The growth of a young frontal wave

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ABSTRACT

In a two-layer model of the atmosphere with the upper warm air moving from the west relative to the cold air below the long waves are unstable and have a rather simple symmetric structure and behavior when the following approximations are used: (*a*) The latitude variation of the depth of the layers is ignored. (*b*) The motion is quasi-geostrophic. (*c*) The planetary β-gradient of the earth's vorticity is ignored. (*d*) The tropopause is a rigid horizontal surface. When these approximations are removed one by one the wave structure and growth rate are modified somewhat, but the essential features of the wave mechanism remain unchanged.

1. The undisturbed balanced state of the model

The model consists of two isentropic layers with a cold lower layer bounded by a level ground, and a warm upper layer also bounded by a rigid surface (tropopause) ten kilometers above the ground. The upper warm air with the potential temperature $\theta = \theta_1$ moves from the west with the speed 2U relative to the cold air ($\theta = \theta_2$) below. We shall consider conditions near a reference latitude where both layers have the same depth $h = 5$ km, and for the moment we ignore the variation of the Coriolis parameter f with latitude. The frontal interface which separates the layers has a balanced meridional Margules slope

$$\alpha = \frac{dh}{dy} = \frac{fU}{sg}; \quad s = \frac{\theta_1 - \theta_2}{\theta_1 + \theta_2}$$

such that the torque of the Coriolis forces across the front is balanced by the proper "overweight" of the sloping front.

If the rotation and the frontal wind shear are removed from the model, the frontal surface is level, and long quasi-static gravity waves propagate along the surface with the phase speed

$$C_L = \sqrt{sgh}.$$

This velocity parameter may be taken as a measure of the static stability of the model. We choose the temperature difference 20 deg K between the layers which in this sense reflects the average observed tropospheric temperature stratification in middle latitudes in winter. The phase speed of the gravity waves in our model is then about 40 ms^{-1}.

An important length parameter is the wave length L_0 of the gravity wave which has the inertial frequency. It has in middle latitudes the value

$$L_0 = 2\pi C_L/f = 2500 \text{ km}.$$

This wave marks the boundary between the short stable and the long unstable waves for all values of the wind shear.

2. The stability boundary

Consider the following long ($L \sim L_0$) non-tilting wave disturbance introduced in this two-layer model. The wave components have no variation with latitude. The meriodional velocity (the vorticity wave) has the same amplitude and phase in both layers, so the wave trough of the resultant motion has no tilt from the lower to the upper layer. The motion is quasi-static with no wind shear inside the isentropic layers. Vertical air columns remain vertical at all times and their potential vorticity is conserved. We give the air columns everywhere the divergence which leaves the frontal interface undeformed: In the region of south winds the warm upper air columns have divergence and shrink as they ascend northward over the sloping non-deformed frontal surface, the

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cold air columns below have convergence and stretch as they move north. In this state of the wave the frontal surface may be solidified without disturbing the motion, and the waves in each layer may be considered separately. They are in fact at the moment simple Rossby waves, driven by the equal and opposite potential vorticity gradients in the layers,

$$h\frac{d}{dy}\left(\frac{f}{h}\right)=\frac{\alpha f}{h}$$

where α denotes the Margules slope. Both waves move intrinsically upwind through the air with the well known Rossby-velocity

$$C_R=\frac{\alpha f}{h}/k^2=\left(\frac{f}{k\sqrt{sgh}}\right)^2 U=\left(\frac{L}{L_0}\right)^2 U.$$

This propagation of the wave is caused by the convergence and vorticity generation upwind from the wave trough. The wave with the wave length L_0 is stationary in this non-tilting state. We note that the Coriolis forces of the meridional motion and the accelerations of the zonal motion have no torque across the front so the front remains undeformed, and hence the $(L=L_0)$-wave remains stationary and neutral at all times when it is put into this non-tilting b-state with an undeformed interface. However, shorter waves move downwind and longer waves upwind from this state.

