Outflow from a circular cylinder under the action of Coriolis force

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ABSTRACT

A simple potential flow describing the evacuation of fluid from a circular cylinder is presented. The effect of Coriolis force on this outflow is considered. For large Rossby number, it is shown that a circulation $2\pi\Omega\sin\alpha(a^2-r^2)G(t)$ develops in a circuit distance r from the axis of the cylinder, where Ω is the earth's angular speed, α the latitude, a the radius of the cylinder and $G(t)$ the fraction of the fluid that has been evacuated from the cylinder.

1. Introduction

When a fluid flows out of a vessel it frequently happens that an appreciable swirling motion develops. This motion arises from the convection of vortex lines to the region above the outlet, the vortex lines in most cases existing in the fluid before the commencement of the outflow. Generation of vorticity by the action of viscosity or a non-conservative force, such as Coriolis force, usually has a negligible effect compared with the gathering of vorticity already existing in the fluid.

For an ideal, incompressible fluid, which is initially quiescent, Coriolis force terms are responsible for whatever swirling motion exists above the outlet; anticlockwise in the Northern hemisphere and clockwise in the Southern hemisphere. In this paper the effect of Coriolis force on a particular outflow from a circular cylinder is considered.

In the next section a particular potential flow is obtained, which represents outflow from a circular cylinder in the absence of Coriolis force. The cylinder is evacuated by the action of a line sink along its axis; different outlet conditions may be simulated by suitable choice of the time dependence of the strength of the sink. The base of the cylinder is a fixed plane and its top a plane which falls at a rate determined by the strength of the sink.

In section 3 the effect of Coriolis force on this outflow is examined. At a latitude of 90° an exact solution is presented which describes the evacuation of the cylinder under the action of Coriolis force. At other latitudes an approximate solution is indicated which shows the same quantitative build up of swirling motion around the axis of the cylinder. The approximate solution is valid for large Rossby number, R, the ratio of a typical outflow velocity to a typical Coriolis induced swirling velocity. A typical outflow from a bath has a Rossby number of $O(10^3)$. To examine outflow from a bath it would be desirable to use an outflow model based on the action of a point sink. However, as kindly pointed out to the author by the referee, the thin viscous core observed above the outlet before the vortex becomes air-cored is a region of high downward velocity and would act very similarly to the line sink considered in this paper.

2. A potential outflow

For irrotational motion of an incompressible fluid a velocity potential ϕ exists. In cylindrical polar co-ordinates (r, θ, z) a possible potential is

$$\phi = -m(t)\log r + m(t)(r^2 - 2z^2)/(2a^2), \qquad (1)$$

t being the time.

This potential has $\partial\phi/\partial r = 0$ on $r = a$ and $\partial\phi/\partial z = 0$ on $z = 0$ and thus represents a flow in the semi-infinite cylinder $r < a$, $z > 0$. The first term in the potential is a uniform line sink situated on the axis of the cylinder; the strength of the sink may vary with time. As $\partial\phi/\partial z$ is independent of r and θ, a horizontal circular disc of radius a, free to move with the fluid, would not interfere with the flow. A solution for outflow from a cylinder evacuated along its axis, with a lid descending on top of the fluid, has thus been obtained.

The velocity components are

$$v_r = -m(a^2 - r^2)/(a^2 r), \quad v_\theta = 0,$$
$$v_z = -2mz/a^2. \quad (2)$$

If the lid is initially at a height Z_0 and is at $z = Z$ at time t, then from (2)

$$\frac{dZ}{dt} = -2mZ/a^2, \quad (3)$$

giving
$$\frac{Z}{Z_0} = \exp\left\{-\frac{2}{a^2}\int_0^t m(t)\,dt\right\}. \quad (4)$$

The $m(t)$ in this solution is arbitrary and may be chosen by an outflow condition. In particular, if the outflow from the cylinder is taken as constant, then $m(t)Z(t) = m_0 Z_0$, m_0 being the initial value of m. For a constant outflow, (3) gives

$$Z = Z_0\left(1 - \frac{2m_0 t}{a^2}\right), \quad m(t) = m_0\left(1 - \frac{2m_0 t}{a^2}\right)^{-1},$$

the outflow ending at $t = a^2/(2m_0)$.

