Three-dimensional oscillations in a jet at high Reynolds numbers

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ABSTRACT

An analysis is made of the three-dimensional periodic oscillations in a jet at high Reynolds numbers. It is shown that there are two physically significant solutions: one is a symmetric and the other an antisymmetric function of the lateral coordinate; both involve a Lommel function. In the case of the symmetric solution, the maximum amplitude of the vertical velocity occurs near the inflection points of the velocity profile of the jet, whereas in the case of the antisymmetric solution, it occurs near the center as well as the inflection points of the jet.

1. Introduction

Analyses of two-dimensional jet flows have been made by many investigators. SCHLICHTING (1933) obtained an approximate solution for a steady jet flow of an incompressible viscous fluid. BICKLEY (1937) showed that an exact solution exists and that the velocity profile of the jet is essentially of the form $u = \text{sech}^2 y$. Later, the periodical solutions of the two-dimensional disturbances in a Bickley jet were analyzed (SAVIC, 1941; SAVIC & MURPHY, 1943). The viscous stability problem in the two-dimensional case has also been studied (CURLE, 1957; TATSUMI & KAKUTANI, 1958; HOWARD, 1959; CLENSHAW & ELLIOTT, 1960). However, no investigation was made of the three-dimensional disturbance in the jet, although in view of Squire's transformation (SQUIRE, 1933) the differential equations for a three-dimensional disturbance can be reduced to those having the same characteristics as those for a two-dimensional disturbance. Benney and Lin have recently examined the problem of the three-dimensional disturbance in a parallel flow with a velocity profile of $\tanh y$ (BENNEY & LIN, 1960; BENNEY, 1961). The purpose of this paper is to extend their study to the case of the three-dimensional disturbance in a jet.

2. The governing equations

We consider a three-dimensional flow of an imcompressible homogeneous fluid at high Reynolds numbers. Let $\mathbf{v}$ denote the velocity, $\mathbf{q}$ the vorticity, p the pressure, g the gravity acceleration, R the Reynolds number. The equations of motion, continuity, and vorticity may respectively be written

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla p + \mathbf{g} + \frac{1}{R}\nabla^2\mathbf{V}, \tag{1}$$

$$\nabla\cdot\mathbf{V} = 0, \tag{2}$$

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{q} = (\mathbf{q}\cdot\nabla)\mathbf{V} + \frac{1}{R}\nabla^2\mathbf{q}, \tag{3}$$

where all quantities are expressed in dimensionless form, ∇^2 is the three-dimensional Laplacian operator.

We may suppose that $\mathbf{v}$, $\mathbf{q}$, and p are expanded as perturbation series of the following form

$$\left.\begin{aligned} \mathbf{V} &= \mathbf{V}^{(0)} + a\mathbf{V}^{(1)} + a^2\mathbf{V}^{(2)} + \dots, \\ \mathbf{q} &= \mathbf{q}^{(0)} + a\mathbf{q}^{(1)} + a^2\mathbf{q}^{(2)} + \dots, \\ p &= p^{(0)} + ap^{(1)} + a^2p^{(2)} + \dots, \end{aligned}\right\} \tag{4}$$

where the numerals in the superscript denote the order of the perturbation motion, the symbol a denotes a perturbation amplitude. Hence, $\mathbf{v}^{(0)}$ and $\mathbf{q}^{(0)}$ are respectively the velocity and vorticity of the basic undisturbed flow, $\mathbf{v}^{(1)}$ and $\mathbf{q}^{(1)}$ are those of the primary flow, etc., $p^{(0)}$, $p^{(1)}$ and $p^{(2)}$ are the pressure distributions respectively associated with the basic, primary and secondary flows.