3. The quasi-geostrophic b-state

To see what happens to a $(L\neq L_0)$-wave after it leaves the non-tilting b-state with the undeformed interface, we assume provisionally that the resultant wind shear across the front is geostrophically balanced at all times, so the torque of the Coriolis force across the front is balanced by the overweight of the proper Margules slope. This means that the wind shear across the front is parallel to the frontal height contours. This approximation, as we shall see, is fairly good if the static stability is larger than the wind shear across the front $(C_L > 2U)$.

If the air columns have the divergence which leaves the front non-deformed, a wave longer than L_0 moves upwind through the non-tilting b-state with the speed $(\lambda-1)\,U$, where

$$\lambda=(L/L_0)^2.$$

A little later the wind shear across the front makes an angle with the non-deformed contours, so this evolution is non-geostrophic. To make it geostrophic the divergence must be reduced so that a warm sector forms in the region of south winds and the contours and streamlines turn at the same rate at the frontal nodes. A simple calculation[1] shows that this condition is satisfied when the divergence amplitude is reduced by the factor $2/(\lambda+1)$ below the value which would leave the frontal surface undeformed. The speed of the wave through the air is reduced by the same factor, so the wave moves upwind through the geostrophic b-state with the speed

$$C_b=\left(\frac{2\lambda}{\lambda+1}-1\right)U=\frac{\lambda-1}{\lambda+1}U.$$

In this evolution the air columns have no divergence at the frontal nodes where they move along the contours. The wave trough moves upwind into the region of stationary convergence and starts growing. At the same time the wave slows down because of the diminishing convergence upwind from the wave trough.

4. The stationary tilt of the growing wave

The final growing state of the unstable quasi-geostrophic wave may now be anticipated if we consider a second non-tilting state (a-state) where the upper and the lower wave have opposite phase, so the streamlines in both layers are parallel to the geostrophically balanced frontal contours.

Here the air columns have no divergence. The wave drifts downwind with the air through the geostrophic a-state,

$$C_a=U. \qquad \text{(all wave lengths)}$$

From either of these non-tilting states the wave approaches asymptotically the same tilting state which is nearer the b-state since $C_b < C_a$. In fact, it can be shown that the asymptotic phase difference 2θ between the upper and the lower wave trough is given by the formula

$$\tan\theta=\sqrt{\frac{C_b}{C_a}}=\sqrt{\frac{\lambda-1}{\lambda+1}}. \qquad (\sec 2\theta=\lambda)$$

[1] See appendix A.

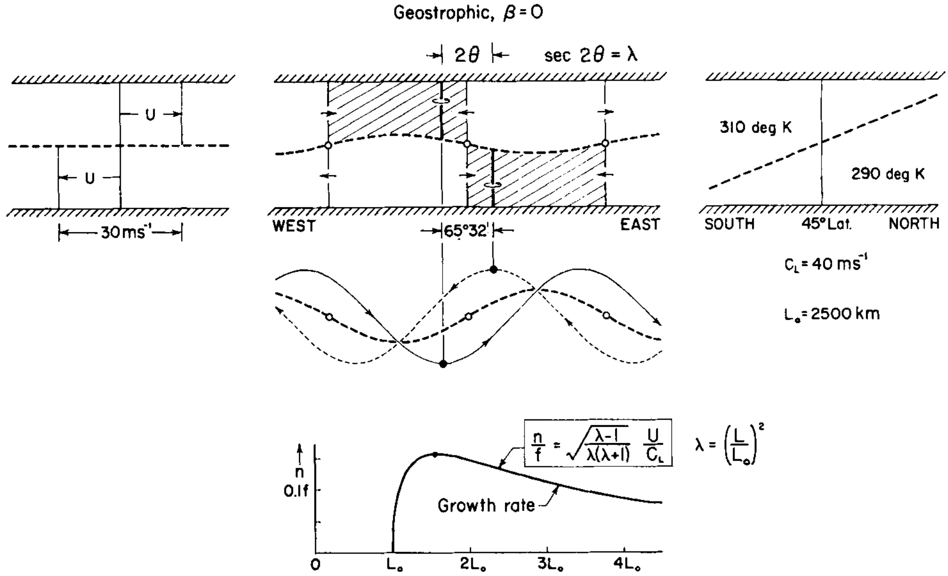

FIG. 1. Fastest growing wave in a two layer atmosphere.

