The concentration of liquid water in the atmosphere

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ABSTRACT

The liquid water content of the atmosphere is studied on the basis of the differential equations of eddy exchange of total heat content (sensible plus latent heat) and total water content (vapour plus liquid water in form of droplets). We arrive at an equation of liquid water content which permits quantitative comparisons between different atmospheric processes of liquid water concentration, some of these being well known —for instance the adiabatic cooling, and the radiative cooling—others being mostly disregarded—such as the turbulent mixing and the vertical variation of the intensity of eddy exchange. We obtain support for the theory, as it relates to the distinction between the effects of mixing and radiation, by means of statistics of fogs. Finally, a classification of fogs is shown on the basis of our conclusions.

Introduction

The discussion of cloud and fog formation in text-books of meteorology is mostly focussed upon the condensation of vapour into drops and upon the role played by condensation nuclei. Much less interest is paid to the equally important question of how vapour attains the saturation pressure, and how the concentration of liquid water found in fogs and clouds is produced. As regards clouds, the prevalent idea about these refers to the effect of adiabatic cooling associated with convection and large scale upward motion. No doubt that is the way of growth of for instance cumulus. In addition we note that the vertical variation of the coefficient of eddy exchange, $\delta K/\delta z$, has the dimension of velocity. From the theoretical point of view this term ranks equal with the mean vertical motion and is effective in stratus as well as fogs. The eddy exchange is concerned with the irregular motions of turbulence. And, to the above-mentioned procedures we add the increase of liquid water content that happens when eddies of different temperatures are mixed up (TAYLOR, 1917; BRUNT, 1935).

Paying my attention to fog formation, I in 1962 presented the idea of the fundamental role played by turbulent mixing in a simple mathematical way. The effect of vertical pressure distribution was disregarded and so the relationship between clouds and fogs did not stand out. The theory is outlined in the following sections, and there indeed appear some progressive aspects on the effect of turbulence in clouds as well as fogs. The magnitude of the effect corresponds with those of convection, radiative cooling as well as fall-out of precipitation.

The equation of eddy exchange

The effects of atmospheric turbulence are mathematically summarised by the differential equation of eddy exchange. Applied to the heat content h per unit mass of dry air, it reads

$$\frac{d\bar{h}}{dt} = \frac{\delta}{\delta z}\left(K_h \frac{\delta \bar{h}}{\delta z}\right) + Q. \tag{1}$$

The last term of the equation, Q, refers to the net gain (positive) or net loss of heat of radiation in a volume containing a unit mass of the dry-air component.

The physical basis of the equation is that there are short-range fluctuations h' of heat content,

$$h = \bar{h} + h', \tag{2}$$

and that the mean value $\bar{h}$ involved in the equation covers a period long enough for numerous fluctuations to occur. The fluctuations appear in the coefficient of eddy exchange which is a product moment of the fluctuations of the vertical velocity component w' and the mixing length l_h,

$$K_h = -\overline{w'l_h}, \; h' = +l_h \frac{\delta \bar{h}}{\delta z}. \qquad (3)$$

The mean of the vertical velocity component is found in the derivative of the left part of equation (1), which we may rewrite in such a form that the terms of horizontal advection and vertical displacements are distinguished,

$$\frac{\delta \bar{h}}{\delta t} - K_h \frac{\delta^2 \bar{h}}{\delta z^2} = -\left[\bar{u}\frac{\delta \bar{h}}{\delta x} + \bar{v}\frac{\delta \bar{h}}{\delta y}\right] - \frac{\delta \bar{h}}{\delta z}\left[\bar{w} - \frac{\delta K_h}{\delta z}\right] + Q. \qquad (4)$$

The horizontal x- and y-components of velocity are signed u and v. They enter in (4) as mean values over the period concerned. Their fluctuations u' and v' should be included in (4) in terms of the corresponding eddy coefficients K_{hx} and K_{hy} if we were not convinced that the effect of the vertical gradient of heat content far exceeds the horizontal one. Mostly we are —but apropos (14) we shall later review the horizontal gradients and find conditions when we might expect them having an apparent effect.

We find in (4) the mean vertical velocity component and the gradient $-\delta K_h/\delta z$ within the same brackets and consequently equally effective on the vertical distribution of heat content. For instance, the upward increase of stratification at the base of an inversion results in cooling in the same way as upward motion in terms of $\bar{w}$.

Applied to the total water content r – i.e. vapour or vapour plus droplets—per unit mass of the dry-air component (in the case of unsaturated air r is called the mixing ratio of vapour and dry air), the differential equation of eddy exchange reads in the same form as (4) as follows,

$$\frac{\delta \bar{r}}{\delta t} - K_r \frac{\delta^2 \bar{r}}{\delta z^2} = -\left[\bar{u}\frac{\delta \bar{r}}{\delta x} + \bar{v}\frac{\delta \bar{r}}{\delta y}\right] - \frac{\delta \bar{r}}{\delta z}\left[\bar{w} - \frac{\delta K_r}{\delta z}\right] + P. \qquad (5)$$

Distinction is here made between the coefficient of heat exchange in (4) and that of water exchange in (5). The term P refers to the fall-out of precipitation and is negative.

There are limitations imposed upon the amount of any property entered like $\bar{h}$ and $\bar{r}$ into the equation of eddy exchange. The quantity must not represent a property whose measures are changed by delay or by transportation to a new level. Thus, the contents of sensible heat and latent heat each satisfy these conditions *in unsaturated air*, but so does only the total content of heat—sensible plus latent—in clouds and fogs as well as *in saturated air*. The same rule is valid to the contents of vapour and liquid water in form of droplets. None of these components alone but well the total content of water—vapour plus liquid water—are retained constant in a closed volume of fog or cloud being subject to vertical transportations.

The total heat content

Following NORMAND's *Proposition I* (BRUNT, 1944) we get the total heat content in terms of the wet-bulb temperature: "The heat content of the air is equal to the heat content of the same air saturated at the wet-bulb temperature, *minus* the heat content of the additional water required so to saturate it."