Let (u, v, w) and (ξ, η, ζ) denote respectively the velocity and vorticity components along the rectangular coordinates (x, y, z). Furthermore, let the z-axis be oriented along the vertical, the velocity of the basic flow be parallel to the x-axis and be denoted by $u_0^{(0)}(y)$, and k denote the wave number of the primary flow associated with the x-direction. We may express

$$\left.\begin{aligned} \mathbf{v}^{(1)}(x,y,z,t) &= \mathbf{v}_1^{(1)}(y,z,t)e^{ikx} + \mathbf{v}_1^{(1)*}(y,z,t)e^{-ikx}, \\ p^{(1)}(x,y,z,t) &= p_1^{(1)}(y,z,t)e^{ikx} + p_1^{(1)*}(y,z,t)e^{-ikx}, \end{aligned}\right\} \quad (5)$$

where an asterisk is used to denote a complex conjugate.

We suppose that the primary flow composes of two- and three-dimensional oscillations; the velocity components of the former will be denoted by capital letters, whereas those of the latter by letters of lower case. Furthermore, we assume a sinusoidal spanwise variation of amplititude associated with the three-dimensional wave. We express

$$\left.\begin{aligned} u_1^{(1)}(y,z,t) &= [\lambda\hat{u}_1(y)\cos\beta z + \gamma\hat{U}_1(y)]e^{-ikct}, \\ v_1^{(1)}(y,z,t) &= [\lambda\hat{v}_1(y)\cos\beta z + \gamma\hat{V}_1(y)]e^{-ikct}, \\ w_1^{(1)}(y,z,t) &= \lambda\hat{w}_1(y)\sin\beta z e^{-ikct}, \\ p_1^{(1)}(y,z,t) &= [\lambda\hat{p}_1(y)\cos\beta z + \hat{P}_1(y)]e^{-ikct}, \end{aligned}\right\} \quad (6)$$

where c is the phase velocity, β is the wave number along z, the ratio γ/λ is a measure of the relative importance of two- and three-dimensional oscillations.

It can be shown that the equations governing the basic undisturbed flow are

$$0 = -\frac{\partial p^{(0)}}{\partial x} + \frac{1}{R}\frac{\partial^2 u_0^{(0)}}{\partial y^2}, \quad (7)$$

$$0 = -\frac{\partial p^{(0)}}{\partial z} - g \quad (8)$$

and that the amplitude equations governing the two- and three-dimensional osciallations in the primary flow are

$$\left.\begin{aligned} ik(u_0-c)\hat{u}_1 + \frac{du_0}{dy}\hat{v}_1 &= -ik\hat{p}_1 + \frac{1}{R}\left(-k^2-\beta^2+\frac{d^2}{dy^2}\right)\hat{u}_1, \\ ik(u_0-c)\hat{v}_1 &= -\frac{d\hat{p}_1}{dy} + \frac{1}{R}\left(-k^2-\beta^2+\frac{d^2}{dy^2}\right)\hat{v}_1, \\ ik(u_0-c)\hat{w}_1 &= \beta\hat{p}_1 + \frac{1}{R}\left(-k^2-\beta^2+\frac{d^2}{dy^2}\right)\hat{w}_1, \\ ik\hat{u}_1 + \frac{d\hat{V}}{dy} + \beta\hat{w}_1 &= 0, \end{aligned}\right\} \quad (9)$$

$$\hat{\xi}_1 = \frac{d\hat{w}_1}{dy} + \beta\hat{v}_1,$$

$$\hat{\eta}_1 = -\beta\hat{u}_1 - ik\hat{w}_1$$

$$\hat{\zeta}_1 = ik\hat{v} - \frac{d\hat{u}_1}{dy},$$

$$\left.\begin{aligned} ik(u_0-c)\hat{U}_1 + \frac{du_0}{dy}\hat{V}_1 &= -ik\hat{P}_1 + \frac{1}{R}\left(-k^2+\frac{d^2}{dy^2}\right)\hat{U}_1, \\ ik(u_0-c)\hat{V}_1 &= -\frac{d\hat{P}_1}{dy} + \frac{1}{R}\left(-k^2+\frac{d^2}{dy^2}\right)\hat{V}_1, \\ ik\hat{U}_1 + \frac{d\hat{V}_1}{dy} &= 0, \end{aligned}\right\} \quad (10)$$

where u_0 is the abbreviation of $u_0^{(0)}$.

