A two-layer frictional model of wind-driven motion in a rectangular oceanic basin

By PIERRE WELANDER, *Woods Hole Oceanographic Institution*

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ABSTRACT

A study is made of the circulation in a two-layer ocean with rectangular boundaries driven by a zonal wind stress. The dynamics is linear, with bottom and interface stresses. A non-linearity arises from finite variations in layer depths. In the case where the lower layer is much deeper than the upper layer the equations can be integrated exactly. The solution gives a directly wind-driven gyre in the upper layer, with an interior Sverdrup regime and a western boundary current. In the lower layer there is no interior motion but a recirculating boundary current develops. The total transport is thereby intensified near the coast while a counter-transport develops at the outer edge. The boundary current width varies with the current depth and the latitude.

The key parameter in the model is $\varepsilon = (D/F)(H/h_0)$, where D and F are the "frictional depths" applicable at the interface and the bottom, H and h_0 are the total depth and the upper layer depth. Numerical examples are worked out for a sinusoidal wind stress distribution in the case $\varepsilon = 1$.

1. Introduction

The works by SVERDRUP (1947), STOMMEL (1948), and MUNK (1950) demonstrated that the mass transport field of the oceans, including boundary currents of Gulf Stream type, could be predicted relatively well by a simple model in which the pressure gradient, the Coriolis force and a dissipating force (side friction or bottom friction) balance for each vertical column of water. Actually, it was shown that the action of the dissipating stress is limited to the western boundary region, and that the mass transport in the interior can be computed from the two equations

$$\beta M_y = \operatorname{curl} \tau^w, \tag{1}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} = 0, \tag{2}$$

with the boundary condition $M_x = 0$ at an eastern wall. The first equation, the "Sverdrup relation", gives the northward mass transport M_y in terms of β (the variation of the Coriolis parameter) and the wind stress curl. It holds under the additional assumption that the bottom topography has no effect (either it is assumed that the ocean has a uniform depth, or that all motions decay below a certain level).

A number of later investigators have tried to improve the theory by adding non-linear acceleration terms, that are expected to play a role in the boundary currents. For a review of some of these papers see, for example, STOMMEL, H., *The Gulf Stream* (1958). As soon as one includes non-linear dynamics one cannot, however, derive a simple mass transport equation, except in the case of a homogeneous ocean. The homogeneous ocean model has been studied in some detail in the last years, both analytically and numerically, and one seems to be close to an understanding of this case. The next step would then be to consider models with stratification.

In the past some studies of stratified models have been carried out, but these are generally restricted to the ideal fluid case. To solve the problem of a wind-forced circulation in a closed oceanic basin friction must, however, be retained, and in the present study an attempt is made to develop a model that includes both stratification and friction in their simplest forms.

The stratification is modelled in terms of a two-layer system. To make this realistic one places the interface about at the thermocline depth. This makes the lower layer essentially deeper than the upper one (in the open ocean the ratio of the depths is of the order 10:1). The crudest model simply considers the two layers immissible and behaving as two different fluids in laminar motion (the surface tension is assumed negligibly small). This allows only for transfer of momentum (normal and tangential) through the interface. More sophisticated models should consider the processes at the interface in a better approximation, considering that this discontinuity really represents the limit of a region of strong stratification in a single fluid, with a possibility of exchange not only of momentum but also of mass and heat.

In the present study the first type of two-layer system is discussed, but it is intended to return later to the system of the second type that will allow us, in a crude way at least, to incorporate features of the thermal circulation in the theory.

Friction is modelled in terms of "Ekman stresses". Due to the rotational constraints the frictional effects are concentrated to thin shear layers near the interface and the bottom. In the vertically integrated equations that are used we need only the stresses at the interface and the bottom, these are easily computed by the theory of EKMAN (1905). No attempt is made to include friction due to lateral shear. While the Ekman modelling (replacing the molecular viscosity by constant eddy viscosity) seems to be at least qualitatively correct, one has strong indications that lateral friction in the ocean cannot at all be described by a constant eddy viscosity.

In the model the role of the interface stress is critical (the deep layer motion is driven by the interface stress, not the pressure action from the upper layer). This may seem unrealistic. However, it can be shown by applying the so-called potential vorticity equation, that the same result holds true also in more general models including both bottom topography and non-linear acceleration terms.

The dynamics in the present model is entirely linear, i.e. a balance is assumed between the pressure gradient, the Coriolis force, and the friction force. The model is, however, still non-linear, through the interface boundary conditions. In a realistic model one must assume that at least the upper layer depth can vary to the same order as the layer depth itself. In the vertically integrated dynamic equations one gets a term that involves the product of the depth and the pressure gradient. This is accordingly considered as an essentially non-linear term.

The case specifically studied in the paper is the circulation set up by a zonal wind stress $\tau_x^w(y)$, acting over an ocean enclosed in a rectangular basin in the β-plane. The case of meridional wind stresses should be studied as a separate problem. The meridional wind stress, in contrast to the zonal wind stress, is directly important in the boundary layer equations at a meridional wall, and these local stresses can drive a boundary circulation.