In mathematical form the proposition reads,

$$h(T,p,r) = h(T_v,p) - c(r_v - r)\,T_v. \qquad (6)$$

T and T_v refer to the dry-bulb and wet-bulb temperatures, p to the atmospheric partial pressure of dry air, r_v to the saturation mixing ratio of vapour and the dry-air component, and c to the specific heat of liquid water. In unsaturated air, r is smaller than r_v and T is greater than T_v. But at supersaturation as well as in clouds and fogs, r exceeds r_v. At supersaturation, T is slightly less than T_v. In clouds and fogs, we might put T_v equal to T. We disregard the effects of the curvatures and the salt-nuclei contents of the droplets upon vapour pressure. So the total heat content of clouds and fogs per unit mass of the dry-air component reads,

$$h(T,p,r) = h(T_v,p) + c(r - r_v)\,T_v. \qquad (7)$$

The equations (6) and (7) are identical. The mathematical formulation of total heat content is thus the same when the air is unsaturated, supersaturated or contains droplets of clouds and fogs.

The liquid water content

In clouds and fogs, the difference $(r - r_v)$ in (7) represents the content of liquid water in form of droplets. Under unsaturated conditions (6), the same difference is found but negative instead of positive, and then it indicates the amount of liquid water required to saturate the air in the psychrometer process. We put for the content of liquid water per unit mass of the dry-air component the symbol r_w,

$$r = r_v + r_w. \tag{8}$$

By definition, r_w is positive in clouds and fogs as well as in supersaturated air but negative in unsaturated air.

The saturation mixing ratio of vapour and dry air depends only upon the wet-bulb temperature and the partial pressure of dry air,

$$r_v = r_v(T_v, p). \tag{9}$$

The sensible heat content of r_w—that is the last term in (7)—is a relatively small part of the total heat content of the air, about 0.01 to 0.05%. If we drop the term there remains

$$h = h(T_v, p), \tag{10}$$

which is a good and reliable approximation of (7).

Thus r_v and h depend on the same variables, and the liquid water content in (8) is a function of total water content and total heat content,

$$r_w = r - r_v(h). \tag{11}$$

In an appendix to this paper, the differential relations between r_v and h are discussed.

The eddy exchange of liquid water content

Now, we replace $\bar{r}$ in the equation of eddy exchange (5) by $\bar{r}_w$ and $\bar{r}_v$ in using (11). We arrive at derivatives of $\bar{r}_v$ which we replace by derivatives of $\bar{h}$ found in part (*b*) of the appendix. In doing so we should refer the formulae (53), (54) and (55) to mean values of the variables and in the case of (55) be reminded that

$$\frac{\delta^2 \bar{r}_v}{\delta z^2} = \overline{\frac{\delta^2 r_v}{\delta z^2}} \tag{12}$$

and

$$\overline{\left(\frac{\delta T_v}{\delta z}\right)^2} = \left(\frac{\delta \bar{T}_v}{\delta z}\right)^2 + \overline{\left(\frac{\delta T_v'}{\delta z}\right)^2}, \tag{13}$$

where $T_v = \bar{T}_v + T_v'$ and $\overline{T_v'} = 0$. The term $\overline{(\delta T_v'/\delta z)^2}$ is the square of the standard deviation of $\delta T_v/\delta z$, $\delta \bar{T}_v/\delta z$ is the mean value.

Proceeding in this way, we obtain the following equation of eddy exchange of liquid water content,

$$\begin{aligned}\frac{\delta \bar{r}_w}{\delta t} - K_r \frac{\delta^2 \bar{r}_w}{\delta z^2} &= -\left(\bar{u}\frac{\delta \bar{r}_w}{\delta x} + \bar{v}\frac{\delta \bar{r}_w}{\delta y}\right) \\ &+ \left[\left(\frac{dr_w}{dz}\right)_{ad} - \frac{\delta \bar{r}_w}{\delta z}\right]\left[\bar{w} - \frac{\delta K_r}{\delta z}\right] \\ &+ K_r\left[A\left(\frac{\delta \bar{T}_v}{\delta z}\right)^2 + A\overline{\left(\frac{\delta T_v'}{\delta z}\right)^2} + B\frac{\delta \bar{T}_v}{\delta z} + C\right] \\ &- \chi \frac{\bar{T}_v}{\bar{T}}\frac{Q}{L} + P \\ &+ \frac{\chi}{L}\frac{\bar{T}_v}{\bar{T}}\frac{\delta}{\delta z}\left[(K_r - K_h)\frac{\delta \bar{h}}{\delta z}\right] \\ &+ K_r \frac{\delta \bar{r}_v}{\delta z}\frac{\delta}{\delta z}\log \bar{T}_v/\bar{T}.\end{aligned} \tag{14}$$

Both terms of the left-hand side of the equation (14) represent the form of the profile of the liquid water content. They are merely descriptive and do not give any insight into the variation of the liquid water content. The "active" terms which "produce" the profile are found on the right-hand side. When the right-hand terms together make a sum that is greater than zero, the first of the left-hand terms indicates that this implies an increase of the liquid water content. In the steady state, $\delta^2 \bar{r}_w/\delta z^2$ preponderates and then a positive sum on the right-hand side implies that there is a maximum of $\bar{r}_w$ between two levels where the values of $\bar{r}_w$ are equal.

At the base as well as at the top of a cloud or a fog the liquid water content is zero. Somewhere in between there is of course a maximum of liquid water content. The right-hand side of (14) should therefore be greater than zero during cloud and fog conditions. Indeed, each term of this side implies a mechanism of concentration of liquid water in the atmosphere. Before going into a discussion of them one by one, we should say something about the magnitude that they should attain in order to be significant.

Say, the magnitudes of the terms are 10^{-6} $\sec^{-1}$. That means that in nascent fog and clouds $\delta\bar{r}_w/\delta t$ corresponds to an increase of 1×10^{-3} grams of droplets per gram of air during 17 minutes. This seems quite realistic in the light of the fact that r_w in fog and stratus is about 0.1×10^{-3}.

Secondly, say that K_r in fog is about 10^3 cm^2 sec^{-1}. Then $-\delta^2\bar{r}_w/\delta z^2$ should be about 10^{-9} cm^{-2}. If we assume the vertical distribution of r_w to vary according to the square of the distance z above the base of the fog and r_w is further assumed to be zero at the base as it is at the top, then the depth of the fog must be about 10 metres in order that the maximum content of liquid water in the middle part of the fog layer becomes the known degree of amount, 0.1×10^{-3}. Even on this basis, therefore, the assumed magnitude of the terms (10^{-6} sec^{-1}) seems to be rather realistic.

