Solitary waves in compressible, stratified fluids

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ABSTRACT

A solitary wave is found in a stratified, compressible fluid in a uniform gravity field. This wave depends for its existence on the compressibility of the medium no matter how small, although the speed of propagation is of the order of an internal gravity wave. The analytical discussion is carried out most fully for small compressibility. Another case, more appropriate for atmospheric problems, is solved by a numerical approach.

1. Introduction

The solitary water wave (LAMB, 1932; URSELL, 1953) has been the subject of study for a long time. It is a single crest on a water surface traveling at a speed close to that of a long wave. It has physical and mathematical interest, especially because its existence depends on non-hydrostatic pressures, no matter how small.

Solitary waves have been found in other fluid systems: at the interface of two fluids of different density (KEULEGAN, 1953; LONG, 1956); in a liquid with continuous density gradient in the vertical (PETERS & STOKER, 1959; LONG, 1965); and as a "Rossby wave" in a non-uniform westerly current on a β-earth (LONG, 1964; LARSEN, 1965).

A common feature of these waves is the dependence on small effects: small non-hydrostatic pressures in the water and two-fluid waves, small inertia effects of density variation in the continuous-density wave, and small shear in the solitary wave in the westerlies. Having found a solitary wave in a given case, we may then change the model slightly and still find the wave. For example, BENJAMIN (1962) derived the properties of the solitary wave on a water surface when there is a variable, instead of a uniform basic current. Similarly, it appears likely that a slightly compressible fluid with continuous *potential density* variation vertically has a solitary wave with properties differing only slightly from the solitary wave in a liquid with the same *density* distribution in the undisturbed state.

Our purpose here, however, is to show that there is another class of solitary waves in a compressible fluid of variable potential density, and that the existence of this wave depends utterly on the compressibility, no matter how slight.

2. Basic equations

The model for the investigation is shown in Fig. 1. There is steady flow in a channel of height h with a uniform gravity field along the vertical. At $|x'| = \infty$ the flow is horizontal with uniform speed u_0. The fluid is compressible with adiabatic exponent σ.

Let ϱ be the density, p be the pressure. Then

$$\frac{p}{\varrho^{\sigma}} = \frac{\bar{p}}{\bar{\varrho}^{\sigma}}, \tag{1}$$

where $\bar{\varrho}$, called the *potential density*, is the density of the parcel if its pressure is increased adiabatically from p to $\bar{p}$. If there is no gain or loss of heat, $\bar{\varrho}$ is a conservative property. Notice that $\bar{p}$ is a constant which we may assign arbitrarily.

With the assumption of steady, two-dimensional motion, and with the definitions

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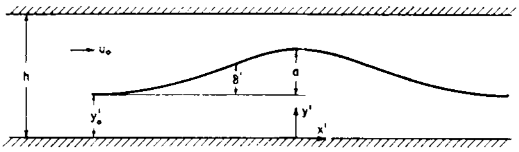

FIG. 1. Model of a solitary wave.

$$P' = \frac{\sigma}{\sigma - 1}\bar{p}^{1/\sigma} p^{1-1/\sigma}, \tag{2}$$

$$\varepsilon = \frac{1}{\sigma - 1} \tag{3}$$

our equations become

$$\bar{\varrho}\left(u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'}\right) = -\frac{\partial P'}{\partial x'}, \tag{4}$$

$$\bar{\varrho}\left(u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'}\right) = -\frac{\partial P'}{\partial y'} - \bar{\varrho}g, \tag{5}$$

$$P'^{\varepsilon}\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) + u'\frac{\partial P'^{\varepsilon}}{\partial x'} + v'\frac{\partial P'^{\varepsilon}}{\partial y'} = 0, \tag{6}$$

$$u'\frac{\partial \bar{\varrho}}{\partial x'} + v'\frac{\partial \bar{\varrho}}{\partial y'} = 0. \tag{7}$$

Let us define

$$P = \frac{P'}{\varrho_1 g h}, \tag{8}$$

where ϱ_1 is a given value of $\bar{\varrho}$ to be specified later. Then eq. (6) implies the existence of a scalar function χ', such that

$$u' P^{\varepsilon} = -\frac{\partial \chi'}{\partial y'}, \tag{9}$$

$$v' P^{\varepsilon} = \frac{\partial \chi'}{\partial x'}. \tag{10}$$

Notice from eq. (7) that $\bar{\varrho} = \bar{\varrho}(\chi')$.