The completely linearized case has been considered by Stommel (private communication). This model produces also a boundary gyre in the lower layer, and the gross features of the two solutions do not differ much. However, the situation will be different when effects of bottom topography, separation of the upper layer from the boundary, etc. come into play.

2. Basic equations

Consider a two-layer fluid in a β-plane, (x, y, z are rectangular coordinates, x eastward, y northward, z upward. The Coriolis parameter is $f = f_0 + \beta y$). The fluid is confined between a horizontal bottom (at depth $z = -H$), two meridional walls ($x = 0$ and a) and two zonal walls ($y = 0$ and b). The fluid is driven by a zonal wind stress τ_x^w that varies with y only. It is assumed that $d\tau_x^w/dy = 0$ at $y = 0$ and b, to avoid a Sverdrup transport at these boundaries.

Let the densities of the upper and lower layers be ϱ and $\varrho + \Delta\varrho$, the depths be $h(x, y)$ and $h'(x,y)$. Assuming a balance of the pressure gradient, Coriolis force, and friction due to vertical shear, the momentum equations, integrated over the respective layer depths, are

$$-fM_y = -h\frac{\partial p}{\partial x} + \tau_x^w(y) - \tau_x^i, \tag{3}$$

$$fM_x = -h\frac{\partial p}{\partial y} - \tau_y^i, \tag{4}$$

$$-fM'_y = -h' \frac{\partial p'}{\partial x} + \tau^i_x - \tau^b_x, \qquad (5)$$

$$fM'_x = -h' \frac{\partial p'}{\partial y} + \tau^i_y - \tau^b_y. \qquad (6)$$

M_x, M_y and M'_x, M'_y are the mass transport components in the upper and lower layers, p and p' the pressures. $\tau^w_x(y)$ is the wind stress, (τ^i_x, τ^i_y) the interface stress and (τ^b_x, τ^b_y) the bottom stress.

For the mass continuity we have

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} = 0, \qquad (7)$$

$$\frac{\partial M'_x}{\partial x} + \frac{\partial M'_y}{\partial y} = 0. \qquad (8)$$

The interface stress (τ^i_x, τ^i_y) must generally be assumed to exist, as the (geostrophic) velocities in the two layers differ. As it turns out, it is also mathematically necessary to have this stress, as otherwise all the side boundary conditions cannot be met. It is natural to assume that the interface stress is proportional to the difference of the geostrophic velocity vector between the layers, allowing also for an angle between stress and the velocity difference. Similarly the bottom stress is assumed to be proportional to the geostrophic velocity in the lower layer, again allowing for an arbitrary angle. The general expressions are

$$\left.\begin{aligned} \tau^i_x &= -C\frac{\partial}{\partial x}(p-p') - D\frac{\partial}{\partial y}(p-p'), \\ \tau^i_y &= -C\frac{\partial}{\partial y}(p-p') + D\frac{\partial}{\partial x}(p-p'); \end{aligned}\right\} \qquad (9\,\text{a, b})$$

$$\left.\begin{aligned} \tau^b_x &= -E\frac{\partial p'}{\partial x} - F\frac{\partial p'}{\partial y}, \\ \tau^b_y &= -E\frac{\partial p'}{\partial y} + F\frac{\partial p'}{\partial x}. \end{aligned}\right\} \qquad (10\,\text{a, b})$$

We assume that C, D, E, F (dimension length) are small compared to h and h', and constant. C, D may be of different order of magnitude than E, F.[1]

[1] In Ekman's theory assuming a constant eddy viscosity C, D, E, F are all equal. They are not constants but vary slowly with latitude.

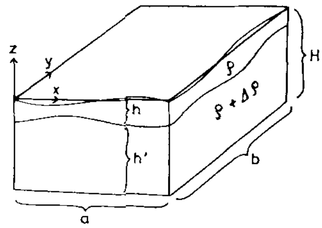

FIG. 1. Coordinate system and geometry of the model.

The pressure gradients can be evaluated hydrostatically, this is justified because we deal with large-scale, steady motion (from this point of view, also our boundary currents are large-scale).

If we define the surface elevation ζ and a new depth variable Z by the relations

$$\left.\begin{aligned} &h+h' = H+\zeta, \quad \zeta - \frac{\Delta\varrho}{\varrho}h = \frac{\Delta\varrho}{\varrho}Z, \\ &(H \text{ is the mean total depth}) \end{aligned}\right\} \qquad (11\,\text{a, b})$$

the pressure gradients in the upper and lower layer are

$$\left.\begin{aligned} \frac{\partial p}{\partial x} &= g\varrho\frac{\partial \zeta}{\partial x} = g\Delta\varrho\frac{\partial}{\partial x}(h+Z), \text{ etc.} \\ \frac{\partial p'}{\partial x} &= g\varrho\frac{\partial}{\partial x}\left(\zeta - \frac{\Delta\varrho}{\varrho}h\right) = g\Delta\varrho\frac{\partial Z}{\partial x}, \text{ etc.} \end{aligned}\right\} \qquad (12)$$