Thirdly, P in (14) denotes the net fall-out of precipitation. We put N for precipitation, of course positive and increasing in downward direction. Then,

$$\varrho P = +\delta N/\delta z \leqslant 0, \tag{15}$$

where ϱ is the density of the dry-air component.

We unfortunately lack data about the vertical variation of fall-out of precipitation. It seems impossible to make reliable measurements of the difference between simultaneous precipitation at two points, one of them being at some height vertically above the other. Say, for instance, a cloud or a fog being 100 metres deep produces 0.5 mm precipitation per hour. This is indeed a rather large amount of drizzle. Or, suppose the depth of the cloud is 1000 metres and the amount of N at the base of the cloud is 5 mm per hour. This is a plausible rate of precipitation. In both examples P is 10^{-6} sec^{-1}.

We shall find that the other terms in (14) except the last two surely attain the mentioned magnitude under conditions which are specific for each term. So we may reasonably base a discussion of production of clouds, fogs and precipitation upon (14). As a matter of fact, P would make clouds and fogs to dissolve if it were not balanced by other terms being positive in (14). It seems apparent that precipitation is neither always nor only a matter of vertical motion of air but may also be due to radiation effects, temperature gradients in inversions and frontal zones and to vertical variations of K_r. All the terms are likely to combine so as to bring about clouds and fogs as well as precipitation.

The term of advection

—e.g. of moving clouds—

$$-\left(\bar{u}\,\frac{\delta\bar{r}_w}{\delta x} + \bar{v}\,\frac{\delta\bar{r}_w}{\delta x}\right) \tag{16}$$

is analogous to the corresponding terms in the equations (4) and (5) of the eddy exchanges of total heat and water contents. It is the product of two vectors, the horizontal gradient of liquid water content and the wind velocity. Suppose that the magnitude of the gradient is 0.1×10^{-3} per 100 metres, and of the wind velocity, one metre per second. Then the effect of advection is about 10^{-6} sec^{-1}.

The term of convection

$$\left[\left(\frac{dr_w}{dz}\right)_{ad} - \frac{\delta\bar{r}_w}{\delta z}\right]\left[\bar{w} - \frac{\delta K_r}{\delta z}\right] \tag{17}$$

includes the well-known effect of vertical displacement upon the liquid water concentration in a closed volume,

$$\left(\frac{dr_w}{dz}\right)_{ad} \times \bar{w}. \tag{18}$$

The derivative in (18) is defined in the appendix (69). It depends on temperature and pressure as shown in Table 1. When $\bar{w} = 1$ m sec^{-1}, (18) is about 10^{-6} sec^{-1}.

Now, (17) reveals that the effect of vertical motion upon the distribution of liquid water

TABLE 1

$$f(T_v, p) = \left(\frac{dr_w}{dz}\right)_{ad}. \text{ (See (69).)}$$

T_v	$p = 1000$ mb	700 mb
+ 30°C	$+2.96 \times 10^{-8}$	$+3.27 \times 10^{-8}$
+ 20	2.47	2.79
+ 10	1.94	2.27
0	1.38	1.70
− 10	0.87	1.12
− 20	0.48	0.65
− 30	0.23	0.32
− 40	0.10	0.14

content depends upon the vertical gradient in terms of

$$\left(\frac{dr_w}{dz}\right)_{ad} - \frac{\delta \bar{r}_w}{\delta z}. \tag{19}$$

We usually say that upward displacements by convection lead to cloud formation. But then we take for granted that the vertical gradient of liquid water content is relatively small,

$$\frac{\delta \bar{r}_w}{\delta z} < \left(\frac{dr_w}{dz}\right)_{ad}.$$

The opposite case,

$$\frac{\delta \bar{r}_w}{\delta z} > \left(\frac{dr_w}{dz}\right)_{ad}$$

implies that clouds are dissolved by upward displacements. This indeed happens when stratus formed at night is dispersed in the morning owing to the drying effect of solar heating near the ground level and the consequent increase of $\delta \bar{r}_w/\delta z$ in the layer up to the cloud.

In (17) the derivative $\delta K_r/\delta z$ is found within the same brackets as the mean vertical velocity $\bar{w}$. We have already seen that this is specific not only to equation (14) of eddy exchange of liquid water content but also to (4) and (5) referring to the total contents of heat and water. It means that the vertical variations of the coefficients of eddy exchange have the same effects as the mean vertical velocity. Surely, cloud sheets at the base of inversions are more likely explained to be caused by the upward increase of stability of stratification and the consequent decrease of turbulent exchange, than by mean vertical motion. A decrease of K_r from 10^5 cm² sec⁻¹ to 10^4 or less in a layer that is 10 metres deep is equivalent to $\bar{w} = 1$ m sec^{-1}. We recall that we in (18) referred the vertical velocity to this magnitude in order that (17) should attain about 10^{-6} sec^{-1}.

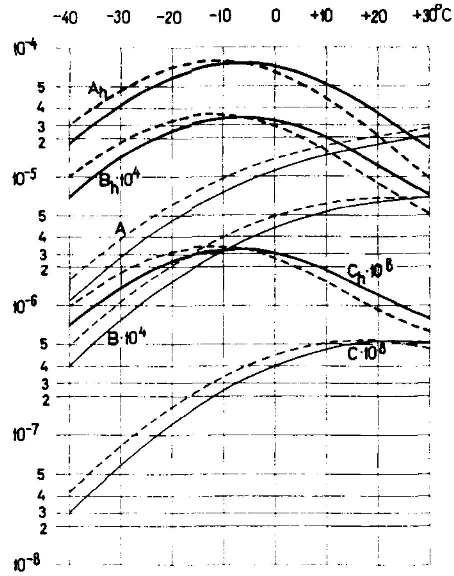

FIG. 1. The variations of the coefficients A, B and C in (20) and A_h, B_h and C_h in (25) with temperature. In order to get the coefficients into a common diagram, B and B_h are multiplied by 10^4, C and C_h by 10^8.

------, the variations at 700 mb;
———, the variations at 1000 mb.

