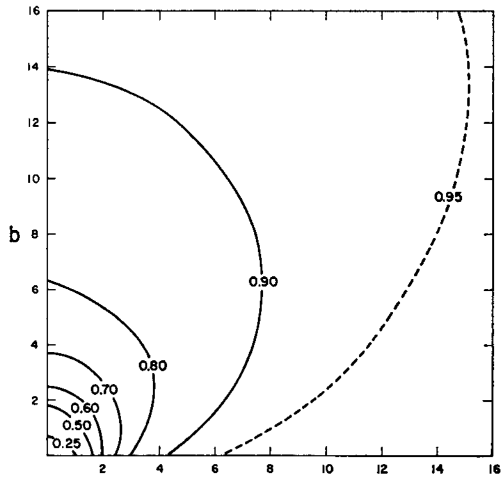

Fig. 3. The steady-state geostrophic energy component $\overline{u_g^2}/2$ as a function of a and b.

and
$$\frac{\overline{u_g^2}}{2} = 1 - 2\left(\frac{\overline{v_s'^2}}{2} + \frac{\overline{\pi_s'^2}}{2}\right). \qquad (5.5)$$

The numerical computations of $\overline{v_s'^2}/2$ and $\overline{u_g^2}/2$ as functions of a and b are displayed in Figs. 2 and 3 respectively. The most striking feature of these diagrams is the large region, in the (a, b) plane, in which the geostrophic kinetic energy $\overline{u_g^2}/2$ is close to unity. Only in a very small region, where a and b are small, does the ageostrophic energy, associated with inertia motion, predominate.

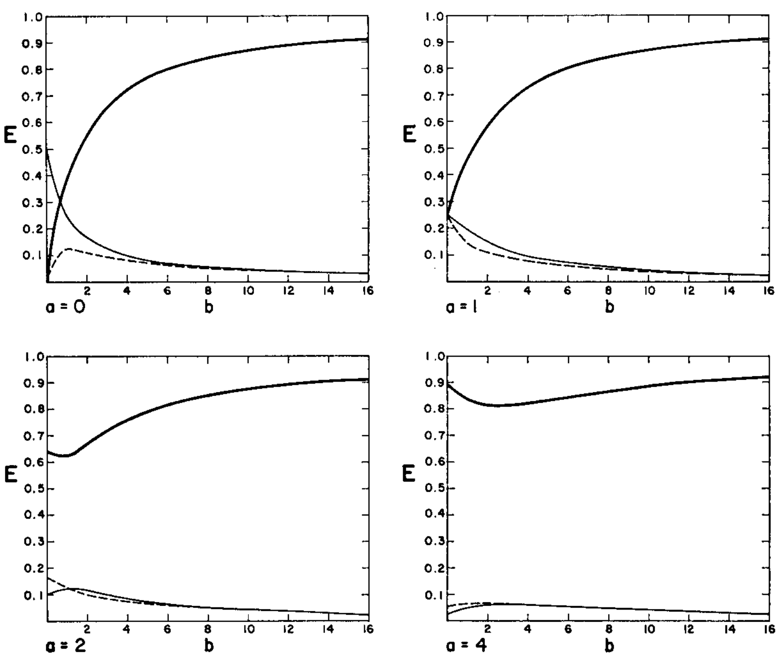

Fig. 4. Steady-state energies (E) as a function of b for given values of a. $\overline{u_g^2}/2$, $\overline{v_s'^2}/2$ and $\overline{\pi_g^2}/2$ are denoted by the heavy solid, thin solid and dashed lines respectively.