The h' occurring in (5) and (6) is expressed as $H-h$, since $\zeta < h$. We have then the six equations (3–8) expressed in M_x, M_y, M'_x, M'_y, h and Z. The boundary conditions are

$$M_x = M'_x = 0 \quad \text{at} \quad x = 0, a, \qquad (13)$$

$$M_y = M'_y = 0 \quad \text{at} \quad y = 0, b. \qquad (14)$$

3. Interior and boundary layer regimes

The equations are simplified further by the assumptions (i) that the frictional stresses only are important in regions close to the western wall, and (ii) that the interior regime is "compensated", meaning that the horizontal pressure gradient in the lower layer vanish. These assumptions will be justified by the subsequent calculations.

It should be noted that these really are two independent assumptions: that the stresses in the interior are negligible does not imply that compensation occurs, conversely, if the interior solution is compensated there may still be important stresses. The two assumptions together imply that there is no motion in the lower layer.

In the boundary layer region $\partial/\partial_x \gg \partial/\partial_y$, and the expressions for the interface and bottom stresses simplify accordingly. It is further possible to neglect the driving force $\tau_x^w(y)$ in the boundary layer. The reason is that in the boundary layer M_y must become large not only compared to M_x but also compared to the typical M_y in the interior solution, in order to balance the total meridional mass transport to zero. The interior M_y is of order $f_0^{-1}\,\tau_x^w$, if the wind field has a global scale (of order f_0/β), it follows that the τ_x^w-term in (3) can be dropped in the boundary layer.

The equations become accordingly:

A. Interior

$$-fM_y = -g\Delta\varrho h\frac{\partial h}{\partial x} + \tau_x^w(y), \tag{15}$$

$$fM_x = -g\Delta\varrho h\frac{\partial h}{\partial y}, \tag{16}$$

$$M_x' = M_y' = 0. \tag{17, 18}$$

B. Boundary layer

$$-fM_y = -g\Delta\varrho\left[h\frac{\partial}{\partial x}(h+Z) - C\frac{\partial h}{\partial x}\right], \tag{19}$$

$$fM_x = -g\Delta\varrho\left[h\frac{\partial}{\partial y}(h+Z) + D\frac{\partial h}{\partial x}\right], \tag{20}$$

$$-fM_y' = -g\Delta\varrho\left[(H-h)\frac{\partial Z}{\partial x} + C\frac{\partial h}{\partial x} - E\frac{\partial Z}{\partial x}\right], \tag{21}$$

$$fM_x' = -g\Delta\varrho\left[(H-h)\frac{\partial Z}{\partial y} - D\frac{\partial h}{\partial x} + F\frac{\partial Z}{\partial x}\right]. \tag{22}$$

Neglections allowed because the frictional depths are small compared to h, h' are not introduced yet.

The interior solution can be directly given. We have to satisfy $M_x = 0$ at $x = a$ (the y-conditions are automatically satisfied with $d\tau_x^w/dy = 0$ at $y = 0$ and b). The interior solution for M_x and M_y fulfilling the condition is

$$M_x^{\text{int.}} = -\int_x^a \frac{\partial M_y}{\partial y}dx = -\frac{a-x}{\beta}\frac{\partial^2\tau_x^w}{\partial y^2}(y), \tag{23}$$

$$M_y^{\text{int.}} = -\frac{1}{\beta}\frac{\partial\tau_x^w}{\partial y}(y). \tag{24}$$

Call the depth at the eastern wall h_0, this is constant by (16), since $M_x = 0$. The equations (15) and (24) give

$$g\Delta\varrho h\frac{\partial h}{\partial x} = \tau_x^w - \frac{f}{\beta}\frac{\partial\tau_x^w}{\partial y} = -\frac{f^2}{\beta}\frac{\partial}{\partial y}\left(\frac{\tau_x^w}{f}\right)$$

with the solution

$$h^2 = h_0^2 + \frac{2f^2}{\beta g\Delta\varrho}(a-x)\frac{\partial}{\partial y}\left(\frac{\tau_x^w}{f}\right). \tag{25}$$

For matching the boundary layer solution we need $h^{\text{int.}}$ at $x = 0$, that we denote $h_1(y)$:

$$h_1^2(y) = h_0^2 + \frac{2af^2}{\beta g\Delta\varrho}\frac{\partial}{\partial y}\left(\frac{\tau_x^w}{f}\right). \tag{25 a}$$

4. The boundary layer equations. The "deep ocean approximation"

Inserting the expressions for M_x, M_y, M_x', and M_y' found from (19–22) into the mass continuity equations (7) and (8) gives the following two equations in h and Z

$$J\left(h+Z, \frac{h}{f}\right) - \frac{D}{f}\frac{\partial^2 h}{\partial x^2} = 0, \tag{26}$$

$$J\left(Z, \frac{H-h}{f}\right) + \frac{D}{f}\frac{\partial^2 h}{\partial x^2} + \frac{F}{f}\frac{\partial^2 Z}{\partial x^2} = 0; \tag{27}$$

(the small terms

$$\frac{\partial}{\partial y}\left(\frac{c}{f}\cdot\frac{\partial h}{\partial x}\right),\quad \frac{\partial}{\partial y}\left(\frac{c}{f}\cdot\frac{\partial h}{\partial x}\right) \text{ and } \frac{\partial}{\partial y}\left(\frac{E}{f}\cdot\frac{\partial Z}{\partial x}\right)$$

are neglected).