Let us now cross-differentiate (4) and (5) to eliminate P'. With use of

$$\zeta' = \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} = \frac{1}{P^{\varepsilon}}\left(\frac{\partial^2 \chi'}{\partial x'^2} + \frac{\partial^2 \chi'}{\partial y'^2}\right) - \frac{1}{P^{2\varepsilon}}\left(\frac{\partial P^{\varepsilon}}{\partial x'}\frac{\partial \chi'}{\partial x'} + \frac{\partial P^{\varepsilon}}{\partial y'}\frac{\partial \chi'}{\partial y'}\right), \tag{11}$$

$$q'^2 = u'^2 + v'^2 = \frac{1}{P^{2\varepsilon}}\left[\left(\frac{\partial \chi'}{\partial x'}\right)^2 + \left(\frac{\partial \chi'}{\partial y'}\right)^2\right] \tag{12}$$

and (9) and (10), we get

$$\frac{d\bar{\varrho}}{d\chi'}\left[v' P^{\varepsilon}\frac{dv'}{dt'} + u' P^{\varepsilon}\frac{du'}{dt'} + g v' P^{\varepsilon}\right] + \bar{\varrho}\frac{d\zeta'}{dt'} + \bar{\varrho}\left[\frac{\partial u'}{\partial x'}\frac{\partial v'}{\partial x'} + \frac{\partial v'}{\partial x'}\frac{\partial v'}{\partial y'} - \frac{\partial u'}{\partial y'}\frac{\partial u'}{\partial x'} - \frac{\partial v'}{\partial y'}\frac{\partial u'}{\partial y'}\right] = 0, \tag{13}$$

where $$\frac{d}{dt'} = u'\frac{\partial}{\partial x'} + v'\frac{\partial}{\partial y'}; \tag{14}$$

eq. (13) may be written

$$\frac{d}{dt'}(\zeta'\bar{\varrho}) + P^{\varepsilon}\frac{d\bar{\varrho}}{d\chi'}\frac{d}{dt'}\left(\frac{q'^2}{2} + gy'\right) - \frac{\bar{\varrho}\zeta'}{P^{\varepsilon}}\frac{dP^{\varepsilon}}{dt'} = 0 \tag{15}$$

or $$\frac{d}{dt'}\left[\frac{\zeta'}{P^{\varepsilon}} + \frac{1}{\bar{\varrho}}\frac{d\bar{\varrho}}{d\chi'}\left(\frac{q'^2}{2} + gy'\right)\right] = 0 \tag{16}$$

whence $$\frac{\zeta'}{P^{\varepsilon}} + \frac{1}{\bar{\varrho}}\frac{d\bar{\varrho}}{d\chi'}\left(\frac{q'^2}{2} + gy'\right) = H(\chi'), \tag{17}$$

where $H(\chi')$ is an arbitrary function.

Another integral of the equations of motion may be obtained by multiplying (4) and (5) by u' and v' and adding. The result is

$$P + \frac{\bar{\varrho}}{\varrho_1 g h}\left(\frac{q'^2}{2} + gy'\right) = L(\chi'), \tag{18}$$

where $L(\chi')$ is also arbitrary.

Let us now introduce the quantity y'_0, which is the height far upstream or downstream of the curve $\chi' =$ constant (Fig. 1). Equation (9) yields

$$u_0 P_0^{\varepsilon} = -\frac{d\chi'}{dy'_0}, \tag{19}$$

where $P_0(y'_0)$ is the value of P on the curve $\chi' =$ constant far upstream.