This situation may be seen more clearly in Fig. 4, where the energies are shown as a function of b for $a = 0$, 1, 2 and 4. The geostrophic potential energy $\overline{\pi_g^2}/2$ (dashed line) has also been included in Fig. 4, but $\overline{\pi_s'^2}/2 = 1/2(\overline{\pi_g^2}/2)$ and $\overline{u'^2}/2 \approx 1/2(\overline{v'^2}/2)$ for $b > 2$, have been omitted. These figures show that there is a tendency for a maximum of ageostrophic kinetic energy (and minimum of geostrophic kinetic energy) for a value $b \sim a$. When $b \sim a$, the wave number of maximum initial energy density (3.21) is $k \sim (a^2 + b^2)^{1/2}$. The figures also show that, for a given a, an increasing amount of energy goes into $\overline{u_g^2}/2$ as b increases. However a large value of b implies an initial spectrum with a very broad maximum or, in other words, the initial field of motion becomes more randomly distributed (uncorrelated). This means that the characteristic scale of motion must decrease, the earth's rotation becomes less important and (2.11) reduces to the classical wave equation for pure gravity waves. Therefore $\overline{u_g^2}/2$ must attain a maximum then drop off monotonically as b is increased further, approaching zero as $f \to 0$. When b is held constant and a increases the wavelength of maximum energy density is approximately $L_{\max} \approx 2\pi/a$ (3.21) and the arguments presented above concerning a maximum $\overline{u_g^2}/2$ again apply.

6. Steady-state spectra

The initial energy density spectrum $\Psi(k)/2$ and the correlation function in (3.3) are Fourier cosine transforms of each other (Yaglom, 1962). Similarly the steady-state energy density spectra of the geostrophic and ageostrophic components may be obtained from (3.6) and (3.9) through (3.12); or more simply, these spectra are just the integrands of the corresponding time-independent integrals in (3.15) through (3.19).

The spectra of $\overline{u_s'^2}/2$, $\overline{v_s'^2}/2$, $\overline{u_g^2}/2$ and $\overline{\pi_g^2}/2 = 2(\overline{\pi_s'^2}/2)$ will be denoted by $\Lambda(k)$, $\Phi(k)$, $\Upsilon(k)$ and $\Pi(k)$ respectively. They are given by:

$$\Lambda(k) = [2^{1/2}(1 + k^2)]^{-2}\Psi(k)/2, \tag{6.1}$$

$$\Phi(k) = [2(1 + k^2)]^{-1}\Psi(k)/2, \tag{6.2}$$

$$\Upsilon(k) = k^4(1 + k^2)^{-2}\Psi(k)/2 \tag{6.3}$$

and

$$\Pi(k) = k^2(1 + k^2)^{-2}\Psi(k)/2, \tag{6.4}$$

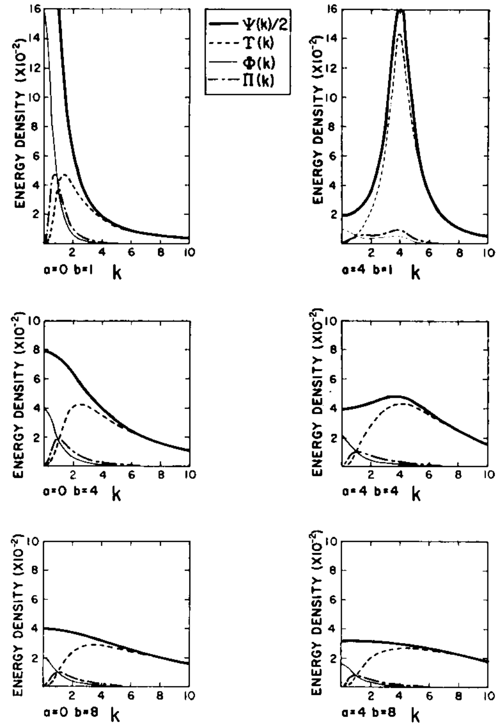

Fig. 5. The energy density spectra, defined in the text, as a function of wave-number k. The initial energy density spectrum $\Psi(k)/2$ has been included for comparison. At $a = 0$, $b = 1$, $\Psi(k = 0)/2 = 2\Phi(k = 0)$.

where $\Psi(k)/2$ (3.20) may be rewritten more compactly as

$$\tfrac{1}{2}\Psi(k) = \frac{b}{\pi} \frac{k^2 + a^2 + b^2}{k^4 + 2(b^2 - a^2)\,k^2 + (a^2 + b^2)^2}. \tag{6.5}$$

Since the mean ($k = 0$) potential energy does not change in this simple model, $\Pi(k = 0) = 0$ and it follows from (2.9) that $\Upsilon(k = 0) = 0$. (In the case of a homogeneous ocean, the mean depth H_0 remains constant.) At $k = 1$, the wave number which arbitrarily separates the process of wave dispersion principally by gravity-waves from that of inertia-type waves, there is an equipartition of energy density

$$\Phi(k = 1) = \Upsilon(k = 1) = \Pi(k = 1) = \Lambda(k = 1) + \tfrac{1}{2}\Pi(k = 1)$$

for all a and b.