These equations have to be solved subject to the x-boundary conditions

$$\left.\begin{aligned} &h\frac{\partial}{\partial y}(h+Z) + D\frac{\partial h}{\partial x} = 0, \\ &(H-h)\frac{\partial Z}{\partial y} - D\frac{\partial h}{\partial x} + F\frac{\partial Z}{\partial x} = 0 \end{aligned}\right\} \text{at } x = 0 \tag{28, 29}$$

and

$$\left.\begin{aligned} &h = h_1(y), \\ &Z = 0 \end{aligned}\right\} \text{at large } x \text{ (in boundary layer scaling).}$$

Instead of the equation (27) and boundary condition (29) we use (26) + (27) and (28) + (29), respectively:

$$\frac{\beta}{f} H \frac{\partial Z}{\partial x} + \frac{\beta}{f} h \frac{\partial h}{\partial x} + F \frac{\partial^2 Z}{\partial x^2} = 0, \qquad (27\,\text{a})$$

$$H \frac{\partial Z}{\partial y} + h \frac{\partial h}{\partial y} + F \frac{\partial Z}{\partial x} = 0, \quad \text{at } x = 0. \qquad (29\,\text{a})$$

$\partial Z/\partial x$ should become small compared to $\partial h/\partial x$, if the lower layer is deep compared to the upper, and some simplification should be possible. This is seen in the following way. As the lower layer is made deeper the boundary current is expected to narrow (the effective friction is decreased in the vertically integrated equations), and $\partial h/\partial x$ will increase. The magnitude of $\partial Z/\partial x$, on the other hand, is fixed by the relation

$$\beta(M_y + M'_y) = -\frac{\partial \tau_x^w}{\partial y}(y) - \operatorname{curl} \tau^b.$$

Inserting the boundary layer form

$$\operatorname{curl} \tau^b = F \frac{\partial^2 p'}{\partial x^2} = F g \Delta \varrho \frac{\partial^2 Z}{\partial x^2}$$

and integrating from $x = 0$ to $x = a$, one finds

$$F g \Delta \varrho \left(\frac{\partial Z}{\partial x}\right)_{x=0} = a \frac{\partial \tau_x^w}{\partial y}(y) \qquad (30)$$

since the total transport integrated across the basin must vanish and $(\partial Z/\partial x)_{x=a} = 0$. By (30) the characteristic value of $\partial Z/\partial x$ in the boundary layer is independent of H, and will not increase when the lower layer is made deep.

The more formal approach is as follows: We make the transformation

$$\left.\begin{aligned} &x \to \frac{f_0 F}{\beta H} \cdot \xi, \quad y \to \frac{f_0}{\beta} \eta, \quad h \to h_0 \cdot h^*, \\ &Z \to \frac{h_0^2}{H} Z^*, \quad h_1(y) \to h_0 \phi_1(\eta), \end{aligned}\right\} \qquad (31)$$

where h_0 is a characteristic value for h and $\phi_1(\eta)$ is an "order unity function". The governing equations (26), (27a) and the boundary conditions (28) and (29a) become, then,

$$\delta \cdot J(Z^*, h^*) - \frac{h^*}{1+\eta} \left(\frac{\partial h^*}{\partial \xi} + \delta \cdot \frac{\partial Z^*}{\partial \xi} \right) - \varepsilon \frac{\partial^2 h^*}{\partial \xi^2} = 0, \qquad (32)$$

$$\frac{1}{1+\eta} \left[\frac{\partial Z^*}{\partial \xi} + h^* \frac{\partial h^*}{\partial \xi} \right] + \frac{\partial^2 Z^*}{\partial \xi^2} = 0, \qquad (33)$$

$$\left.\begin{aligned} &h^* \left(\frac{\partial h^*}{\partial \eta} + \delta \cdot \frac{\partial Z^*}{\partial \eta} \right) + \varepsilon \cdot \frac{\partial h^*}{\partial \xi} = 0, \\ &\frac{\partial Z^*}{\partial \eta} + h^* \frac{\partial h^*}{\partial \eta} + \frac{\partial Z^*}{\partial \xi} = 0 \end{aligned}\right\} \text{at } \xi = 0, \qquad (34, 35)$$

$$\left.\begin{aligned} &h^* = \phi_1(\eta), \\ &Z^* = 0 \end{aligned}\right\} \text{for large } \xi. \qquad (36, 37)$$

The only parameters occurring are

$$\delta = \frac{h_0}{H}, \qquad (38\,\text{a})$$

$$\varepsilon = \frac{D H}{F h_0}. \qquad (38\,\text{b})$$

We restrict ourselves to the case where δ is small while ε is at most of order unity.