If we now introduce

$$\eta' = P - P_0, \quad \delta' = y' - y'_0 \tag{20}$$

eqs. (17) and (18) may be written

$$\frac{u_0 P_0^\varepsilon}{(\eta'+P_0)^{2\varepsilon}}\left(\frac{\partial^2\delta'}{\partial x'^2}+\frac{\partial^2\delta'}{\partial y'^2}\right)-\frac{1}{(\eta'+P_0)^{2\varepsilon}}\frac{d}{dy_0'}(u_0P_0^\varepsilon)$$
$$\times\left[1-2\frac{\partial\delta'}{\partial y'}+\left(\frac{\partial\delta'}{\partial x'}\right)^2+\left(\frac{\partial\delta'}{\partial y'}\right)^2\right]+\frac{\varepsilon u_0 P_0^\varepsilon}{(\eta'+P_0)^{2\varepsilon+1}}$$
$$\times\left[\frac{\partial\eta'}{\partial y'}-\frac{\partial\delta'}{\partial x'}\frac{\partial\eta'}{\partial x'}-\frac{\partial\delta'}{\partial y'}\frac{\partial\eta'}{\partial y'}+\frac{dP_0}{dy_0'}\right.$$
$$\left.\times\left(1-2\frac{\partial\delta'}{\partial y'}+\frac{\partial\delta'}{\partial x'}\frac{\partial\delta'}{\partial x'}+\frac{\partial\delta'}{\partial y'}\frac{\partial\delta'}{\partial y'}\right)\right]-\frac{1}{\bar\varrho u_0 P_0{}^\varepsilon}\frac{d\bar\varrho}{dy_0'}$$
$$\times\left\{\frac{u_0^2P_0^{2\varepsilon}}{2(\eta'+P_0)^{2\varepsilon}}\left[1-2\frac{\partial\delta'}{\partial y'}+\left(\frac{\partial\delta'}{\partial x'}\right)^2+\left(\frac{\partial\delta'}{\partial y'}\right)^2\right]\right.$$
$$\left.+g\delta'-\frac{u_0^2}{2}\right\}=0, \tag{21}$$

$$\eta'+\frac{\bar\varrho}{\varrho_1 gh}\left\{\frac{u_0^2P_0^{2\varepsilon}}{2(\eta'+P_0)^{2\varepsilon}}\left[1-2\frac{\partial\delta'}{\partial y'}+\left(\frac{\partial\delta'}{\partial x'}\right)^2\right.\right.$$
$$\left.\left.+\left(\frac{\partial\delta'}{\partial y'}\right)^2\right]+g\delta'-\frac{u_0^2}{2}\right\}=0, \tag{22}$$

where we have evaluated H and L as

$$H=-\frac{1}{u_0P_0^\varepsilon\bar\varrho}\frac{d\bar\varrho}{dy_0'}\left(\frac{u_0^2}{2}+gy_0'\right), \tag{23}$$

$$L=P_0+\frac{\bar\varrho}{\varrho_1 gh}\left(\frac{u_0^2}{2}+gy_0'\right). \tag{24}$$

Let us now adopt a model in which the potential density decreases slowly and linearly with height in the undisturbed state, i.e.,

$$\bar\varrho=\varrho_1(1-by_0'). \tag{25}$$

Let a be characteristic of the wave amplitude and λ be characteristic of the wave length (they will be specified more precisely later). We now introduce non-dimensional quantities

$$\delta=\frac{\delta'}{a},\quad x=\frac{x'}{\lambda},\quad y=\frac{y'}{h},\quad \eta=\frac{\eta'}{\alpha},$$
$$k^2=\frac{g\beta h}{u_0^2},\quad \beta=bh,\quad \gamma=\frac{h^2}{\lambda^2},\quad \alpha=\frac{a}{h}. \tag{26}$$