The simplicity of the flow described in this section makes it a suitable model for consideration of the effect of Coriolis force on outflow.

3. Effect of Coriolis force on the outflow

The inclusion of a Coriolis force term/unit mass of $2\mathbf{v} \wedge \mathbf{\Omega}$ in the equation of motion of an ideal incompressible fluid, leads to the modified Helmholtz equation

$$\frac{D\boldsymbol{\omega}^*}{Dt} = (\boldsymbol{\omega}^* \cdot \nabla)\mathbf{v} \quad (5)$$

for $\boldsymbol{\omega}^* = \boldsymbol{\omega} + 2\mathbf{\Omega}$, where $\boldsymbol{\omega}$ is the vorticity and $\mathbf{\Omega}$ the earth's angular velocity. In cylindrical polar co-ordinates (r, θ, z) with the z-axis vertically upwards and the half-plane $\theta = 0$ containing the $\mathbf{\Omega}$ vector,

$$\mathbf{\Omega} = \Omega(\cos\alpha\cos\theta, \ -\cos\alpha\sin\theta, \ \sin\alpha)$$

at a latitude α. Variations in α throughout the fluid are unimportant in the present problem and α is taken as constant.

We first note an exact solution of eq. (5) for the case $\alpha = \frac{1}{2}\pi$. In this special case Coriolis force leaves the r and z components of velocity in the irrotational flow (2) unaltered and builds up an azimuthal velocity. The exact solution of (5) is

$$\left.\begin{aligned} v_r &= -m(a^2 - r^2)/(a^2 r), \\ v_\theta &= \Omega\left(1 - \frac{Z}{Z_0}\right)(a^2 - r^2)/r, \\ v_z &= -2mz/a^2, \quad \boldsymbol{\omega}^* = 2\Omega\frac{Z}{Z_0}\mathbf{k}, \end{aligned}\right\} \quad (6)$$

where $\mathbf{k}$ is a unit vector in the z-direction. This solution is valid for any $m(t)$ and Z is given by (4). A circulation

$$K = 2\pi\Omega\left(1 - \frac{Z}{Z_0}\right)(a^2 - r^2) \quad (7)$$

is produced in a circuit distance r from the axis of the cylinder.

For the general case, $\alpha \neq \frac{1}{2}\pi$ a corresponding exact solution is not available in simple terms. However, the important result, analogous to eq. (7), may be easily deduced for large Rossby number, $R = m/(\Omega a^2)$ in the present context. The effect of Coriolis force is to introduce velocity perturbations of $O(R^{-1})$ to the velocity vector given by (2). For large R we may therefore assume that a circuit moving with the fluid moves in a manner determined by (2). A circuit initially at a distance r_0 from the axis will at time t be a circuit distance r from the axis, where r, r_0, t are related by the expression for v_r in (2). We have, by integrating this v_r expression

$$\int_0^t m(t)\,dt = \tfrac{1}{2}a^2 \log\frac{(a^2 - r^2)}{(a^2 - r_0^2)}. \quad (8)$$

In an inertial frame, in which the earth is rotating, the circulation remains unaltered and is seen to be initially $2\pi\Omega r_0^2 \sin\alpha$. At time t an amount $2\pi\Omega r^2 \sin\alpha$ of this circulation is due to the earth's rotation, leaving the circulation in the circuit of radius r observed in an earth frame as

$$\begin{aligned} K &= 2\pi\Omega\sin\alpha(r_0^2 - r^2) \\ &= 2\pi\Omega\sin\alpha\left(1 - \frac{Z}{Z_0}\right)(a^2 - r^2) \end{aligned} \quad (9)$$

by use of (8) and then (4). Although it is possible to obtain the velocity perturbations to $O(R^{-1})$ they are of secondary importance compared with the result (9) and the details are not presented.