It is of value to have the ranges of validity of the different approximations stated at this stage. We use dimensional quantities. In this case the boundary layer scales entering are $X_1 = f_0 F/\beta H$ and $X_2 = X_1 \varepsilon = f_0 D/\beta h_0$. First we note that the concept of a boundary layer is correct only if $X < \left\{ \begin{smallmatrix} a \\ f_0/\beta \end{smallmatrix} \right.$. The neglect of the non-linear acceleration terms is justified when the "geostrophic Rossby numbers"

$$R_0 = \frac{\frac{1}{f \varrho_0} \left| \frac{\partial p}{\partial x} \right|}{f_0 X}, \qquad R'_0 = \frac{\left| \frac{1}{f \varrho_0} \frac{\partial p'}{\partial x} \right|}{f_0 X}$$

are small compared to unity. The conditions are found to reduce to $g(\Delta \varrho/\varrho) h_0 : f_0^2 X^2 < 1$. If we let a be of planetary scale f_0/β, and use the values for X estimated above, the conditions can be summarized in the form

$$\left.\begin{aligned} F/H \\ D/h_0 \end{aligned}\right\} < 1, \quad \left.\begin{aligned} (F/H)^2 \\ (D/h_0)^2 \end{aligned}\right\} \gg \frac{g(\Delta\varrho/\varrho) h_0 \beta^2}{f_0^4} = \left(\frac{R_d}{R}\right)^2, \qquad (39)$$

where R_d is the internal radius of deformation, and R the radius of the earth. In the ocean $(R_d/R)^2$ has a value of the order 10^{-4} to 10^{-5}. It is therefore obvious that values for F and D that justify the model can always be chosen.

The δ-terms can then be neglected. It will be found that the simplified system has a solution satisfying all boundary conditions.

If ε is of order unity we have a uniform scaling across the boundary layer, but if ε is small two scales enter. The equation (32) gives a scale ε for h^*, while the equation (33) gives two scales for Z^*, the scale ε as "forced" by the term $h^*(\partial h^*/\partial\xi)$ and the scale unity as the scale of the "free part". Since Z^* enters also in the upper layer pressure gradient, the double boundary layer will obviously show up in the currents of both layers.

That ε is restricted while H becomes large means that the interface friction D must be chosen sufficiently small compared to F. Note, however, that we cannot neglect the interface friction altogether, since we then lower the order of our system, and all the x-boundary conditions cannot be satisfied.

5. Solution to the boundary layer equations

The mathematical problem to be solved is given by the equations (32) and (33) subject to the boundary conditions (34–37). The parameter δ is put equal to zero.

The equations can be integrated once with respect to ξ, to give

$$\tfrac{1}{2}h^{*2}+\varepsilon(1+\eta)\frac{\partial h^*}{\partial\xi}=\tfrac{1}{2}\phi_1^2, \tag{40}$$

$$\tfrac{1}{2}h^{*2}+Z^*+(1+\eta)\frac{\partial Z^*}{\partial\xi}=\tfrac{1}{2}\phi_1^2, \tag{41}$$

were (36) has been used.

From (40) we get

$$\frac{d\left(\dfrac{h^*}{\phi_1}\right)}{1-\left(\dfrac{h^*}{\phi_1}\right)^2}=\frac{\phi_1}{2\varepsilon(1+\eta)}\cdot d\xi.$$

where the differential is in ξ (η occurs only parametrically), and integrating,

$$\tanh^{-1}\left(\frac{h^*}{\phi_1}\right)=\frac{\phi_1}{2\varepsilon(1+\eta)}+A(\eta)\quad(\text{for } h^*\leqslant\phi_1), \text{ or}$$

$$\coth^{-1}\left(\frac{h^*}{\phi_1}\right)=\frac{\phi_1}{2\varepsilon(1+\eta)}+B(\eta)\quad(\text{for } h^*>\phi_1).$$

The solution for h^* is thus of the form

$$h^*=\begin{cases}\phi_1(\eta)\tanh\left[\dfrac{\phi_1(\eta)}{2\varepsilon(1+\eta)}\cdot\xi+A(\eta)\right] & (h^*\leqslant\varphi_1),\\[2ex] \phi_1(\eta)\coth\left[\dfrac{\phi_1(\eta)}{2\varepsilon(1+\eta)}\cdot\xi+B(\eta)\right] & (h^*>\phi_1).)\end{cases} \tag{42}$$

The boundary condition (34) (with $\delta=0$) becomes, by substituting $\varepsilon(\partial h^*/\partial\xi)$ from (40):

$$\frac{\partial(h^{*2})}{\partial\eta}=\frac{1}{(1+\eta)}(h^{*2}-\phi_1^2), \tag{43}$$

This equation, that can easily be solved, determines the variation of h^* along the wall. One arbitrary constant occurs, but this is given by the y-boundary conditions. We are going to put $h^*=\phi_1(0)$ and $Z^*=0$ along the southern boundary, thereby fulfilling the conditions $M_y=M_y'=0$ (see (19) and (21)). Similar conditions are applied at the northern boundary.