Obviously the function $P_0(y_0)$ may now be determined. If we integrate eq. (5) far upstream, we get

$$P_0=-\left(y_0-\frac{\beta y_0^2}{2}\right)+\pi, \tag{27}$$

where π is the value of P_0 upstream at $y_0=0$. It is a considerable simplification if we now choose $\bar p$ to be

$$\bar p^{1/\sigma}\equiv\frac{\varrho_1 gh p_{00}^{1/\sigma-1}}{1+\varepsilon}, \tag{28}$$

where p_{00} is the *pressure* upstream at $y=0$. Then from eq. (2)

$$\varrho_1 gh\pi=\frac{\sigma}{\sigma-1}p_{00}^{1-1/\sigma}\frac{\varrho_1 gh}{1+\varepsilon}p_{00}^{1/\sigma-1} \tag{29}$$

so that $\pi=1$ and

$$P_0=-\left(y_0-\frac{\beta y_0^2}{2}\right)+1. \tag{30}$$

Equations (21) and (22) now become

$$\frac{P_0^\varepsilon}{(\eta\alpha+P_0)^{2\varepsilon}}\left[\alpha\frac{\partial^2\delta}{\partial y^2}+\alpha\gamma\frac{\partial^2\delta}{\partial x^2}\right]-\frac{1}{(\eta\alpha+P_0)^{2\varepsilon}}\frac{dP_0^\varepsilon}{dy_0}$$
$$\times\left[1-2\alpha\frac{\partial\delta}{\partial y}+\alpha^2\left(\frac{\partial\delta}{\partial y}\right)^2+\alpha^2\gamma\left(\frac{\partial\delta}{\partial x}\right)^2\right]$$
$$+\frac{\varepsilon P_0^\varepsilon}{(\eta\alpha+P_0)^{2\varepsilon+1}}\left\{\alpha\frac{\partial\eta}{\partial y}-\alpha^2\frac{\partial\delta}{\partial y}\frac{\partial\eta}{\partial y}-\alpha^2\gamma\frac{\partial\delta}{\partial x}\frac{\partial\eta}{\partial x}\right.$$
$$\left.+\frac{dP_0}{dy_0}\left[1-2\alpha\frac{\partial\delta}{\partial y}+\alpha^2\left(\frac{\partial\delta}{\partial y}\right)^2+\alpha^2\gamma\left(\frac{\partial\delta}{\partial x}\right)^2\right]\right\}$$
$$+\frac{\beta}{(1-\beta y_0)P_0^\varepsilon}\left\{\frac{P_0^{2\varepsilon}}{2(\alpha\eta+P_0)^{2\varepsilon}}\left[1-2\alpha\frac{\partial\delta}{\partial y}+\alpha^2\left(\frac{\partial\delta}{\partial y}\right)^2\right.\right.$$
$$\left.\left.+\alpha^2\gamma\left(\frac{\partial\delta}{\partial x}\right)^2\right]+\frac{\alpha k^2\delta}{\beta}-\frac{1}{2}\right\}=0. \tag{31}$$

$$k^2\alpha\eta+\beta(1-\beta y_0)\left\{\frac{P_0^{2\varepsilon}}{2(\alpha\eta+P_0)^{2\varepsilon}}\left[1-2\alpha\frac{\partial\delta}{\partial y}+\alpha^2\left(\frac{\partial\delta}{\partial y}\right)^2\right.\right.$$
$$\left.\left.+\alpha^2\gamma\left(\frac{\partial\delta}{\partial x}\right)^2\right]+\frac{\alpha k^2\delta}{\beta}-\frac{1}{2}\right\}=0. \tag{32}$$