A study of the secondary flow of the fluid will be given in a later paper. In this paper we shall analyze the periodic solutions of the three dimensional disturbances in the primary flow at high Reynolds numbers.

For flows at high Reynolds numbers the velocity field may be approximated by the inviscid equations, except within the critical layer regions, where viscous correction would be needed (Lin, 1955). In this case $\hat{v}_1$ can be shown to satisfy the following equation

$$(u_0-c)\left(\frac{d^2}{dy^2}-k^2-\beta^2\right)\hat{v}_1 - \frac{d^2u_0}{dy^2}\hat{v}_1 = 0. \quad (11)$$

For given boundary conditions, the above differential equation may be solved for $\hat{v}_1$, $\hat{u}_1$, $\hat{w}_1$, and $\hat{p}_1$ may then be expressed in terms of $\hat{v}_1$ as follows:

$$\hat{u}_1 = \frac{i}{k(k^2+\beta^2)}\left[\frac{\beta^2}{(u_0-c)}\frac{du_0}{dy}\hat{v}_1 + k^2\frac{d\hat{v}_1}{dy}\right], \quad (12)$$

$$\hat{w}_1 = \frac{\beta}{(k^2+\beta^2)}\left[\frac{1}{(u_0-c)}\frac{du_0}{dy}\hat{v}_1 - \frac{d\hat{v}_1}{dy}\right], \quad (13)$$

$$\hat{p}_1 = \frac{ik}{(k^2+\beta^2)}\left[\frac{du_0}{dy}\hat{v}_1 - (u_0-c)\frac{d\hat{v}_1}{dy}\right]. \quad (14)$$

3. Flows in a jet current

We consider a three-dimensional motion in a basic flow of a jet of the following expression

$$u_0(y) = \operatorname{sech}^2 y \tag{15}$$

which is symmetric with respect to the y-axis.

For a finite jet Kuo (1949) has shown that if $u_0 = c$ at some point, $y = y_c$ for neutral waves, then $d^2u_0/dy^2 = 0$ at that point. We shall first find the phase velocities of the neutral waves for which $c \geq 0$.

$$\frac{d^2u_0}{dy^2} = -6 \operatorname{sech}^2 y\,(\operatorname{sech}^2 y - \tfrac{2}{3})$$

$$= -6(u_0 - c_1)(u_0 - c_2), \tag{16}$$

where $c_1 = 0, \quad c_2 = \tfrac{2}{3}$.

The differential equation (11) becomes

$$\frac{d^2\hat{v}_1}{dy^2} + [6(\operatorname{sech}^2 y - c_j) - k^2 - \beta^2]\hat{v}_1 = 0, \quad j = 1, 2, \tag{17}$$

either subscript on c may be valid, depending on the value of the phase velocity.

We consider the following boundary conditions:

$$\hat{v}_1(\pm\infty) = 0, \quad \hat{v}_1(0) = 0, \tag{18}$$

$$\hat{v}_1(\pm\infty) = 0, \quad \left(\frac{d\hat{v}_1}{dy}\right)_{y=0} = 0, \tag{19}$$

respectively for the antisymmetric and symmetric solutions of $\hat{v}_1$.

It can be shown that for the antisymmetric solution (for example, SAVIC & MURPHY, 1943)

$$\left.\begin{aligned} &\hat{v}_1 = \operatorname{sech} y \tanh y, \\ &k^2 + \beta^2 = 1, \\ &c = \tfrac{2}{3} \end{aligned}\right\} \tag{20}$$

and that there are two symmetric solutions (for example, SAVIC, 1941; HOLMBOE, 1962):

$$\left.\begin{aligned} &\hat{v}_1 = \operatorname{sech}^2 y, \\ &k^2 + \beta^2 = 4, \\ &c = \tfrac{2}{3} \end{aligned}\right\} \tag{21}$$

and

$$\left.\begin{aligned} &\hat{v} = \operatorname{sech}^2 y, \\ &k^2 + \beta^2 = 0, \\ &c = 0. \end{aligned}\right\} \tag{22}$$

Since the last solution has no physical significance, we shall consider the solutions (20) and (21).