It is of interest to note that the h-solution (42) is valid independent of the second governing equation (33). This is a consequence of the assumption that the lower layer is deep. The pressure gradient in the lower layer is small, and the upper layer solution can be computed as if the lower layer is at rest. The interface stress acts similarly to the bottom stress in the homogeneous models. Some motion must, however, occur in the lower layer, as only bottom stresses can give the overall balance of the wind stress curl. This small motion furthermore, becomes important when we want to consider transports. The formal solution for Z^* in terms of h^* is, from (41),

$$Z^*=\frac{1}{2}\int_{\xi'=0}^{\xi}e^{(\xi'-\xi)/(1+\eta)}[\phi_1^2(\eta)-h^{*2}(\xi',\eta)]\,d\xi' + G(\eta)\,e^{-\xi/(1+\eta)}. \tag{44}$$

$G(\eta)$ is uniquely determined by the boundary condition (35) and the condition $Z^*=0$ at $\eta=0$.

As will be seen later, Z^* is actually zero along the entire boundary $\xi=0$, i.e. $G(\eta)=0$.

The nature of the solution is most easily discussed with reference to Figs. 2 and 3, that show the relations between $h^*_{\xi=0}$ and $\phi_1(\eta)$ on

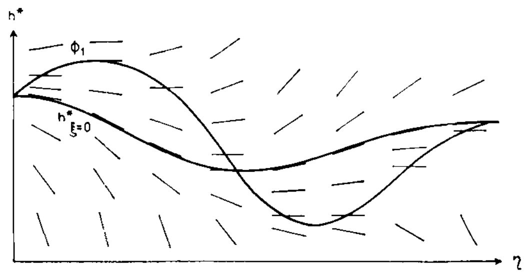

FIG. 2. Construction of $h^*_{\xi=0}$ from (43) (schematic example).

one hand, and between Z^* and h^* on the other hand.

In Fig. 2 is drawn a schematic example of a curve $h^* = \phi_1(\eta)$ and the field at slopes $(\partial h^*/\partial\eta)_{\xi=0}$ as determined by (43). $h^*_{\xi=0}$ starts at the same point as ϕ_1 at $\eta = 0$; it is also known, from the mass continuity, that it will end up at the same point. Obviously, $h^*_{\xi=0}$ decreases when smaller than ϕ_1, increases when larger than ϕ_1, which means that it always decreases in the direction of the boundary current. It may happen that h^* becomes zero, in which case the interface intersects the surface. The present solution cannot be used from there on. It is also possible that the interface intersects the bottom, with a similar diffculty. (The case where the interface reaches the surface is of great practical interest as this phenomenon often occurs in the ocean, and this case deserves further study. However, it is not considered at present.)

With $h^*_{\xi=0}$ computed we know $A(y)$ or $B(y)$ in (42). They have the form $\tanh^{-1}[h^*_{\xi=0}/\phi_1]$ and $\coth^{-1}[h^*_{\xi=0}/\phi_1]$ respectively, and the h^*-profile across the current can be constructed. h^* changes monotonically from the wall to the edge, hence no countercurrents occur in the upper layer. The scale width varies with y, as $(1+\eta)\phi_1^{-1}$. Thus the current becomes wider when its depth decreases, more narrow when its depth increases, and there is also an overall widening proportional to $1+\eta$.

Z^* can also be determined graphically, representing the equations (40) and (41) in a "phase-plane". Solving for $\partial Z^*/\partial\xi$ and $\partial h^*/\partial\xi$ and dividing these expressions we have

$$\frac{dZ^*}{dh^*} = \varepsilon \frac{\frac{1}{2}(\phi_1^2 - h^{*2}) - Z^*}{\frac{1}{2}(\phi_1^2 - h^{*2})}. \tag{45}$$

The field of slopes dZ^*/dh^* and examples of "trajectories", for a given η, are shown schematically in Fig. 3.

The parabola $Z^* = \frac{1}{2}(\phi_1^2 - h^{*2})$ and the line $h^* = \phi_1$, along which $dZ^*/dh^* = 0$ and ∞, respectively, are marked out.