These equations may be simplified to yield a single equation in δ if, as a special case, we assume that the quantity β in these two equations is negligibly small, i.e., if we make the Boussinesq approximation (Boussinesq, 1903). As shown by Long (1965), this will eliminate certain classes of solitary waves, but it does not affect the basic nature of the wave discussed in this paper. We get $\eta=-\delta$ and

$$\frac{1}{(1-y)^{2\varepsilon}}\left[\alpha\frac{\partial^2\delta}{\partial y^2}+\alpha\gamma\frac{\partial^2\delta}{\partial x^2}\right]+\frac{\varepsilon}{(1-y)^{2\varepsilon}(1-y+\alpha\delta)}\times\left[1-2\alpha\frac{\partial\delta}{\partial y}+\alpha^2\left(\frac{\partial\delta}{\partial y}\right)^2+\alpha^2\gamma\left(\frac{\partial\delta}{\partial x}\right)^2\right]+\frac{\varepsilon}{(1-y)^{2\varepsilon+1}}\left[+\alpha\frac{\partial\delta}{\partial y}-1\right]+\frac{\alpha k^2\delta}{(1-y+\alpha\delta)^{2\varepsilon}}=0 \tag{33}$$

or

$$\frac{\partial^2\delta}{\partial y^2}+\gamma\frac{\partial^2\delta}{\partial x^2}+\frac{k^2(1-y)^{2\varepsilon}}{(1-y+\alpha\delta)^{2\varepsilon}}\delta+\frac{\varepsilon}{(1-y)(1-y+\alpha\delta)}\times\left[-\delta-(1-y)\frac{\partial\delta}{\partial y}+\alpha\delta\frac{\partial\delta}{\partial y}+\alpha(1-y)\left(\frac{\partial\delta}{\partial y}\right)^2+\alpha\gamma(1-y)\left(\frac{\partial\delta}{\partial x}\right)^2\right]=0. \tag{34}$$

3. Slight compressibility

The simplest approach to the wave under investigation in this paper is to assume that the fluid is only slightly compressible, i.e., that ε is small. We assume also that $\gamma=\alpha\varepsilon$ and expand the solution of (34) in a power series in α and ε, i.e.,

$$\delta=\delta_{00}+\alpha\delta_{10}+\varepsilon\delta_{01}+\alpha\varepsilon\delta_{11}+\ldots, \tag{35}$$

$$k^2=k_{00}^2+\alpha k_{10}^2+\varepsilon k_{01}^2+\alpha\varepsilon k_{11}^2+\ldots \tag{36}$$

In the next section we will discuss a case, $\varepsilon=2$, close to that of an atmospheric solitary wave.

Equation (34) yields for the lowest approximation

$$\frac{\partial^2\delta_{00}}{\partial y^2}+k_{00}^2\delta_{00}=0. \tag{37}$$

A solution which vanishes at the rigid, horizontal boundaries $y=0$, $y=1$ is

$$\delta_{00}=f_{00}\sin n\pi y, \tag{38}$$

$$k_{00}^2=n^2\pi^2, \tag{39}$$

where f_{00} is an arbitrary function of x.

The next approximation is

$$\frac{\partial^2\delta_{01}}{\partial y^2}+n^2\pi^2\delta_{01}=-k_{01}^2f_{00}\sin n\pi y+\frac{f_{00}n\pi\cos n\pi y}{1-y}+\frac{f_{00}\sin n\pi y}{(1-y)^2}; \tag{40}$$

hence

$$\delta_{01}=f_{01}\sin n\pi y+\frac{k_{01}^2}{2n\pi}f_{00}y\cos n\pi y+f_{00}K_n(y), \tag{41}$$

where f_{01} is an arbitrary function of x, and

$$K_n(y)=\frac{\sin n\pi y}{n\pi}\int_0^y\left[\frac{\sin n\pi y^*}{(1-y^*)^2}+\frac{n\pi\cos n\pi y^*}{(1-y^*)}\right]\times\cos n\pi y^*\,dy^*-\frac{\cos n\pi y}{n\pi}\int_0^y\left[\frac{\sin n\pi y^*}{(1-y^*)^2}+\frac{n\pi\cos n\pi y^*}{(1-y^*)}\right]\sin n\pi y^*. \tag{42}$$

Notice that $K_n(0)=0$, and that the solution for δ_{01} satisfies the kinematic condition at the bottom. The kinematic condition at the top requires that

$$k_{01}^2=-2n\pi(-1)^nK_n(1). \tag{43}$$

This result is shown for a few values of n in Table 1.