For the antisymmetric solution of $\hat{v}_1$, it can be shown that the solution for an inviscid fluid is

$$\left.\begin{aligned} \hat{u}_{II} &= \frac{i}{k}\left[\operatorname{sech} y(2\operatorname{sech}^2 y - 1) + \frac{\beta}{3}\frac{\operatorname{sech} y(\operatorname{sech}^2 y - 2)}{(\operatorname{sech}^2 y - \frac{2}{3})}\right], \\ \hat{v}_{II} &= \operatorname{sech} y \tanh y, \\ \hat{w}_{II} &= \frac{\beta}{3}\frac{\operatorname{sech} y(\operatorname{sech}^2 y - 2)}{(\operatorname{sech}^2 y - \frac{2}{3})}, \\ \hat{p}_{II} &= i\frac{k}{3}\operatorname{sech} y(\operatorname{sech}^2 y - 2), \\ \hat{U}_{II} &= i\operatorname{sech} y(2\operatorname{sech}^2 y - 1), \\ \hat{V}_{II} &= \operatorname{sech} y \tanh y, \\ \hat{P}_{II} &= i\frac{1}{3}\operatorname{sech} y(\operatorname{sech}^2 y - 2). \end{aligned}\right\} \tag{23}$$

For the symmetric solution of $\hat{v}_1$, the solution for an inviscid fluid is

$$\left.\begin{aligned} \hat{u}_{II} &= -\frac{i}{2k}\operatorname{sech}^2 y \tanh y \times\left[4 + \beta^2\left(\frac{\operatorname{sech}^2 y}{\operatorname{sech}^2 - \frac{2}{3}} - 1\right)\right], \\ \hat{v}_{II} &= \operatorname{sech}^2 y, \\ \hat{w}_{II} &= -\frac{\beta}{3}\frac{\operatorname{sech}^2 y \tanh y}{(\operatorname{sech}^2 y - \frac{2}{3})}, \\ \hat{p}_{II} &= -i\frac{k}{3}\operatorname{sech}^2 y \tanh y, \\ \hat{U}_{II} &= -i\operatorname{sech}^2 y \tanh y, \\ \hat{V}_{II} &= \operatorname{sech}^2 y, \\ \hat{P}_{II} &= -i\frac{1}{3}\operatorname{sech}^2 y \tanh y. \end{aligned}\right\} \tag{24}$$

It may be noted that, although $\hat{v}_1$ is a regular function, however, for inviscid fluid

$$\hat{u}_{II} = \frac{i}{k(k^2+\beta^2)}\left[\frac{\beta^2}{(u_0 - c)}\frac{du_0}{dy}\hat{v}_{II} + k^2\frac{d\hat{v}_{II}}{dy}\right],$$

$$\hat{w}_{II} = \frac{\beta}{(k^2+\beta^2)}\left[\frac{1}{(u_0 - c)}\frac{du_0}{dy}\hat{v}_{II} - \frac{d\hat{v}_{II}}{dy}\right] \tag{25}$$

have poles at $y = y_c$ where $u_0(y_c) = c$. The subscript I has been added to emphasize the fact that these solutions are purely inviscid solutions, and therefore will be reliable solutions for flow of a real fluid at high Reynolds numbers, provided we are not close to the critical layers ($y = y_c$). It is important to note that the amplitude functions u_{1I} and w_{1I} exhibit singularities at $y = y_c$, which would be absent in any real fluid. That is, when viscosity is included these poles would not be present.

Since in the problem at hand $\hat{v}_1$ does not contain a pole, to consider $\hat{u}_{1I}$ and $\hat{w}_{1I}$, it is convenient to consider the y-component of the vorticity which contains both $\hat{u}_{1I}$ and $\hat{w}_{1I}$. Let

$$\eta_{1I}^{(1)} = \frac{\partial u_{1I}^{(1)}}{\partial z} - \frac{\partial w_{1I}^{(1)}}{\partial x} = -\hat{\Phi}_I(y)\lambda \sin \beta z e^{-ikct}, \quad (26)$$

where $$\hat{\Phi}_I = i\frac{\beta}{k}\frac{1}{(u_0 - c)}\frac{du_0}{dy}\hat{v}_{1I} \quad (27)$$

which has a pole at $u_0(y_c) = c$.