In the stationary growing state all wave amplitudes grow at the same exponential rate. The growth rate depends upon the position of the wave trough inside the region of convergence. It is given by the simple formula[2]

$$n = kU \tan \theta.$$

With the value of the phase substituted, the growth rate, in units of $f = k_0 C_L$, is

$$\frac{n}{f} = \sqrt{\frac{\lambda-1}{\lambda(\lambda+1)}}\frac{U}{C_L}, \quad \lambda = \left(\frac{L}{L_0}\right)^2.$$

If we choose values which reflect observed conditions, namely the temperature difference 20 deg K ($C_L = 40$ ms^{-1}) and the wind shear $2U = 30$ ms^{-1}, the wavelength of the gravity wave which has the frequency f in middle latitude is about 2500 km. The fastest growing wave is about 50 per cent longer (4000 km). Its growth rate, $n = 0.15\ f$, doubles the wave amplitude every 13 hours.

The growth mechanism is clearly exhibited in Fig. 1. The wave keeps on growing because the air columns in the wave trough, which have the fastest rotation, are inside the region of convergence. They are stretched, and their rotation speeds up. As the wave grows, the frontal deformation grows at the same relative rate to readjust to the geostrophic balance at the frontal nodes, such that the streamlines and contours remain parallel at all times at these nodes.

[2] See appendix B.

A number of approximations have been made to obtain this simple symmetric theoretical model of a frontal wave, namely: (*a*) The latitude variations of the depth of the layers is ignored. (*b*) The motion is assumed to be quasi-geostrophic. (*c*) The planetary gradient of the earth's vorticity is ignored. (*d*) The tropopause is replaced by a rigid horizontal surface. We shall remove these approximations one by one and examine how these factors modify the structure of the model.

5. The latitude variation of the depth of the layers

Some idea of the importance of this factor may be obtained by comparing the two layer model with an "equivalent" model with continuous distribution of temperature and wind, which have no latitude variation. The continuous model was studied by EADY 1949. The models may be regarded as equivalent when they have the same static stability in the sense that the long gravity waves propagate with the same speed, and the kinetic energy of the zonal wind is the same. The simple two layer model