The effect of turbulent mixing

is shown by the term

$$K_r\left[A\left(\frac{\delta \bar{T}_v}{\delta z}\right)^2 + A\overline{\left(\frac{\delta T_v'}{\delta z}\right)^2} + B\frac{\delta \bar{T}_v}{\delta z} + C\right]. \tag{20}$$

It is always greater than zero and has nothing corresponding in the equations (4) and (5) of the eddy exchange of the total heat and total water contents. It depends upon the vertical gradient of wet-bulb temperature and upon the coefficient of eddy exchange. The coefficients A, B and C depend mainly upon temperature and only slightly upon pressure. Their variations are shown in Fig. 1. The term that includes A and the square of the temperature gradient, predominates when the lapse or the inversion is greater than 10^{-3} °C cm^{-1}. Suppose that the vertical gradient of temperature is about 1°C per metre and that K_r is about 10^3 cm^2 sec^{-1}. Then, the effect of turbulent mixing is about 10^{-6} sec^{-1}.

Of course, K_r may happen to be smaller than 10^3 cm^2 sec^{-1}. But a decrease of K_r is generally accompanied by an increase in the inversion. Since (20) varies with the square of the temperature gradient but only with the first power of K_r, the effect of K_r-variation upon (20) is expected to be counterbalanced by the co-variation of the temperature gradient. It should additionally be noted that in the steady state K_r drops out,

$$-\frac{\delta^2 \bar{r}_w}{\delta z^2} = A\left(\frac{\delta \bar{T}_v}{\delta z}\right)^2 + A\overline{\left(\frac{\delta T_v'}{\delta z}\right)^2} + B\frac{\delta \bar{T}_v}{\delta z} + C. \tag{21}$$

Thus, fogs and clouds may prevail independently of stratification in a temperature transition zone.

Equation (21) implies that there is a level of maximum liquid water content in a layer of inversion or temperature lapse,

$$\bar{r}_w\ (\max) \sim \tfrac{1}{8} A (\bar{T}_1 - \bar{T}_0)^2. \tag{22}$$

$\bar{T}_1$ and $\bar{T}_0$ are the temperatures at the top and the base of the fog or cloud layer, and there $r_w = 0$. Say, the temperature difference between the top and the base is 8°C. Then $\bar{r}_w$(max) is about 0.1×10^{-3}. (At 0°C and 1000 mb, A is 1.12 10^{-5}.)

The mean square of the fluctuations of the temperature gradient appears in (20) and (21). So it is very interesting to note that any vertical variation of the turbulent fluctuations of the total heat content operates to increase the mean concentration of liquid water. Unfortunately we do not know much about the magnitude of these fluctuations and their vertical variation but we may suppose the latter to be considerable in inversions and zones of temperature lapse, for example in cold air above a warm water surface.

The effect of the term $\overline{(\delta T_v'/\delta z)^2}$ seems to help us to explain why fog is formed not only in the lowermost part of the surface boundary layer where $\delta \bar{T}/\delta z$ is mostly greater than elsewhere, but also in the upper part where this mean gradient often seems insufficient for fog formation by course of (20). Due to eddies, the square of the standard deviation, $\overline{(\delta T_v'/\delta z)^2}$ presumably attains the conditional magnitude for fog formation even in the upper part of the surface boundary layer.

Furthermore, it is interesting to note that if we had not disregarded the terms of K_{rx} and K_{ry} in (5), we should in the right-hand sides of (20) and (21) find also

$$A\left[K_{rx}\overline{\left(\frac{\delta T_v'}{\delta x}\right)^2} + K_{ry}\overline{\left(\frac{\delta T_v'}{\delta y}\right)^2}\right]. \tag{23}$$

The magnitudes of these terms of the horizontal fluctuations may be considerably greater than those of the mean horizontal conditions, which one had in mind when the horizontal turbulent transfer was excluded from the equation of eddy exchange. Maybe (23) has the same magnitude as (20) under specific conditions and consequently sometimes is essential to the process of fog formation.

Experimental verification of these ideas about the effect of turbulent fluctuations on liquid water concentration in the atmosphere is highly desirable.

When $\delta \bar{T}_v/\delta z$ is $-B/2A$, the effect of turbulence (20) has a minimum which reads

$$K_r[A\overline{(\delta T_v'/\delta z)^2} - (B^2/4A - C)]. \tag{24}$$

We give in Table 2 the magnitudes of the terms concerned. It is obvious firstly that the minimum occurs very close to isothermal conditions, secondly that it differs so very little from zero that we may disregard this difference and persist in our statement above where we pointed out that (20) does not attain any significant value below zero.

The sensible heat content of the air varies in linear proportion to the dry-bulb temperature. The total heat content, which involves the sensible heat plus latent heat contents, is related to the wet-bulb temperature, instead. But, this dependence is not a linear one. The reason is that the specific heat written in terms of the wet-bulb temperature,

$$c_p + T_v \frac{\partial}{\partial T_v}\left[\frac{Lr_v}{T_v}\right]$$

TABLE 2

$f_1(T_v, p) = -B/2A, \quad f_2(T_v, p) = -(B^2/4A - C).$

T_v	f_1 1000 mb	f_1 700 mb	f_2 1000 mb	f_2 700 mb
+30°	-0.16×10^{-4}	-0.15×10^{-4}	-0.35×10^{-15}	-0.50×10^{-15}
+20	−0.18	−0.17	−0.32	−0.30
+10	−0.18	−0.18	−0.28	−0.28
0	−0.18	−0.18	−0.22	−0.26
−10	−0.18	−0.18	−0.15	−0.18
−20	−0.17	−0.17	−0.08	−0.11
−30	−0.16	−0.16	−0.04	−0.06
−40	−0.16	−0.15	−0.02	−0.00

(see (48) in the appendix) depends upon temperature and is considerably greater the higher the temperature. This is shown in Table 3. Consequently, the vertical distribution of wet-bulb temperature in (20) does not indicate how this term (20) varies under constant conditions of vertical gradient of total heat content with temperature, say, from warm weather conditions to cold.