It is well known that the critical layer has a thickness of order $(kR)^{-1/3}$ and it is within this region that the solution $\hat{\Phi}_I$ will be modified by a correction of the boundary layer type.

In the critical layer, we take into account of the frictional effect and consider the complete differential equation for $\hat{\Phi}$, which may be obtained from the general equation of motion. We have

$$\frac{d^2\hat{\Phi}}{dy^2} - [ikR(u_0 - c) + 1]\hat{\Phi} = \beta R\frac{du_0}{dy}\hat{v}_1. \quad (28)$$

It may be noted that formally putting R infinity in the above equation we have differential equation for $\hat{\Phi}_I$, (27).

For the solution (20) or (21), equation (28) becomes

$$\frac{d^2\hat{\Phi}}{dy^2} - [ikR(\operatorname{sech}^2 y - \tfrac{2}{3}) + 1]\hat{\Phi} = -2\beta R\hat{v}_1 \operatorname{sech}^2 y \tanh y. \quad (29)$$

Formally putting R infinity, we have the inviscid solution

$$\hat{\Phi}_I(y) = -\left[i\frac{2\beta}{k}\frac{\operatorname{sech}^2 y \tanh y}{(\operatorname{sech}^2 y - \tfrac{2}{3})}\hat{v}_1\right].$$

First we shall consider the general solution of the homogeneous equation

$$\frac{d^2\hat{\psi}}{dy^2} - [ikR(\operatorname{sech}^2 y - \tfrac{2}{3}) + 1]\hat{\psi} = 0, \quad (30)$$

then extend it to the non-homogeneous differential equation (29).

To find an asymptotic representation of the solution of (30) we shall apply the Langer method (LANGER, 1949). First consider the following differential equation

$$\frac{d^2\hat{\psi}_1}{dy^2} - \left[ikR(\operatorname{sech}^2 y - \tfrac{2}{3}) + \frac{1}{S}\frac{d^2 S}{dy^2}\right]\hat{\psi}_1 = 0 \quad (31)$$

which has a solution

$$\hat{\psi}_1(y) = S(y)h(Z), \quad (32)$$

where

$$S(y) = \left[\frac{3}{2}\int_{y_c}^{y}(\operatorname{sech}^2 y - \tfrac{2}{3})^{1/2}dy\right]^{1/6}(\operatorname{sech}^2 y - \tfrac{2}{3})^{-1/4}, \quad (33)$$

$$Z(y,kR) = iY(y,kR) = i(kR)^{1/3} \times \left[\frac{3}{2}\int_{y_c}^{y}(\operatorname{sech}^2 y - \tfrac{2}{3})^{1/2}dy\right]^{2/3}, \quad (34)$$

$$\frac{d^2 h}{dZ^2} + Zh = 0. \quad (35)$$

Equation (31) may be considered as the approximating differential equation for (30) and is correct to order $(kR)^{-1/3}$; we may write $\hat{\psi} \equiv \hat{\psi}_1$.

The solution of (35) are the modified Hankel functions of order one-third. That is

$$h_1(Z) = (\tfrac{2}{3}Z^{3/2})^{1/3}H_{1/3}^{(1)}(\tfrac{2}{3}Z^{3/2}),$$
$$h_2(Z) = (\tfrac{2}{3}Z^{3/2})^{1/3}H_{1/3}^{(2)}(\tfrac{2}{3}Z^{3/2}). \quad (36)$$

The general solution is therefore

$$\hat{\psi} = [A_1 h_1(Z) + A_2 h_2(Z)]S(y), \quad (37)$$

where A_1 and A_2 are arbitrary constants.