The next approximation is

$$\frac{\partial^2\delta_{10}}{\partial y^2}+n^2\pi^2\delta_{10}=-k_{10}^2f_{00}\sin n\pi y. \tag{44}$$

The only solution satisfying conditions at top and bottom is

$$\delta_{10}=f_{10}\sin n\pi y, \tag{45}$$

$$k_{10}^2=0. \tag{46}$$

Finally

$$\frac{\partial^2\delta_{11}}{\partial y^2}+n^2\pi^2\delta_{11}=-(k_{11}^2f_{00}+f_{00}'')\sin n\pi y-k_{01}^2f_{10}\times\sin n\pi y+\frac{2n^2\pi^2}{(1-y)}f_{00}^2\sin^2 n\pi y-\frac{f_{00}^2n^2\pi^2\cos^2 n\pi y}{1-y}+\frac{f_{10}n\pi\cos n\pi y}{1-y}-\frac{2n\pi f_{00}^2\cos n\pi y\sin n\pi y}{(1-y)^2}-\frac{f_{00}^2\sin^2 n\pi y}{(1-y)^3}+\frac{f_{10}\sin n\pi y}{(1-y)^2}, \tag{47}$$

where f_{00}'' is the derivative of f_{00} with respect to x.

The solution is

TABLE 1

n	k_{01}^2
1	+ 4.4553
2	− 9.3755
3	+ 14.3072
4	− 19.2410
5	+ 24.1755

$$\delta_{11} = f_{11} \sin n\pi y + f_{10} K_n(y) + \frac{k_{01}^2}{2n\pi} f_{10} y \cos n\pi y + \frac{(f_{00}'' + k_{11}^2 f_{00})}{2n\pi} y \cos n\pi y + P_n(y) f_{00}^2, \quad (48)$$

where

$$P_n(y) = \frac{\sin n\pi y}{n\pi} \int_0^y M(y^*) \cos n\pi y^* dy^* - \frac{\cos n\pi y}{n\pi} \int_0^y M(y^*) \sin n\pi y^* dy^*, \quad (49)$$

where

$$M(y) = \frac{3n^2\pi^2 \sin^2 n\pi y - n^2\pi^2}{(1-y)} - \frac{2n\pi \cos n\pi y \sin n\pi y}{(1-y)^2} - \frac{\sin^2 n\pi y}{(1-y)^3}.$$

The solution (48) vanishes at the bottom of the channel. The condition that it vanish at the top yields

$$f_{00}'' + k_{11}^2 f_{00} + 2n\pi(-1)^n P_n(1) f_{00}^2 = 0. \quad (50)$$

This has a first integral

$$f_{00}'^2 + k_{11}^2 f_{00}^2 + \tfrac{4}{3} n\pi(-1)^n P_n(1) f_{00}^3 = 0. \quad (51)$$

Notice that if we let the amplitude factor a in (26) represent the maximum value of the vertical displacement of the potential density surfaces, i.e.,

$$a = |\delta'|_{\max} \quad (52)$$

we may then set $f_{00} = \pm 1$ at the crest, $x = 0$. Since f_{00}' vanishes there, (51) yields

$$k_{11}^2 = \mp \tfrac{4}{3} n\pi(-1)^n P_n(1). \quad (53)$$

The solution of (51) is

$$f_{00} = \pm \operatorname{sech}^2 [x\sqrt{\pm \tfrac{1}{3} n\pi(-1)^n P_n(1)}]. \quad (54)$$

