Note on the effect of rotation on diffusion processes

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ABSTRACT

A gas in solid rotation that is subjected to a radial temperature gradient develops a component of heat diffusion at right angles to the gradient. The magnitude of this diffusive flux, relative to the radial component, is $2\lambda\Omega/\bar{v}$, *where* λ *and* $\bar{v}$ *are the mean free path and the mean radial speed of the molecules, respectively, and* Ω *is the angular velocity of the system. Corresponding effects occur in diffusion of mass and momentum. It is suggested that the same effect becomes important in the large-scale turbulent transports in the atmosphere and the oceans. The prediction is that a general eastward eddy heat flux should occur.*

1. Effect of rotation on molecular motion

A gas in solid rotation and of uniform temperature is in an equilibrium state, and the distribution of (relative) molecular velocities is Maxwellian. This is proved by general statistical-mechanical arguments. The molecular paths are, however, affected by the rotation. Consider the case of a rarefied gas where the molecules travel without influence of intermolecular forces except during short moments of "collisions". Without rotation and in the absence of outer forces the paths are straight lines. In a rotating system the (relative) paths are deviated by the action of the centrifugal and Coriolis forces. The centrifugal force makes the molecules drift radially outward until a number density gradient is built up creating the balancing inward flux. The Coriolis force causes a drift of the molecules perpendicular to their direction of motion. When the period of the rotation T is small compared to the characteristic time τ between two successive collisions of a molecule the deviation is small, but when $T \sim \tau$ the effect is strong. When $T \gg \tau$ finally, the paths get completely distorted, and the molecules travel in a spiralling motion (Fig. 1). This simulates the motion of a charged particle in a strong magnetic field. Obviously the effective mean free path is strongly decreased in the plane perpendicular to the rotation vector, and the gas is anisotropic. It should be noted, however, that this anisotropy only manifests itself when we look "Lagrangian" on the motion; the instantaneous distribution of molecules is still Maxwellian and thus isotropic at every point.

Let us follow a single molecule over a longer time. As a specific case, consider a gas between two radial boundaries (radii r_A and r_B), the whole system being in solid rotation (angular velocity Ω). We look first on a molecule starting at the outer boundary (an "A-molecule"). At a later time this molecule will be found somewhere in the interior, at a smaller radial distance r. This displacement is not due to outer forces but simply a result of "random walk". Associated with a radial velocity v and a mass m is now a perpendicular Coriolis force $-2\Omega mv$ (directed zonally with the rotation for inward radial motion). The total impulse that the molecule has experienced by the Coriolis force is $-\int 2\Omega mv dt = 2\Omega m(r_A - r)$. This will cause a zonal drift with the rotation. Normally this drift is very small because of the effect of molecular collisions (as one finds that the drift due to the gravity force is very small under normal conditions), but it is certainly present. A corresponding drift but with opposite sign will occur for a molecule starting at the inner wall (a "B-molecule"). The Coriolis force has thus a differential effect on the molecular motion, different from the action of the centrifugal force, or the gravity force, that are equal for all molecules.

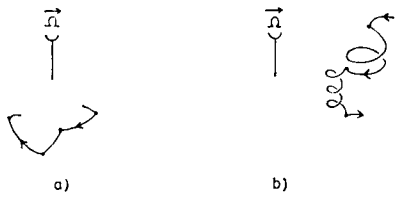

FIG. 1. Molecular paths between collisions in the case of weak (a) and strong (b) Coriolis effect.

2. Effect on molecular transports

From the previous discussion it is clear that the Coriolis force creates a zonal diffusive flux between molecules that leave the outer and the inner walls of the system shown in Fig. 2, the A-molecules diffusing with the rotation and the B-molecules diffusing against the rotation. There is, of course, no net zonal motion (the net radial mass flux is zero, hence the net zonal Coriolis force is zero, summed over both A- and B-molecules). In the case of a real two-component system with radial concentration gradients one finds similarly real zonal diffusive fluxes.

It is also seen that such an effect must occur in the case of heat diffusion. If, in the system shown in Fig. 2, we keep the inner boundary at a higher temperature than the outer (in which case the system is gravitationally stable, i.e. no convective motion occurs) we have not only a radial but also a zonal heat flux, in a direction opposite to the rotation. All molecules with outward motion have systematically a higher temperature than the molecules moving inward in the same region, and since the former have

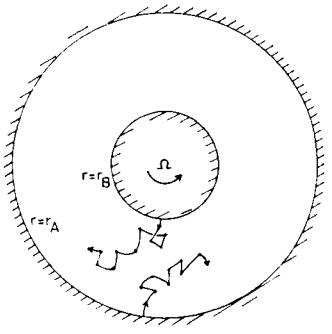

FIG. 2. Molecular paths starting at the outer and inner boundaries and showing the "Coriolis drift" (schematic drawing).

a mean Coriolis drift against the rotation, the latter a mean Coriolis drift with the rotation, a net zonal heat flux against the rotation is created.