Returning to the non-homogeneous problem,

$$\frac{d^2\hat{\Phi}}{dy^2} - \left[ikR(\operatorname{sech}^2 y - \tfrac{2}{3}) + \frac{1}{S}\frac{d^2 S}{dy^2}\right]\hat{\Phi} = -2\beta R \operatorname{sech}^2 y \tanh y\hat{v}_1 \quad (38)$$

the solution $\hat{\Phi}(y)$ is required to approach the inviscid solution $\Phi_I(y)$ for $(kR)^{1/3}$ large. An application of the method of variation of parameter shows that this solution is

$$\Phi = -i\frac{2\beta}{k}\frac{\operatorname{sech}^2 y \tanh y\hat{v}_1}{(\operatorname{sech}^2 y - \frac{2}{3})} ZL(Z), \tag{39}$$

where $L(Z)$ is a Lommel function, and is the solution of the differential equation

$$\frac{d^2L}{dZ^2} + ZL = 1, \tag{40}$$

$$L(Z) = \frac{h_2(Z)\int_{-\infty}^{Z} h_1(\zeta)\,d\zeta - h_1(Z)\int_{-\infty}^{Z} h_2(\zeta)\,d\zeta.}{\begin{vmatrix} h_1(Z) & h_2(Z) \\ \dfrac{dh_1}{dZ} & \dfrac{dh_2}{dZ} \end{vmatrix}}, \tag{41}$$

provided $-\frac{2}{3}\pi < \arg(Z) < \frac{2}{3}\pi$. In the above equations

$$L(Z) = L_r(Y) + iL_i(Y), \tag{42}$$

where

$$Y = (kR)^{1/3}\left\{\frac{3}{2}\int_{y_c}^{y} (\operatorname{sech}^2 y - \tfrac{2}{3})^{1/2} dy\right\}^{2/3}. \tag{43}$$

In dealing with the case of high Reynolds numbers we shall retain the leading terms in kR for all quantities in the subsequent analysis. The viscous corrections, which apply for $Y = 0(1)$ or $y = 0(kR)^{-1/3}$, remove the singularities obtained by the inviscid analysis.

From (40) and (42) the differential equations for the Lommel function may be written

$$\left.\begin{aligned} \frac{d^2L_r}{dY^2} + YL_i &= -1, \\ \frac{d^2L_i}{dY^2} - YL_r &= 0. \end{aligned}\right\} \tag{44}$$

It can be shown that L_r is an even and L_i an odd function of Y, and that

$$L = \sum_{\substack{r=1 \\ n=0,1,2}}^{\infty} i^{r-n}\frac{n!}{(3r+n)!}(n+1)(n+4) \times (n+7)\ldots(n+3r-2)\,|a_k|\,Y^{3r+n}$$

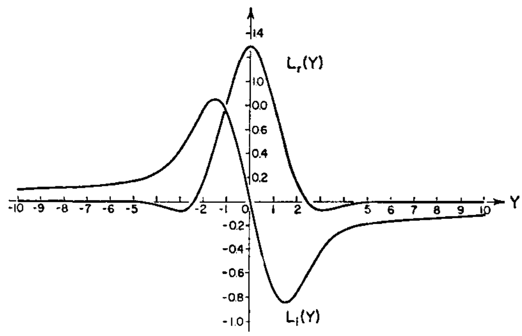


FIG. 1. The real and imaginary parts of Lommel function.

with

$$\begin{cases} a_0 = i\,3^{-2/3}\Gamma(\frac{1}{3}), \\ a_1 = \dfrac{2\pi 3^{-5/6}}{\Gamma(\frac{1}{3})}, \\ a_2 = -\dfrac{i}{2}, \\ a_n = i\,\dfrac{a_{n-3}}{n(n-1)} \quad \text{for} \quad n \geq 3. \end{cases}$$

The graphs of $L_r(Y)$ and $L_i(Y)$ are shown in Fig. 1.

It can also be shown that for large values of Z, $L(Z) \sim Z^{-1}$.