We may show that $P_n(1)$ may be reduced to

$$P_n(1) = -\frac{3n\pi(-1)^n}{2} \int_0^1 \frac{\sin^3 n\pi y^*}{(1-y^*)} dy^* \quad (55)$$

or

$$P_n(1) = \tfrac{3}{2} n\pi \int_0^{n\pi} \frac{\sin^3 z dz}{z}. \quad (56)$$

Clearly this is positive for all n, so that

$$\delta_{00} = -\operatorname{sech}^2 \left[x\sqrt{\frac{n\pi}{3} P_n(1)}\right] \sin n\pi y, \quad n \text{ odd}, \quad (57)$$

$$\delta_{00} = \operatorname{sech}^2 \left[x\sqrt{\frac{n\pi}{3} P_n(1)}\right] \sin n\pi y, \quad n \text{ even}. \quad (58)$$

Table 2 contains computed values of $P_n(1)$ and k_{11}^2 for a number of values of n.

Notice that the wave length is inversely proportional to $(\alpha\varepsilon)^{\frac{1}{2}}$. It becomes longer and longer as the amplitude decreases and longer and longer as the compressibility decreases. The wave ceases to exist if the fluid is considered to be incompressible. To the first order the speed of propagation is

$$c_{00} = \frac{1}{n\pi} \sqrt{g\beta h}$$

so that it moves at a speed close to that of an internal long wave rather than the speed of a sound wave.

4. $\sigma = 3/2$

Successive approximations to the solution of (34) are expressible in terms of elementary functions (rather than Bessel functions) in the

TABLE 2

n	$P_n(1)$	k_{11}^2
1	4.5723	− 19.1524
2	6.4475	− 54.0156
3	12.0809	− 151.8132
4	13.8176	− 231.5163
5	19.4969	− 408.3427

TABLE 3

n	k_0
1	4.4934
2	7.7252
3	10.9041
4	14.0662
5	17.2227

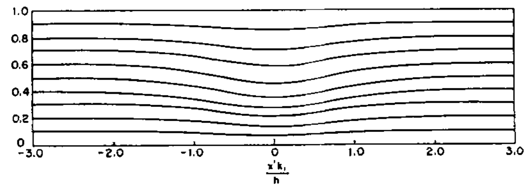

FIG. 2. Solitary wave in compressible fluid, $\sigma = 1.50$, $k_0 = 4.4934$.

case $\sigma = 3/2$ or $\varepsilon = 2$. Moreover, this is close to the value of $\sigma = 1.41$ for the atmosphere of the earth. If we set $\gamma = \alpha$ in this case and expand, we get

$$\delta = \delta_0 + \alpha\delta_1 + \dots, \tag{59}$$

$$k^2 = k_0^2 + \alpha k_1^2 + \dots. \tag{60}$$

Equation (34) yields for the first two approximations

$$\frac{\partial^2 \delta_0}{\partial y^2} - \frac{2}{(1-y)}\frac{\partial \delta_0}{\partial y} + \delta_0\left[k_0^2 - \frac{2}{(1-y)^2}\right] = 0 \tag{61}$$

and

$$\frac{\partial^2 \delta_1}{\partial y^2} - \frac{2}{(1-y)}\frac{\partial \delta_1}{\partial y} + \delta_1\left[k_0^2 - \frac{2}{(1-y)^2}\right] = -\frac{\partial^2 \delta_0}{\partial x^2} - k_1^2\delta_0 + \frac{4k_0^2\delta_0^2}{(1-y)} - \frac{4\delta_0}{(1-y)^2}\frac{\partial \delta_0}{\partial y} - \frac{2}{(1-y)}\left(\frac{\partial \delta_0}{\partial y}\right)^2 - \frac{2\delta_0^2}{(1-y)^3}. \tag{62}$$

The solution of (61), satisfying conditions at top and bottom, is

$$\delta_0 = f_0(x)\left\{\frac{\sin k_0(y-1)}{k_0^2(y-1)^2} - \frac{\cos k_0(y-1)}{k_0(y-1)}\right\}, \tag{63}$$

where $f_0(x)$ is arbitrary, provided

$$\tan k_0 = k_0. \tag{64}$$

The roots are shown in Table 3. These waves travel slower than those in the case of slight compressibility.