If we impose a radial flux of momentum one similarly finds an associated zonal momentum transport. If, for example, we rotate the outer wall in Fig. 2 at a slightly higher angular velocity than the inner wall, a shearing stress occurs, and momentum is transported radially inwards. The molecules with inward motion thus carry more momentum than those with outward motion in the same region, and the Coriolis drift will produce a zonal momentum transport in the direction of rotation. It is easily seen that this manifests itself as a normal stress. To solve the problem correctly requires a discussion of the complete stress tensor that is not attempted here, we only point out that the main effect is the creation of a zonal momentum transport.

The magnitude of these zonal transports, relative to the radial part, can be found by a simple "mean free path calculation". Consider the molecules crossing a small area element ΔA normal to a radial direction (Fig. 3). We assume that the mean free path is small compared to the radius; the first order effect can then be computed for a plane surface element using local rectangular coordinates (see Fig. 3b). We also assume that the Coriolis effect is small, i.e. only small deviations from a straight line molecular path occur between collisions. Now, if we have n molecules per unit volume the number of molecules crossing ΔA in time Δt in an inward direction is $N = n\bar{v}\Delta A\Delta t$, where $\bar{v}$ is the mean radial speed of the molecules (in a volume element). The same number of molecules are crossing outwards in a steady state. We assume now that a certain macroscopic variable (concentration, momentum, temperature, etc.) is made to vary radially. We call this quantity $Q(y)$. The molecules crossing ΔA are assumed to carry with them a value of Q representative for the point of their last collision. Since Q will only vary slightly over the length of a mean free path we can use the expression $Q(y) = Q_0 + y(dQ/dy)_0$, where Q_0 and $(dQ/dy)_0$ are the values evaluated at $y = 0$. The net flux of Q across ΔA in time Δt is $N(\overline{Q}_+ - \overline{Q}_-)$ where $\overline{Q}_+$ and $\overline{Q}_-$ stand for the mean value of Q evaluated for molecules crossing outward and inward, respectively. We have $\overline{Q}_+ = Q_0 - \overline{\lambda}_+(dQ/dy)_0$,

$\overline{Q}_- = Q_0 + \overline{\lambda}_-(dQ/dy)_0$ where $\overline{\lambda}_+$ and $\overline{\lambda}_-$ are the mean distances from the last collision to the surface element, for molecules crossing outwards and inwards, respectively. With sufficient accuracy one can put $\overline{\lambda}_+ = \overline{\lambda}_-(=\lambda)$, where λ has a value close to the mean free path, as ordinarily defined. The radial flux is thus

$$F_r = -2N\lambda(dQ/dy)_0 = -2n\bar{v}\lambda(dQ/dy)_0 \Delta A \Delta t.$$

This derivation of the flux is the one used in the "simple" kinetic theory of viscosity and heat conduction for a gas. We have now to consider the zonal flux created by the Coriolis force. For a molecule crossing inwards and having its last collision at a distance y the Coriolis force will have added a zonal velocity $2\Omega y$. The mean value of this velocity, for all molecules crossing inwards, is $2\Omega\lambda$. The number of molecules crossing another area element ΔA normal to the zonal direction (see Fig. 3a) in time Δt in a positive direction is therefore $n2\Omega\lambda\Delta A\Delta t$, the same number must cross in the opposite direction in a steady state. The molecules crossing in the positive zonal direction carry with them $\overline{Q}_-$, the ones crossing in the negative direction $\overline{Q}_+$, where $\overline{Q}_-$ and $\overline{Q}_+$ have the values given previously. The net zonal flux of Q is accordingly

$$F_c = n2\Omega\lambda\Delta A\Delta t(\overline{Q}_- - \overline{Q}_+) = 4n\Omega\lambda^2(dQ/dy)_0 \Delta A \Delta t.$$

The ratio of the magnitude of the two fluxes is

$$\varepsilon = \left|\frac{F_c}{F_r}\right| = \frac{2\Omega\lambda}{\bar{v}}. \qquad (1)$$

Under normal conditions in a gas this quantity is, of course, very small. For $\lambda = 10^{-6}$ cm, $\bar{v} = 5 \cdot 10^4$ cm s^{-1} and $\Omega = 1$ s^{-1}, we find $\varepsilon = 4 \cdot 10^{-11}$. At a pressure of one millionth of an atmosphere and with an angular velocity of order 10^4 s^{-1}, that can be achieved under laboratory conditions, the effect is certainly noticeable.