For the antisymmetric solution of $\hat{v}_1$ the longitudinal, lateral and vertical components of the primary oscillation are, therefore,

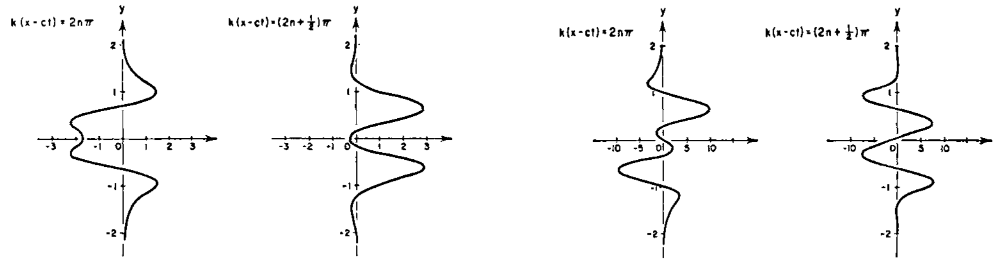
$$\begin{aligned} u^{(1)} = 2\frac{\lambda}{k}\Bigg[&\operatorname{sech} y(2\operatorname{sech}^2 y - 1) \\ &+ \frac{\beta \operatorname{sech} y(\operatorname{sech}^2 y - 2)}{3(\operatorname{sech}^2 y - \frac{2}{3})}\Bigg][L_i(Y)\sin k(x-ct) \\ &- L_r(Y)\cos k(x-ct)]\,Y\cos\beta z - 2\gamma\operatorname{sech} y \\ &\times (2\operatorname{sech}^2 y - 1)\sin k(x-ct), \end{aligned}$$

$$v^{(1)} = 2\operatorname{sech} y\tanh y(\lambda\cos\beta z + \gamma)\cos k(x-ct),$$

$$\begin{aligned} w^{(1)} = -\frac{2}{3}\frac{\lambda\beta\operatorname{sech} y(\operatorname{sech}^2 y - 2)\,Y}{(\operatorname{sech}^2 y - \frac{2}{3})}&[L_r(Y)\sin k \\ \times (x-ct) + L_i(Y)\cos k(x-ct)]&\sin\beta z, \end{aligned}$$

$$p^{(1)} = -\tfrac{2}{3}\operatorname{sech} y(\operatorname{sech}^2 y - 2)(\lambda k\cos\beta z + \gamma)\sin k \times (x-ct).$$

For the antisymmetric solution of $\hat{v}_1$ the



Amplitude of $\omega^{(1)}$ for antisymmetric solution of $\hat{v}_1$ at spanwise position $\beta z = (2n+\frac{1}{2})\pi, \lambda = 1$

Amplitude of $\xi^{(1)}$ for antisymmetric solution of $\hat{v}_1$ at spanwise position $\beta z = (2n+\frac{1}{2})\pi, \lambda = 1$

FIG. 2. The amplitudes of the vertical velocity and the x-component of the vorticity for the antisymmetric solution of $\hat{v}_1$.

amplitudes of the vertical velocity and the x-component of the vorticity are shown in Fig. 2. It may be noted that the amplitude of the vertical velocity is symmetric with respect to the jet axis with its maximums occurring near the points of inflection of the velocity profile of the jet.

For the symmetric solution of $\hat{v}_1$ the longitudinal, lateral and vertical components of the primary oscillation are

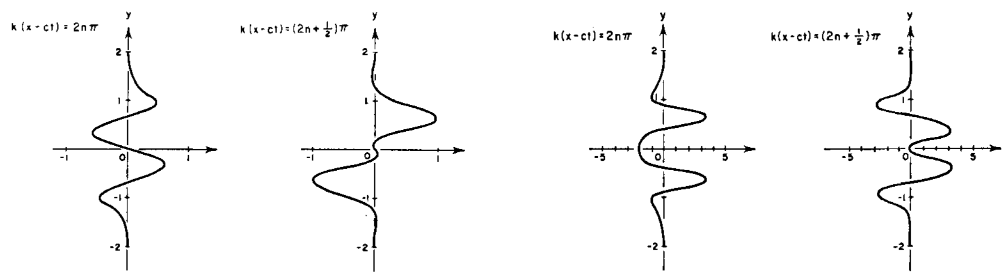
$$u^{(1)} = \frac{\lambda}{k}\operatorname{sech}^2 y \tanh y\left[4+\beta^2\left(\frac{\operatorname{sech}^2 y}{\operatorname{sech}^2 y - \frac{2}{3}} - 1\right)\right]$$
$$\times [L_r(Y)\cos k(x-ct) - L_i(Y)\sin k(x-ct)]$$
$$\times Y\cos\beta z + 2\gamma\operatorname{sech}^2 y \tanh y \sin k(x-ct),$$