Some of the spectra are displayed in Fig. 5. The range of values of a and b chosen are representative of both ageostrophic and geostrophic steady-state motion (see Fig. 4). These diagrams show quite clearly that when the initial energy is concentrated in a small band of wave numbers (b small) the wave number of maximum energy density is practically the same as the wave number of maximum initial energy density. When a is small the maximum energy density appears in the energy of the ageostrophic motion; and when a is large it appears in the energy of the geostrophic motion. However, in both cases the energy density of the geostrophic motion and initial motion are practically equal for large k. The magnitude of these latter spectra would begin to diverge from each other for sufficiently large k, when the earth's rotation is of secondary importance. We note also that the wave number of maximum $\Pi(k)$ near or at $k=1$ remains relatively insensitive to a and/or b while the wave number of maximum $\Upsilon(k)$ is quite sensitive to changes in a and to a lesser extent to changes in b, when $a=0$. This circumstance arises because the $\pi(y)$ field is the space integral of the velocity field, which means that small scale variations in the velocity field are essentially smoothed out. Finally, the maximum value of $\Phi(k)$ is always at $k=0$, except for an additional maximum in the case of $a=4$, $b=1$. Since the motion can only be of the inertia-type at $k=0$, a maximum of energy density at this wave number would be expected.

7. Concluding remarks

On the basis of the present results some additional statements may be made concerning the adjustment of an initially unbalanced current in a barotropic atmosphere. Obukhov's (1949) conclusion concerning the rapidity of the adjustment process (three to four hours) and the amount of initial energy going into the balanced geostrophic current (approximately 95 per cent) depends critically on the characteristic scale of the initial motion and to a lesser extent on the wave number of maximum initial energy density. Over a wide range of initial conditions Obukhov's conclusions are valid. However, if the energy density is concentrated in a fairly narrow band in the low wave number part of the initial spectrum the adjustment process is slow (Fig. 1) and the final state contains a significant amount of inertia-wave energy. The potential energy remains small for most initial conditions and vanishes for pure inertia-motion. Likewise the energy contributed by gravity-type motion is less than about 10 per cent of the total energy. Therefore, even in the steady-state, the total motion is never purely geostrophic since the initial conditions are not confined to a limited region.

The limitations on this model imposed by the linearization could be quite significant, since non-linear energy transfer between wave numbers is precluded. Meecham & Siegel (1964) have investigated Burgers' (1950) one-dimensional non-rotating model for large Reynolds numbers, i.e. when the non-linear inertial transfer is important. Their results indicate that the energy density spectrum of the velocity field approaches an equilibrium value, independent of very different initial conditions. If the non-linear terms were taken into account in the present inviscid model it is felt that the steady-state spectrum would be modified but that the dependence on initial conditions would still be retained since energy dissipation does not occur. In order to understand the effect of non-linear processes in this model, a further investigation is being undertaken.

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РАСПРЕДЕЛЕНИЕ ЭНЕРГИИ В БАРОТРОПНОЙ АТМОСФЕРЕ

Исследуется реакция неограниченной вращающейся баротропной атмосферы на заданный в начальный момент случайно распределенный подвод энергии. Начальное условие определено заданием функции плотности энергии и найдены решения (кинетическая и потенциальная энергии) для одномерной, зависящей от времени линеаризированной системы. Результаты показывают, что приближение к преобладающему установившемуся геострофическому распределению энергии является быстрым, если начальный спектр энергии широкий и (или) волновое число, соответствующее максимуму начального распределения энергии, относительно велико ($k \gtrsim 4$). Если начальная плотность энергии сконцентрирована в узкой линии длинноволновой части спектра, то значительная часть энергии переходит в агеострофическое инерционное движение и приближение к установившемуся распределению происходит медленно.