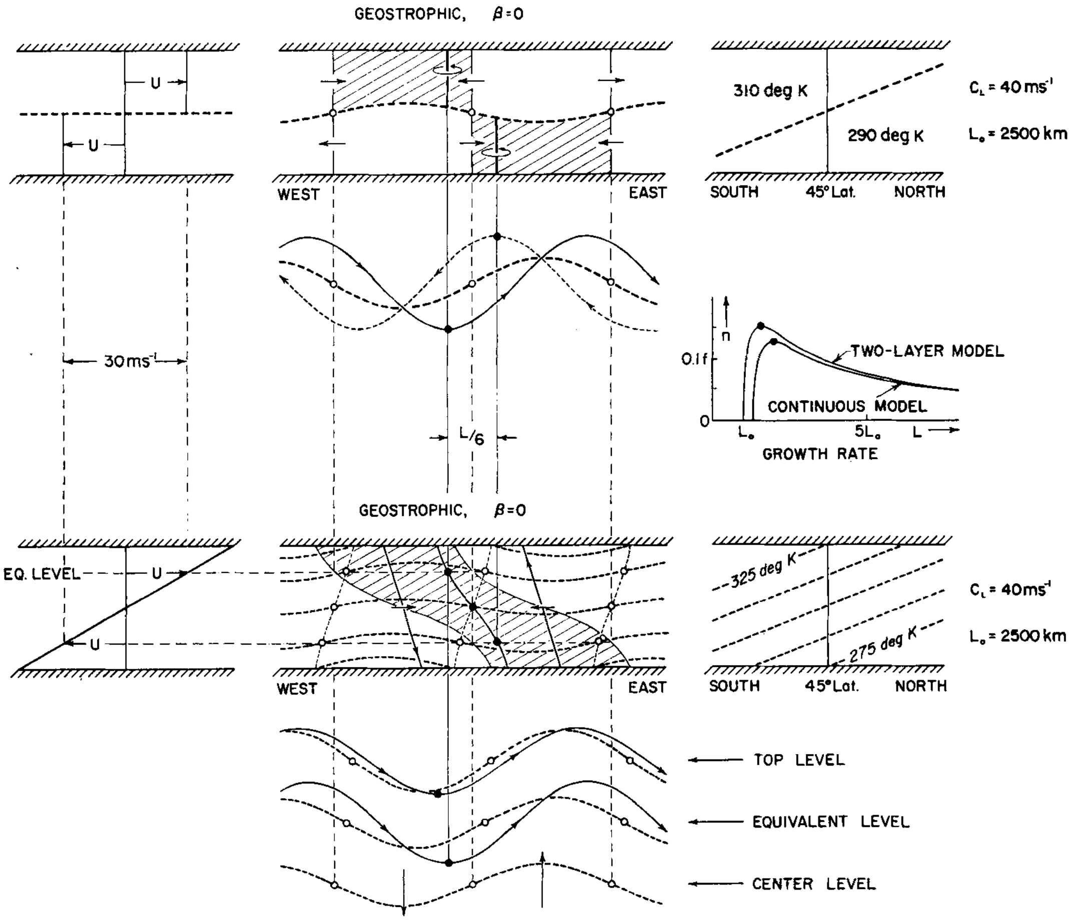

FIG. 2. Fastest growing wave in a two layer model (above) and in the equivalent continuous model (below)

from Fig. 1 and the corresponding equivalent continuous model of Eady are shown in Fig. 2. The longest waves have precisely the same growth rate in the two models. The wave which marks the boundary between the stable and the unstable waves is about 30 per cent longer, and the maximum growth rate is about 30 per cent less in the continuous model. The fastest growing wave in each model are shown in the figure. Their structures are similar. At the "equivalent levels" where the zonal wind in the continuous model is the same as in the two layer model, the wave structure is the same in both models. The wave trough tilts about a sixth of a wave length upwind from the lower to the upper equivalent level, and the divergence wave tilts half a wave length upwind. A new feature in the continuous model is the gentle downwind tilt of the temperature wave. This is necessary to provide for the stretching of the stationary air columns in the wave trough at the center level and the resulting inflation of the isentropic lamella at this point. (A similar downwind tilt is found in a three layer model where the frontal interface is replaced by a transition layer.) The growth mechanism is in principle the same in the two models.

6. The non-geostrophic model

Next let us examine the error which is introduced by the geostrophic approximation. It will be recalled that this approximation assumes balance between the torque of the Coriolis force and the overweight of the sloping front, and hence does not allow for the growth of the