In order to illustrate this we replace the derivatives of temperature in (20) by those of heat content and write for the new coefficients the symbols A_h, B_h and C_h,

$$K_r \left[A_h \left(\frac{\delta \bar{h}}{\delta z}\right)^2 + A_h \overline{\left(\frac{\delta h'}{\delta z}\right)^2} - B_h \frac{\delta \bar{h}}{\delta z} + C_h \right] \tag{25}$$

The relations between the coefficients in (20) and (25) are given in the appendix, (63) to (65).

TABLE 3

$$f(T_v, p) = c_p + T_v \frac{\partial}{\partial T_v} (Lr_v/T_v)$$

$$c_p = 0.240 \text{ cal g}^{-1}\ ^\circ\text{C}^{-1}.$$

T_v	$p = 1000$ mb	700 mb
+30°	1.136	1.571
+20	0.758	0.995
+10	0.531	0.660
0	0.397	0.466
−10	0.321	0.355
−20	0.279	0.296
−30	0.258	0.265
−40	0.247	0.250

The coefficients A_h, B_h and C_h depend upon temperature as is shown in Fig. 1. Each coefficient has a maximum value. It is remarkable that the maxima occur at almost the same temperature, as is shown in Table 4. This means also that (25) has a maximum that occurs at a definite temperature mostly independently of the gradients $\delta \bar{h}/\delta z$ and $\delta h'/\delta z$. In Figs. 2–4 curves are presented which show how (25) varies with temperature under different conditions of vertical heat-content distribution and pressure. For $\delta \bar{h}/\delta z$ we have substituted $\pm 10^{-4}$ in Fig. 2, $\pm 10^{-3}$ in Fig. 3 and $\pm 10^{-2}$ cal g^{-1} cm^{-1} in Fig. 4. Further, we have put $p = 1000$ and 700 mb, and deleted $\overline{(\delta h'/\delta z)^2}$.

We see that (25) mainly depends upon A_h, much less on B_h and C_h when the vertical gradient of heat content or the lapse is greater than 10^{-3}. For information some data on the relation between $\delta \bar{h}/\delta z$ and $\delta \bar{T}_v/\delta z$ are presented in Table 5. The basic formula is found in the appendix, (50).

The fact that (25) depends upon temperature because of the coefficients A_h, B_h and C_h, suggests that there may be a correlation between fog frequency and temperature. We shall now verify this by classifying observations of fog

TABLE 4. *Wet-bulb temperatures at the maximum points of A_h, B_h and C_h.*

p	A_h	B_h	C_h
1000 mb	− 6.4	− 6.7	− 6.8°C
700	− 11.5	− 11.6	− 12.0

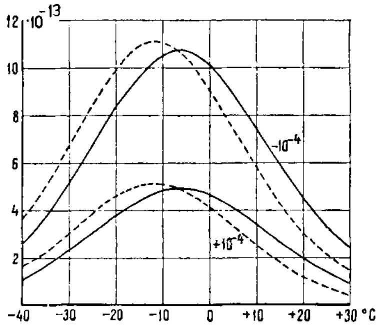

FIG. 2. $f(T_v, p)$

$$= \left[A_h\left(\frac{\delta\bar{h}}{\delta z}\right)^2 - B_h\frac{\delta\bar{h}}{\delta z} + C_h\right] \text{cm}^{-2}$$

when $\delta\bar{h}/\delta z = \pm 10^{-4}$ cal g^{-1} cm^{-1}, and $p = 700$ mb (-----) and 1000 mb (——).

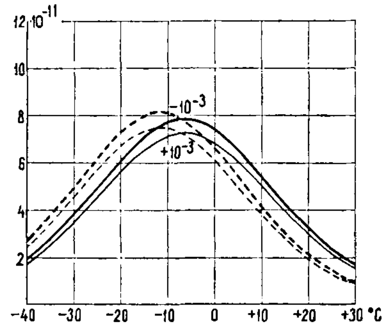

FIG. 3. $f(T_v, p)$

$$= \left[A_h\left(\frac{\delta\bar{h}}{\delta z}\right)^2 - B_h\frac{\delta\bar{h}}{\delta z} + C_h\right] \text{cm}^{-2}$$

when $\delta\bar{h}/\delta z = \pm 10^{-3}$ cal g^{-1} cm^{-1}, and $p = 700$ mb (-----) and 1000 mb (——).

in five-degree intervals $-40°$ to $-35°$, $-35°$ to $-30°$, etc. to $+25°$ to $+30°$C. For each day when fog occurred, a mark is assigned to all temperature intervals that prevailed during the fog. In addition we sample the daily temperature frequency F within each interval by means of daily readings of maximum and minimum temperatures. We thus obtain for each interval the probability of fog, f/F by dividing the fog frequency f by the temperature frequency.

We group the places of fog observations into two classes, let us call them I and II. Class I includes ordinary places but in class II, the places are near rivers which do not freeze over,

TABLE 5

$\delta\bar{T}_v/\delta z = f[(\delta\bar{h}/\delta z), \bar{T}_v, p = 1000$ mb]. h in cal/g, z in cm.

$\bar{T}_v$	$\delta\bar{h}/\delta z = +10^{-2}$	$+10^{-3}$	$+10^{-4}$
+30°	$+0.88 \times 10^{-2}$	$+0.84 \times 10^{-3}$	$+0.51 \times 10^{-4}$
+20	+1.31	+1.27	+0.88
+10	+1.88	+1.83	+1.34
0	+2.51	+2.45	+1.86
−10	+3.11	+3.05	+2.35
−20	+3.58	+3.50	+2.72
−30	+3.88	+3.79	+2.96
−40	+4.04	+3.94	+3.09

$\bar{T}_v$	$\delta\bar{h}/\delta z = 0$	-10^{-4}	-10^{-3}	-10^{-2}
+30°	-0.37×10^{-4}	-1.25×10^{-4}	-0.92×10^{-3}	-0.88×10^{-2}
+20	−0.44	−1.76	−1.36	−1.32
+10	−0.55	−2.43	−1.94	−1.89
0	−0.66	−3.18	−2.58	−2.52
−10	−0.77	−3.89	−3.20	−3.13
−20	−0.86	−4.45	−3.67	−3.59
−30	−0.92	−4.81	−3.97	−3.89
−40	−0.95	−5.00	−4.14	−4.06

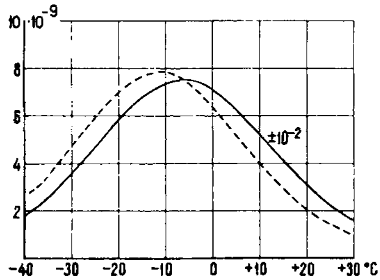

FIG. 4. $f(T_v, p)$

$$= \left[A_h\left(\frac{\delta \bar{h}}{\delta z}\right)^2 - B_h\frac{\delta \bar{h}}{\delta z} + C_h\right] \mathrm{cm}^{-2}$$

when $\delta \bar{h}/\delta z = \pm 10^{-2}$ cal g^{-1} cm^{-1}, and $p = 700$ mb (- - - - -) and 1000 mb (——).