The effect is linear in Ω according to (1). This can, however, only be true for small ε. When the Coriolis effect becomes large the effective mean free path is also influenced by the rotation, as mentioned in Sect. 1. What happens in the limit of infinite Ω is not clear, but most likely ε tends to zero again.

The effects discussed here can be seen in the "higher-order theories" for a rarefied gas, developed by Burnett and others. In CHAPMAN and COWLING (1939, p. 266) it is pointed out that heat fluxes of the form $\Delta T \mathbf{x}$ curl $\mathbf{c}_0$ do occur (T is the temperature and $\mathbf{c}_0$ the mean velocity) in the so-called "third order approximation". The astonishing result that these extra fluxes can be produced by a solid rotation of the system is, however, not commented on further. The present discussion has used a simple "mean free path calculation", to demonstrate that the effect can be associated directly with the Coriolis force.

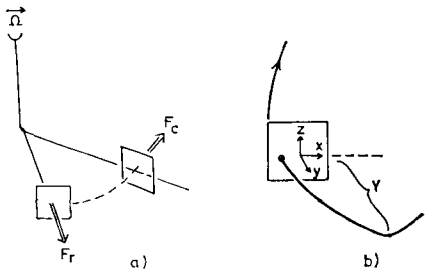

FIG. 3. Radial and zonal transfers (*a*), and geometry of radial transfer calculation (*b*).

3. Application to turbulent diffusion

It is tempting to apply the above mean free path calculation to a turbulent fluid, in a qualitative way at least. In a turbulent case λ and $\bar{v}$ have to be interpreted as a "mixing-length" and a "characteristic turbulent velocity", respectively. The former is quite large and the latter quite small compared to the values in the molecular case, and one can arrive at values on ε of order unity in any simple laboratory experiment with a rotating fluid. So far no experiment has been carried out, to the knowledge of the author, but it would certainly be of interest to try it.

In the turbulent case it is, of course, difficult to derive precise theoretical results. Instead of the molecular paths one has now to consider paths of elements of fluids. The separation of the motion in collisions and "force-free paths" does not work. At least one has to consider the effect of the pressure and frictional forces during the "path". It seems very plausible that their effect is "passive" in the present problem. They tend to oppose the effect of the Coriolis drift but they cannot eliminate it. For strong Coriolis effect the role of the pressure probably

increases, however. The spiralling paths of the molecular case will probably not be seen, but more straight line "geostrophic paths", in which the Coriolis force is closely balanced by a transverse pressure gradient, will develop.

In the atmosphere and the ocean one has a case in nature where the effect can be looked for. In the large-scale atmospheric and oceanic turbulence one finds very active eddies with periods of the order of a day, for which the effect should be noticeable. The quantity to look for would be the heat flux, in the ocean also possibly the salt flux. The flux of momentum is more difficult to study since momentum is also transported by the pressure field.[1]

[1] There is another basic difference between the transport of heat and the transport of momentum that should be noted. In the molecular case it is known that zero momentum transport occurs for solid rotation, i.e. in a case where a radial momentum gradient actually exists. The transport effect we discuss is related to the momentum gradient after solid rotation is subtracted out. Now, in a turbulent fluid the momentum transport must not necessarily vanish in the case of solid rotation. There is, at least, no general proof for this from statistical-mechanical principles. No violation of the second law of thermodynamics occurs if a solid rotation produces momentum fluxes and hence energy possibly can be drawn from the system, for there must anyway be a steady input of mechanical energy to sustain the turbulence.

No similar problem occurs for the heat. The equilibrium case is always the one where the temperature is uniform throughout the gas or fluid, whether the stirring mechanism is molecular or turbulent.

Since both the atmosphere and the ocean are heated at low latitudes and cooled at high latitudes the secondary heat flux should be zonal and eastward, in accordance with the discussion in Sect. 2. For the ocean some computations of the zonal heat transport have been made by OORT (1964) in the western Atlantic (the Gulf Stream region). It was found that the eddy fluxes were systematically directed in the eastern direction although the temperature increases in the same direction!

Such a situation would be possible in terms of the mechanism discussed in the present paper, one would only have to request that the secondary heat flux created by the Coriolis force is larger than the direct westward flux driven by the zonal temperature gradient in the Gulf Stream region.

It is hoped that more computations of the zonal eddy fluxes will be made in the future both for the atmosphere and the ocean so that the idea of a general eastward heat diffusion driven by the Coriolis force can be thoroughly tested.

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