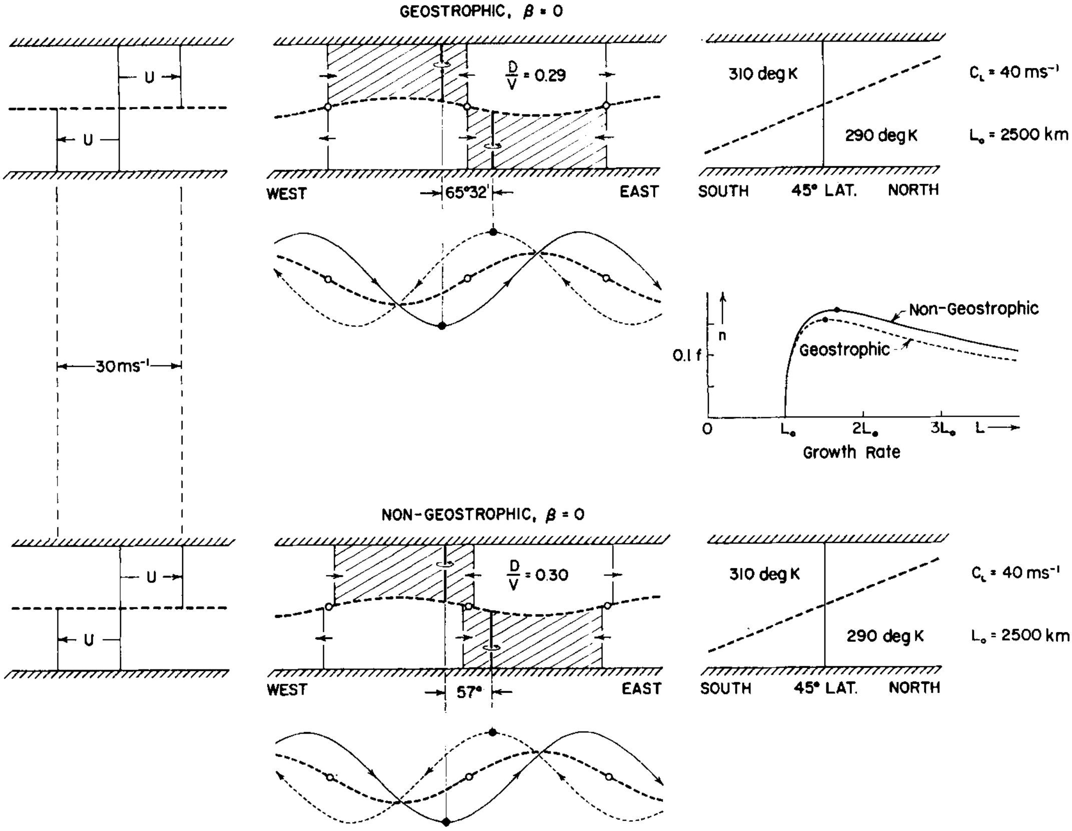

FIG. 3. Fastest growing wave from geostrophic theory (above) and from non-geostrophic theory (below).

zonal wind shear across the frontal nodes, see Fig. 3. The overweight must evidently be a little stronger than the Coriolis torque to provide for this growth. According to the non-geostrophic theory this requirement is met very simply by a reduction of the overall upwind tilt of the wave with no appreciable change of the wave amplitudes. The reduced tilt means that both the Coriolis torque and the zonal wind shear at the frontal node are reduced, while the overweight is unchanged. The geostrophic approximation underestimates the growth rate a little. According to the theory the error depends upon the ratio between static stability and wind shear C_L/U, and becomes insignificant when this ratio is much larger than unity. In the example shown in the figure the error is about seven per cent for the fastest growing wave. If the wind shear is reduced to 20 ms^{-1} the error is less than three per cent. However if the ratio C_L/U approaches unity the results derived from the geostrophic theory are even qualitatively wrong. But so large values of the wind shear are never observed in the atmosphere.

7. The planetary vorticity gradient

The dynamic importance of this so-called β-effect was first recognized by J. BJERKNES 1937. The principle was developed further by ROSSBY 1939 who, as mentioned already, showed that the β-gradient drives the wave *westward* through the air with the speed β/k^2. The modification of the simple symmetric wave in figure 1 when the β-gradient is introduced is shown in Fig. 4. The adjustments may be anticipated by qualitative reasoning: When the β-gradient is suddenly introduced in the ($\beta = 0$)-model both the upper and the lower wave has at the moment an additional westward propagation with the speed β/k^2. As a consequence the upper wave trough penetrates further into