In terms of the function

$$g = -\frac{\sin k_0(y-1)}{k_0^2(y-1)} \tag{65}$$

eq. (62) may be written

$$\frac{\partial^2 \delta_1}{\partial y^2} + \frac{2}{y-1}\frac{\partial \delta_1}{\partial y} + \delta_1\left[k_0^2 - \frac{2}{(y-1)^2}\right] = -g'(f_0'' + k_1^2 f_0) + G f_0^2, \tag{66}$$

where

$$G = \frac{2k_0^4 g^2}{(y-1)} + \frac{12k_0^2 g g'}{(y-1)^2} + \frac{18 g'^2}{(y-1)^3} - \frac{4k_0^2 g'^2}{(y-1)}. \tag{67}$$

The solution of (66) satisfying the kinematic condition at the top[1] is

$$\delta_1 = f_1 g'(y) + Q(y)(f_0'' + k_1^2 f_0) + R(y) f_0^2, \tag{68}$$

where f_1 is an arbitrary function of x and

$$Q(y) = -g'(y)k_0 \int_1^y g'(y^*)h'(y^*)(y^*-1)^2 dy^* + h'(y)k_0\int_1^y g'^2(y^*)(y^*-1)^2 dy^*, \tag{69}$$

$$R(y) = g'(y)k_0\int_1^y G(y^*)h'(y^*)(y^*-1)^2 dy^* - h'(y)k_0\int_1^y G(y^*)g'(y^*)(y^*-1)^2 dy^* \tag{70}$$

and

$$h = -\frac{\cos k_0(y-1)}{k_0^2(y-1)}. \tag{71}$$

The kinematic condition at the bottom requires that

$$Q(0)\{f_0'' + k_1^2 f_0\} + R(0) f_0^2 = 0. \tag{72}$$

The first integral of eq. (72) is

$$Q(0)\{f_0'^2 + k_1^2 f_0^2\} + \tfrac{2}{3}R(0) f_0^3 = 0. \tag{73}$$

[1] Certain quantities such as h' are singular at $y = 1$, but it is easily verified that δ_1 as expressed in (68) vanishes at $y = 1$.

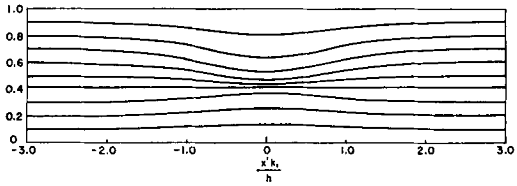

FIG. 3. Solitary wave in compressible fluid, $\sigma = 1.50$, $k_0 = 7.7252$.

We again set $f_0 = \pm 1$ at the crest, $x = 0$. Since $f_0' = 0$ there,

$$k_1^2 = \mp \frac{2}{3} \frac{R(0)}{Q(0)}.$$

The values of k_1^2 are shown in Table 4.

The solution of eq. (73) is

$$f_0 = \pm \operatorname{sech}^2 \left\{ x \sqrt{\pm \frac{1}{6} \frac{R(0)}{Q(0)}} \right\}. \tag{74}$$

TABLE 4

n	k_1^2
1	-27.3760
2	-68.9320
3	-143.8579
4	-231.6480
5	-352.5647

Since $R(0)/Q(0)$ is always positive, we must choose the upper sign in eq. (74) so that

$$f_0 = \operatorname{sech}^2 \left\{ x \sqrt{\frac{1}{6} \frac{R(0)}{Q(0)}} \right\}. \tag{75}$$

The wave is shown for the first two values of k_0 in Figs. 2 and 3.

Notice that these waves move at a speed close to that of long internal gravity waves in an incompressible medium. As a consequence this type of solitary wave may have meteorological importance.

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