The solution curve is fixed if we know $h^*_{\xi=0}$ and $Z^*_{\xi=0}$. The value of $h^*_{\xi=0}$ is found by our previous construction (shown in Fig. 2). For $\partial\tau_x^w/\partial y > 0$ we have $h^*_{\xi=0} < \phi_1$ (southwardin terior transport and northward boundary current in the upper layer), and the trajectory starts to the left of the "equilibrium point" $h^* = \phi_1$, $Z^* = 0$. Correspondingly the trajectory starts to the right for $\partial\tau_x^w/\partial y < 0$. Further the trajectories must start from the h^*-axis ($Z^* = 0$). This is proved as follows: in (21) the terms $C(\partial h/\partial x)$ and $E(\partial Z/\partial x)$ are of order C/h_0 and E/H, respectively, relatively to the term $(H-h)(\partial Z/\partial x)$, using the scales given by (31). These two terms are thus negligible, in our approximation, i.e. M_y' is geostrophic. Putting $H - h \sim H$ and integrating across the boundary layer the condition that $\int M_y' \, dx = 0$ (no interior flow in the lower layer) gives $Z^*_{\xi=0} = Z^*_{\xi=\infty} = 0$.

It can be immediately seen that the lower layer circulation has the same sign as the upper, adding to the current near the wall and subtracting from it at the outer edge.

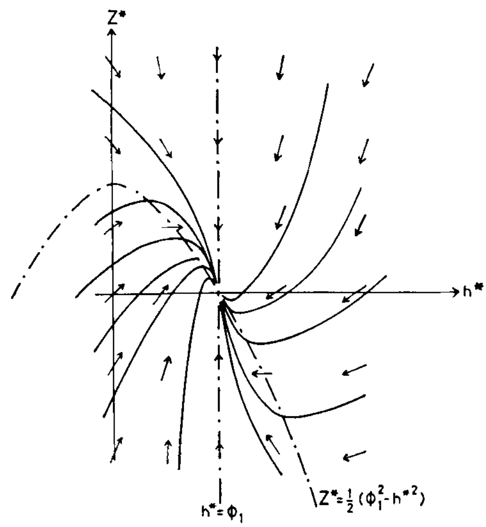

FIG. 3. Field of slopes and trajectories in the h^*-Z^* plane (schematic picture).

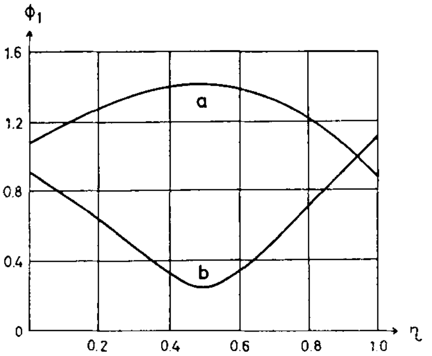

FIG. 4*a–b*. Variation of the interface depth ϕ_1 in the interior solution along the western wall. *a* is the case of negative and *b* the case of positive wind-stress curl. For the parameter values see the text.

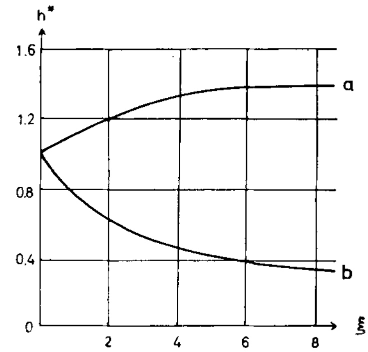

FIG. 6*a–b*. Variation of h^* across the boundary layer at $\eta = \frac{1}{2}$. *a* is the case of negative and *b* the case of positivewind stress curl.

6. Numerical examples

In the numerical examples we take $a = b = f_0/\beta$, $\tau_x^w(y) = \tau_0 \sin \pi(f_0 y/\beta - \frac{1}{2})$. Thus the wind stress changes sign at a mid-latitude. In one case we choose τ_0 positive, in a second case negative. These correspond to a negative and positive wind-stress curl, respectively. The magnitude of the non-dimensional wind stress $a\tau_0/g\Delta\varrho h_0^2$ is 0.1. The value of the parameter $\varepsilon = (D/F)(H/h_0)$ is 1. The variation of the interface depth along the boundary layer edge, determined by the interior solution (25a), is shown in Fig. 4*a–b*. The actual construction of $h^*_{\xi=0}$, is shown in Figs. 5*a–b* (in this case h^{*2} rather than h^* is represented).

The cross-stream variations of h^*, as given by (42) are shown in Figs. 6*a–b*, at the mid-latitude where the wind stress vanishes and the boundary current has its maximum transport. The actual phase-plane constructions relating Z^* to h^* are given in Figs. 7*a–b*.

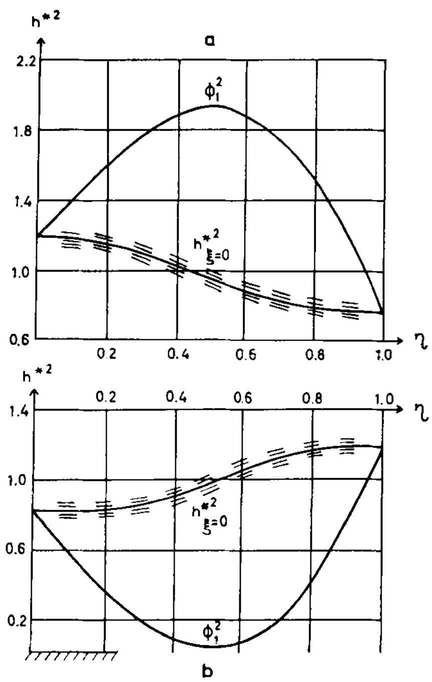

FIG. 5*a–b*. Determination of the interface depth in the boundary layer solution along the western wall ($\xi = 0$). *a* is the case of negative and *b* the case of positive wind-stress curl.