TABLE 6. *Relation between fog probability and temperature.*

I: Places other than those of class II.
II: Places adjacent to ice-free rivers.

Interval	Class I		Class II	
−40°— −35°	—		99.1 %	±0.9
−35 — −30	—		85.0	2.8
−30 — −25	—		63.9	1.9
−25 — −20	3.4 %	±0.5	52.1	1.9
−20 — −15	3.6	0.5	33.3	1.7
−15 — −10	4.8	0.5	18.7	1.2
−10 — − 5	7.4	0.8	10.3	0.9
− 5 — 0	11.8	1.2	9.2	1.5
0 — + 5	13.5	1.0	9.2	1.5
+ 5 — +10	10.0	0.7	6.7	1.0
+10 — +15	5.9	0.5	3.5	0.2
+15 — +20	1.5	0.2	0.7	0.2

even in cold winters, because of the rapid currents. In class I we have 45 places, in class II eight.

The fog probabilities within each class are given in Table 6. The first figures of each column are mean values within the classes, the second figures are the standard errors of the means. In Figs. 5 and 6 are plotted the data of Table 6. The curve of Fig. 5 appears to be composed of two parts separated at 0°C. Above the freezing point the form of the curve shows a similarity to the variation of (25), Figs. 2–4. The other part of the curve shows a sudden decrease just below 0°C. This decrease is due to the fact that below 0°C the ground surface is either frozen or covered by snow or ice, and therefore not wet. Close to a snow or ice surface the vapour pressure is lower than close to a water surface under equal temperature conditions. Consequently, the boundary condition of fog ($r_w = 0$) is not fulfilled there. PETTERSSEN (1940) points out this fact as a reason for fog dissipation. He proves his idea in the statistical way shown in Fig. 5.

Over the surface of a river that is prevented from freezing by rapid currents, we find conditions of steam fog formation. The temperature and also consequently the vapour pressure of such a water surface does not decrease below the freezing point of water. The lapse rate, $-\delta T_v/\delta z$, therefore increases as the air temperature decreases below 0°C. The consequent increase of (20) indicates a parallel increase of fog frequency. In column II of Table 6 mean values of the probabilities of fog are given for eight places along rivers which do not become covered by ice even in very cold winters. The resulting curve is shown in Fig. 6. It shows that the

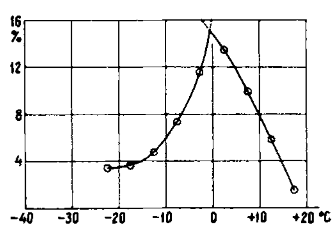

FIG. 5. Fog-probability curve referring to places which are not in the vicinity of rivers which do not freeze over in winter.

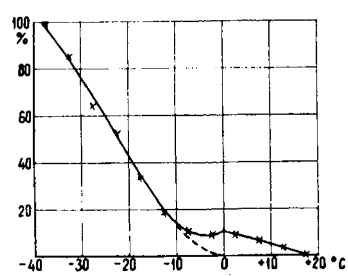

FIG. 6. Fog-probability curve referring to places adjacent to water which remains ice-free even in cold spells during the winter.

probability of the occurrence of fog abruptly increases when the temperature drops below $-10°C$. At higher temperatures, the form of the curve is similar to that in Fig. 5. In order to ascertain that the part of the curve below $-10°C$ is related to (20), we examine the mean values of class II in relation to the function

$$f/F = \frac{A[\frac{1}{2}(T_v - 273), p]}{A[\frac{1}{2}(T_{vm} - 273), p]} \left[\frac{T_v - 273}{T_{vm} - 273}\right]^2, \quad (26)$$

which is based upon the variation of the coefficient

$$A = A[T_v, p]$$

in (20).

We put $p = 1000$ mb. The temperature T_{vm} is a parameter to which we may attribute a value so that (26) fits the empirical data in Table 6. $\frac{1}{2}(T_v - 273)$ is the mean in degrees centigrade of the air temperature and the water surface temperature (assumed to be close to the freezing point). Using the method of least squares, we find that $T_{vm} - 273$ is 39°C. We see that the curve in Fig. 6 which follows (26) for the conditions mentioned, runs very close to all the points which mark the empirical values. Thus it supports the theoretical approach which leads to (20).

The term of radiation

$$-\chi \frac{T_v Q}{T\ L} \quad (27)$$

is one of the very important in (14). It refers to the effect of radiative emission in the atmosphere, or when the sign is reversed, to absorption. For explanation of χ, see (59) in the appendix. L is the latent heat of evaporation of water.

We put I for the net flux of radiation, positive in the upward direction. Then

$$\varrho Q = -\delta I/\delta z, \quad (28)$$

where ϱ is the density of dry air. Usually I is noted in terms of mcal cm^{-2} min^{-1}. Measuring L in cal/g, (27) reads

$$\frac{+\chi}{60\varrho L} \frac{T_v}{T} \frac{\delta I}{\delta z} \times 10^{-3}. \quad (29)$$

The variation of the coefficient $\chi/60\varrho L$ is shown in Table 7. It greatly depends upon temperature and pressure, so the effect of radiative cooling is much smaller at low than at high temperatures. Moreover, it is greater in high than in low altitudes. In order that (29) should attain 10^{-6} sec^{-1} under temperature conditions above 0°C, it seems necessary that $\delta I/\delta z$ is at least 0.1 mcal cm^{-3} min^{-1}. Such a vertical change of net flux of radiation does occur in the region of long-wave radiation during conditions of great temperature inversions. It is most likely to occur at the top of clouds and fogs because water droplets have a greater emissivity than water vapour. Measurements made by KRAUS (1958) give some information on the variation of (27) during a night of fog formation, see Fig. 7. I have discussed KRAUS' measurements in a previous paper (1962) and therefore refer for details to this discussion. The diagram of Fig. 7 verifies that there is a maximum of (27) in the upper part of the fog layer, which in this case was two metres deep. The maximum attains 0.2 to 0.6×10^{-6} sec^{-1}. Radiation measurements are unfortunately not reliable under fog conditions because the deposit of a water film on the instrument greatly affects incoming and outgoing radiation. KRAUS was forced to base his conclusions upon estimations as regards the effect of fog droplets upon radiation.