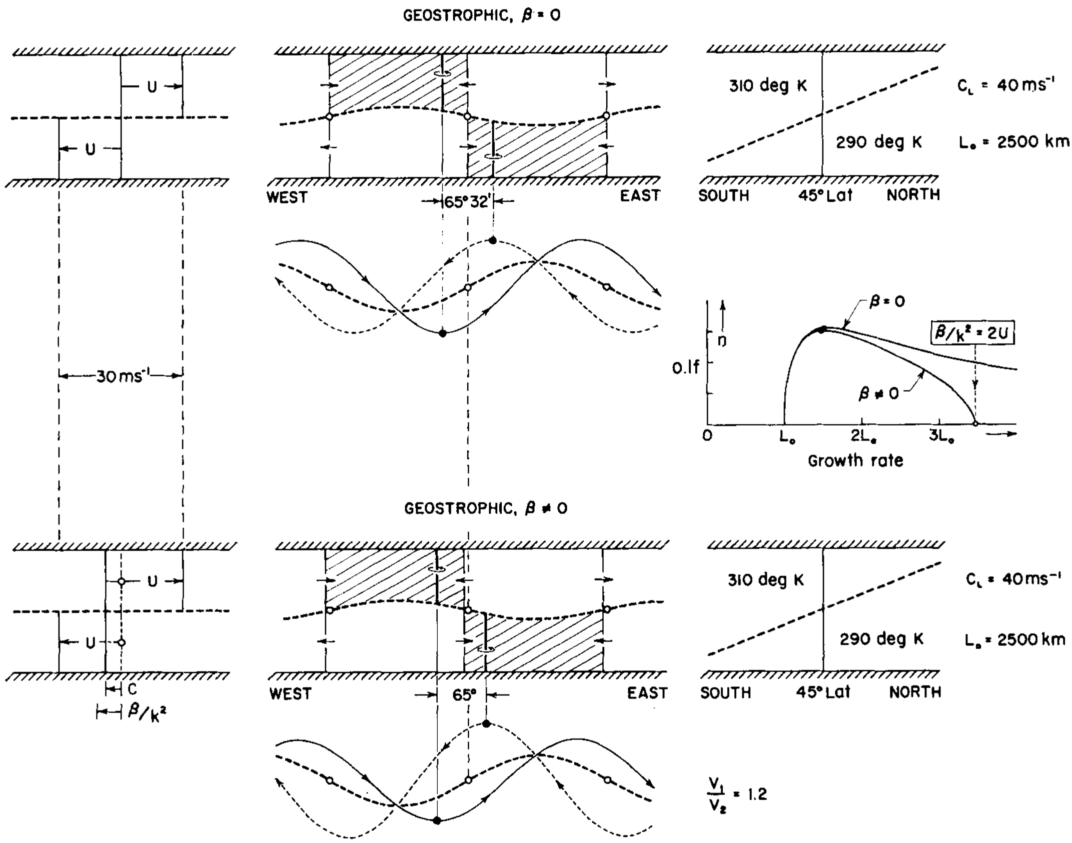

Fig. 4. Modification of the fastest growing wave by the β-gradient.

the region of convergence, so it grows faster and its westward propagation slows down. The lower wave trough moves toward weaker convergence. Its growth is retarded and its westward propagation slows down. After this adjustment, as predicted from geostrophic theory with the β-effect included, the upper wave is about 20 per cent stronger than the lower wave, and the wave as a whole moves west with a somewhat slower speed than the Rossby speed. The fastest growing wave is a little shorter and grows a little slower. The β-effect is a stabilizing factor which becomes increasingly important the longer the wave. It is the dominating factor and the wave is stable if the β-gradient moves the wave westward faster than the wind shear across the front ($\beta/k^2 > 2U$). Fig. 5 shows the stability boundaries as a function of the wind shear in a middle latitude model with the static stability $C_L = 40$ ms^{-1}. The spectral interval of instability decreases with decreasing wind shear, and the model is stable for all wave lengths when the wind shear is less than the critical value $U = \beta/k_0^2$. This stability criterion has a simple physical interpretation in terms of the potential vorticity gradients in the layers:

$$h\frac{d}{dy}\left(\frac{f}{h}\right) = \beta \pm k_0^2 U. \quad \begin{matrix}(+)\text{ Upper layer;}\\ (-)\text{ Lower layer.}\end{matrix}$$

The β-ascendent is directed north in both layers, while the ascendent from the frontal slope is directed north in the upper layer, and south in the lower layer. The two-layer model is stable when the frontal slope is so gentle that the β-gradient dominates and the vorticity gradient has the same direction in both layers. This is a generalization of the stability criterion of Lord Rayleigh for two-dimensional shear flows.