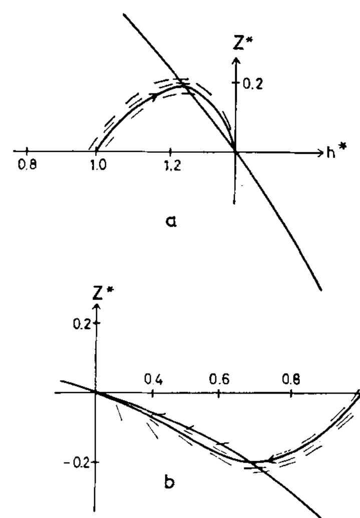

FIG. 7*a–b*. Determination of Z^* as function of h^* from a phase-plane construction. *a* is the case of negative and *b* the case of positive wind stress curl.

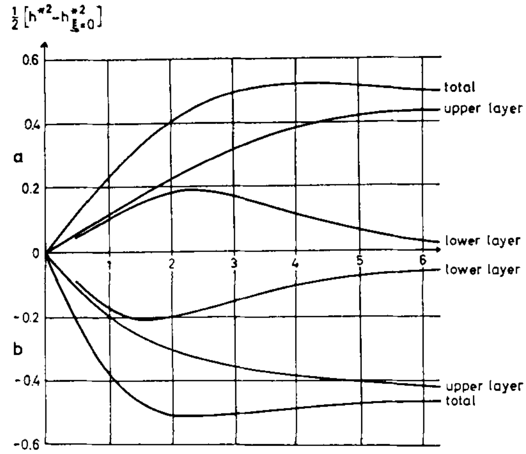

FIG. 8*a–b*. Transports in the boundary current at $\eta=\frac{1}{2}$, integrated from the wall ($\xi=0$). *a* is the case of negative and *b* the case of positive wind stress curl.

The boundary current transports, integrated from the coast, are shown in Figs. 8*a–b*. Figs. 9*a–b*, finally, give a perspective sketch of the interface variations and of the transport streamline patterns in the two cases.

Quantitatively, one is mostly interested in the values of the transports. As can be seen from Fig. 8, the addition of the lower layer gyre makes the total current become more narrow, and a weak countercurrent develops at the outer edge. However, the lower gyre is not strong enough to rise the maximum transport value appreciably.

7. Relation to the real ocean

The result of the present calculation is of interest in connection with recent Gulf Stream observations. The transports in the Stream, after it leaves the coast, seem to increase more than has been anticipated (transport figures of 150 million tons per second have been mentioned). Worthington & Warren (private communications) have suggested that a deeper recirculation occurs in the Western Atlantic (how much occurs on the inshore side and the outshore side of the Stream is still disputed). The theoretical model predicts a boundary recirculation in the lower layer. However, it is too weak to influence the maximum transport value much. Adding lateral friction would probably strengthen the countercurrent. It is felt that the topography is a still more important factor. Its effect will be discussed in a second paper. The topography will probably be important not only in the boundary layer region but also in the interior. The Sverdrup relation is then out of play, and one has much greater freedom in the transport field.

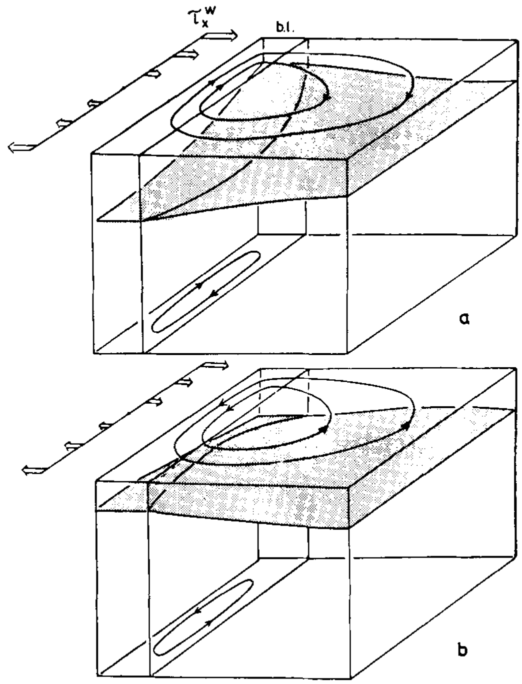

FIG. 9*a–b*. Perspective picture of the interface and transport stream lines (schematic drawing). *a* is the case of negative and *b* the case of positive wind stress curl.

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