Special attention should be paid to the fact that Q does not refer to the radiative cooling of the ground surface but to the direct radiative cooling of the atmosphere. The radiative cooling of the ground surface brings about a temperature gradient next to the surface and then a convective cooling of the atmosphere. However, this kind of cooling is not implicit in Q but in

$$K_h \frac{\delta^2 \bar{h}}{\delta z^2}$$

in (4). On the other hand, the temperature gradient thus formed next to the ground is a direct factor in fog formation, but in terms of (20). Consequently, in radiation fog, Q and (20) are both closely related to outgoing long-wave radiation. Because of this common relationship, they are so highly correlated that it is not possible to get separate evidence on their effects by means of statistics.

Unlike that in radiation fog, the temperature gradient in steam fog over a warm water surface

TABLE 7

$f_1(T_v,p) = \chi$. (See (59) in the appendix.)

$f_2(T_v,p) = \chi/60\varrho L$.

T_v	f_1 p = 1000 mb	f_1 700 mb	f_2 1000 mb	f_2 700 mb
+30°C	0.848	0.910	2.117×10^{-2}	3.247×10^{-2}
+20	0.733	0.814	1.754	2.782
+10	0.586	0.681	1.340	2.227
0	0.422	0.517	0.922	1.615
−10	0.268	0.345	0.560	1.028
−20	0.148	0.199	0.295	0.565
−30	0.072	0.100	0.137	0.270
−40	0.031	0.044	0.055	0.111

is not a direct result of radiation. The separate effects of Q and (20) are consequently expected to be easily distinguished in the case of steam fog. Figs. 8 and 9 prove this.

In Fig. 8, the diagram of fog probability against air temperature is found in the form described above, but there are also two other curves, one of them referring to clear sky conditions, the other to overcast conditions. The diagram is based upon observations made at Lycksele in winter from November to February. Lycksele is situated in the northern part of Sweden at latitude 64.6° N. During the months mentioned the sun there does not rise more than 17° above the horizon. The conditions of clear sky consequently imply a tendency for a net loss of radiative heat.

There is nowadays at Lycksele a hydro-electric power station which has thoroughly changed the character of the river Umeälv. The series of observation refer to the original conditions, before the station was built. In that time there was in the river a cataract with such a rapid current that the water was prevented from freezing even in very cold weather situations. As extremely cold conditions do not occur in overcast weather, the curve which refers to overcast sky, ends at about −25°C.

The actual point of interest in Fig. 8 is that fog is much more frequently formed under

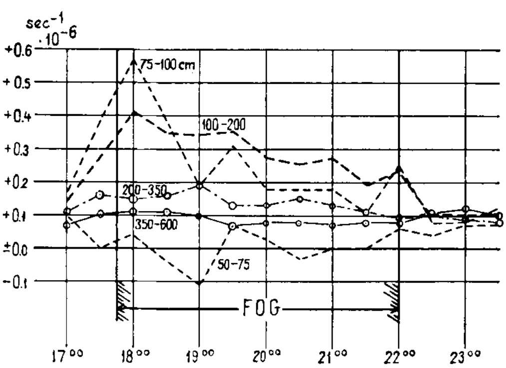

FIG. 7

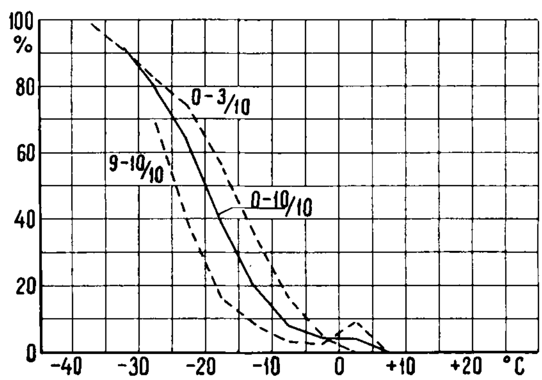

FIG. 8

FIG. 7. The variation of the net loss of radiation (27) according to KRAUS (1958) during a night of radiation-fog formation, October 12, 1956. The curves refer to the following heights above the ground: 50–75 cm, 75–100 cm, 100–200 cm, 200–350 cm and 350–600 cm.

FIG. 8. The fog-probability curve under clear-sky (0–3/10) and overcast-sky (9–10/10) conditions. *Lycksele*, Nov.–Febr. in 1956–62.

clear-sky than under overcast conditions. Under overcast-sky conditions, Q is negligible. The fog is then mostly attributable, in terms of (20), to the temperature difference between the atmosphere and the ice-free water surface. But under clear-sky conditions, the net radiative heat loss, Q, takes an active part with such a high efficiency that for instance between -5 and $-15°C$ the fog probability is more than three times greater than under overcast conditions.

Since we do not find this remarkable increase of the fog probability under cold weather conditions in places away from ice-free rivers, the fog-producing efficiency of net radiative heat loss seems to be caused by the black-body emissivity of the droplets initially formed in terms of (20) in the temperature-gradient zone immediately above the ice-free water surface. Thus, (20) initiates the fog in the zone next above the water surface, but Q reinforces the production of droplets and increases the height of the fog above this often very narrow zone —thanks to the radiative emissivity of the droplets themselves.

Fig. 9, too, proves this interpretation of the different effects of (20) and Q. The histograms of the figure are based on 71 observations of steam-fog conditions at different places in northern Sweden. All the observations refer to ice-free water conditions at air temperatures of $-10°C$ or lower. They are grouped firstly in rows with reference to the air temperature, secondly in classes according to the sky conditions, clear, cloudy, and overcast. Thirdly the height and the density of the fog is noted in columns, involving the categories

1. Dense and deep steam fog
2. Moderate steam fog
3. Light steam fog
4. Patches of steam fog only close to the water surface.