$$v^{(1)} = 2\operatorname{sech}^2 y(\lambda\cos\beta z + \gamma)\cos k(x-ct),$$

$$w^{(1)} = \frac{2}{3}\frac{\lambda\beta\operatorname{sech}^2 y\tanh y}{(\operatorname{sech}^2 y - \frac{2}{3})}[L_r(Y)\sin k(x-ct)$$
$$+ L_i(Y)\cos k(x-ct)]\,Y\cos\beta z,$$

$$p^{(1)} = \frac{2}{3}\operatorname{sech}^2 y\tanh y(\lambda k\cos z + 2\gamma)\sin k(x-ct).$$

The amplitudes of the vertical component of the velocity and the x-component of the vorticity for the symmetric solution of $\hat{v}_1$ are shown in Fig. 3. The former appears to be antisymmetric with respect to the jet axis with its maximums occurring near the inflection points of the velocity profile of the jet.



Amplitude of $\omega^{(1)}$ for symmetric solution of $\hat{v}_1$ at spanwise position $\beta z = (2n+\frac{1}{2})\pi, \lambda = 1$

Amplitude of $\xi^{(1)}$ for symmetric solution of $\hat{v}_1$ at spanwise position $\beta z = (2n+\frac{1}{2})\pi, \lambda = 1$

FIG. 3. The amplitudes of the vertical velocity and the x-component of the vorticity for the symmetric solution of $\hat{v}_1$.

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REFERENCES

BENNEY, D. J., and C. C. LIN, 1960, On the secondary motion induced by oscillations in a shear flow. *Phys. of Fluids*, **3**, p. 656.

BENNEY, D. J., 1961, A nonlinear theory for oscillations in a parallel flow. *J. Fluid Mech.*, **10**, p. 209.

BICKLEY, W., 1937, The plane jet. *Phil. Mag.* Ser. **7**, **23**, p. 156.

CLENSHAW, C. W., and D. ELLIOTT, 1960, A numerical treatment of the Orr-Sommerfeld equation in the case of a laminar jet. *Q. J. Mech. and Applied Math.*, **13**, p. 300.

CURLE, N., 1957, On hydrodynamic stability in unlimited fields of viscous flow. *Proc. Roy. Soc.*, A, **238**, p. 489.

HOLMBOE, J., 1962, On the behavior of symmetric waves in stratified shear layers. *Geofys. Publikasjoner*, 24, p. 67.

HOWARD, L. N., 1959, hydrodynamic stability of a jet. *J. Math. Phys.*, **37**, p. 283.

KUO, H. L., 1949, Dynamic instability of two dimensional nondivergent flow in a barotropic atmosphere. *J. of Meteor.*, **6**, p. 105.

LANGER, R. E., 1949, The asymptotic solutions of ordinary linear differential equations of the second order. *Trans. Amer. Math. Soc.*, **67**, p. 461.

LIN, C. C., 1955, *Theory of Hydrodynamic Stability.* Cambridge University Press, 197 pp.

SAVIC, P., 1941, On acoustically effective vortex motion in gaseous jets. *Phil. Mag.*, **32**, p. 245.

SAVIC, P., and J. W. MURPHY, 1943, The symmetric vortex in sound-sensative plane jet. *Phil. Mag.*, **34**, p. 139.

SCHLICHTING, H., 1933, Laminare Strahlausbreitung. *Z. angew. Math. Mech.*, **13**, p. 260.

SQUIRE, H. B., 1933, On the stability of the three-dimensional disturbances of viscous flow between parallel walls. *Proc. Roy. Soc. A*, **142**, p. 621.

TATSUMI, T., and T. KAKUTANI, 1958, The stability of a two-dimensional laminar jet. *J. Fluid Mech.*, 4, p. 261.