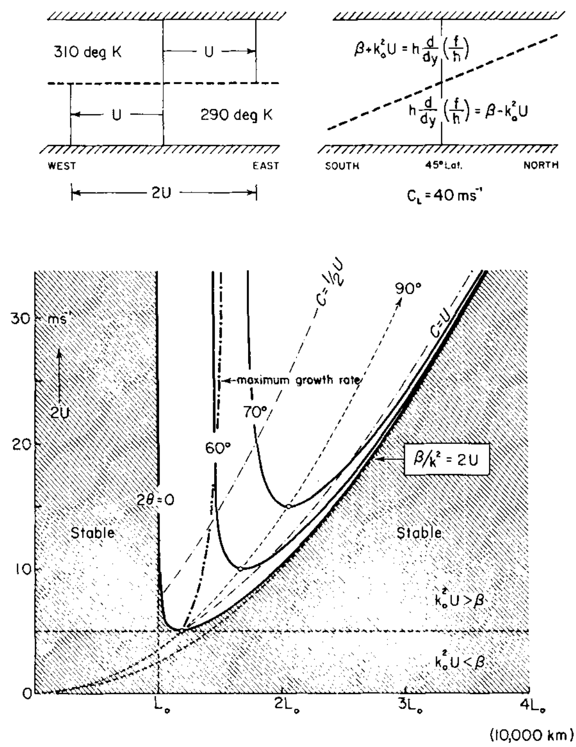

FIG. 5. Stability diagram for a geostrophic two layer atmosphere.

8. Statically soft tropopause

Finally let the rigid tropopause in the two layer model be replaced by a statically softer boundary in a three layer model with the stratosphere represented by a third isentropic layer which has the same zonal motion as the upper tropospheric layer. In this three layer model a certain amount of the unstable wave energy will penetrate into the stratospheric layer, so the waves should be expected to grow at a slower rate. This is confirmed by the geostrophic theory for the three layer model. The reduction of the growh rate from the two layer value depends upon the static stability of the tropopause and the height of the stratosphere. For example if the frontal interface and the tropopause have the same static stability and the stratosphere and the troposphere have the same heights we find that the stability boundary wave is about ten per cent longer than in the two layer model, and the maximum growth rate is about eighteen per cent less.

APPENDIX A

With the notations: A = streamline amplitude, D = divergence amplitude, D_0 = divergence amplitude which leaves frontal surface undeformed, and C_b = upwind phase velocity of vorticity wave through the b-state, the local rates of turning at the contour nodes are for the streamline Ak^2C_b and for the contour $Ak^2U(1-D/D_0)$. These are the same when the divergence amplitude has the value $D/D_0 = 1-C_b/U$. But this divergence gives the vorticity wave the upwind speed $C_b/U = \lambda D/D_0 - 1 = 1 - D/D_0$, which give $D/D_0 = 2/(\lambda+1)$.

APPENDIX B

In an arbitrary state of the wave the velocity components in the upper layer are

$$u = -D \cos kx; \quad v = V \sin (kx + \theta).$$

The local vorticity change relative to the air, namely

$$\frac{\partial}{\partial t}\frac{\partial v}{\partial x} = -f\frac{\partial u}{\partial x},$$

gives $d(V \cos \theta)/dt = 0$. In the stationary growing state ($d\theta/dt = kU$) the vorticity wave grows at the rate $d(\ln V)/dt = n = kU \tan \theta$.

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РОСТ ЗАРОЖДАЮЩЕЙСЯ ФРОНТАЛЬНОЙ ВОЛНЫ

В двухслойной модели атмосферы с теплым воздухом наверху, движущимся с запада относительно холодного воздуха, находящегося внизу, длинные волны неустойчивы и имеют довольно простую симметричную структуру и поведение, если используются следующие приближения: (а) широтным изменением глубины слоев пренебрегают, (б) движение квазигеострофично, (в) планетарным β-градиентом вихря пренебрегают, (г) тропопауза считается твердой горизонтальной поверхностью. Если эти приближения исключать последовательно одно за другим, то волновая структура и скорость роста несколько модифицируются, но существенные черты волновой картины остаются неизменными.