The first column of Fig. 9 refers to fog frequency in all the categories 1 to 4. In the second column, the observations of low patches are excluded, i.e. this column involves steam fog in the categories 1 to 3. In the third column, only the observations of moderate and dense fog in the categories 1 and 2 are sampled. The last column illustrates the frequency of dense and deep steam fog, category 1. The histograms of Fig. 9 confirm the information just obtained from the preceding Fig. 8, as follows.

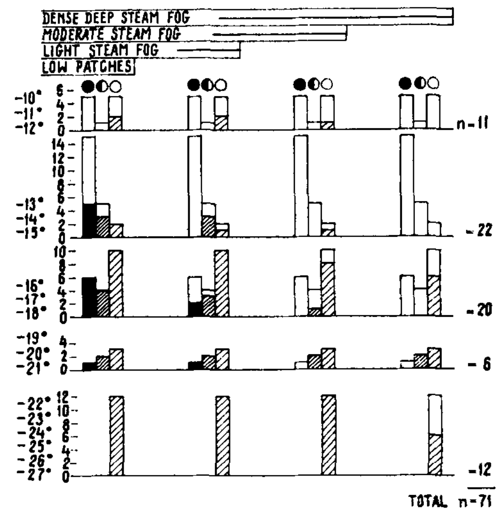

FIG. 9. Histograms of steam-fog development under clear-sky (○), cloudy (◐) and overcast (●) conditions at various temperatures. The heights of the columns represent the numbers of observations within the categories, the shaded areas refer to the occurrence of fog as specified in the top row. (n = The number of observations.)

1. *Under overcast-sky conditions*, only patches of steam fog are produced, in very cold weather at the most, light steam fog.

2. *Under clear-sky conditions*, light and moderate steam fog is formed already at about $-10°C$. When the temperature is below $-16°$, the steam fog becomes dense and deep.

The difference between clear-sky and overcast-sky conditions is very outstanding. There seems to be no doubt that the atmosphere is brought to saturation primarily because of the effect of the temperature gradient in (20). Because of the high long-wave emissivity of the droplets there follows an effective net loss of radiative heat, Q, so resulting in dense and deep steam fog under clear-sky conditions. The combined contribution of Q and (20) surely exists in radiation fog as well, but in this case we cannot prove it by means of statistics as is done in Figs. 8 and 9. But we may permit ourselves to generalize our conclusions and suppose that the temperature gradient in (20) is the initial cause of fog formation thus bringing the atmosphere to saturation in a very narrow zone next to the ground surface. Long-wave radiation immediately becomes very effective in terms of Q when the first droplets are condensed, and then the production of fog droplets is accelerated and the height of the fog is increased above the surface boundary layer within which the original temperature gradient is found. A temperature inversion is established at the top of the fog while the temperature distribution within a deep fog layer becomes moist-adiabatic. The role preliminarily played in (20) by the temperature gradient next to the ground surface is taken over by the temperature gradient in the inversion at the top of the fog.

The term involving $(K_r - K_h)$

$$\frac{\chi}{L}\frac{\bar{T}_v}{\bar{T}}\frac{\delta}{\delta z}\left[(K_r - K_h)\frac{\delta \bar{h}}{\delta z}\right]. \tag{30}$$

Several investigations indicate a difference to exist between K_r and K_h next to the ground, e.g. PASQUILL (1949). Others lead to the conclusion that the eddy coefficients of heat and moisture are the same (CRAWFORD, 1965). All refer to the surface boundary layer under unsaturated conditions. PASQUILL found K_h to be significantly greater than K_r under unstable conditions, which should imply that (30) is negative when $\delta^2\bar{h}/\delta z^2 > 0$. The results of CRAWFORD's investigation prove on the contrary that (30) is of no account in (14). Maybe that is true also in fogs and clouds, we do not surely know.

The term involving $\bar{T}_v/\bar{T}$,

$$K_r\frac{\delta \bar{r}_v}{\delta z}\frac{\delta}{\delta z}[\log\ \bar{T}_v/\bar{T}]. \tag{31}$$

Both derivatives of this term are discussed in (70) to (73) in the appendix. Say, $K_r = 10^3$ cm^2 sec^{-1}. Even under very extreme vertical distributions of $\bar{r}_v$ and $\bar{T}_v/\bar{T}$, we may conclude from the results of the discussion that (31) hardly exceeds 10^{-8} sec^{-1}, which is a very small part of the other terms in (14). In clouds and fogs it is zero because there is $\bar{T}_v/\bar{T} = 1$. Thus, this term may be deleted in (14).

The effect of the boundary conditions

The differential equation (14) points out the terms which operate to increase or decrease the liquid water content within a layer of the atmosphere. The question of whether clouds and fogs are formed or not, is settled not only by the intensities of the terms in (14) but even by the boundary conditions at the base and the top of the layer concerned. In the case of fog it is a matter of the temperature and moisture conditions at the ground or sea level and in the atmosphere. If the boundary conditions are insufficient, $\bar{r}_w$ *remains negative and clouds or* fogs are not formed even when the terms in (14) show a strong evidence to be positive.

Conclusions

We finish the paper in proposing a classification of fogs on the basis of the terms in (14). We recall that fog is generally a product of more than one of the processes enumerated in (14). Mostly, two or more act simultaneously. A term in (14) is not specific to one type of fog only. For example, the effect of long-wave radiation is specific to radiation fog, but radiative heat loss in the top layer of fog plays a very important role in maintaining advection fog, too.

Most of the terms in (14) refer to fog-producing as well as fog-dissolving processes. The alternative is settled by the gradient conditions. Turbulent mixing (20) has, however, only a fog-producing effect. The last two items of the classification—sublimation on a snow surface and fall-out of precipitation—are only fog-dissolving.

Classification of fogs

(P = fog-producing processes, D = fog-dissolving processes.)

1. Maintenance of a temperature gradient, (20),

$$K_r\left[A\left(\frac{\delta \bar{T}_v}{\delta z}\right)^2 + A\overline{\left(\frac{\delta T_v'}{\delta z}\right)^2} + B\frac{\delta \bar{T}_v}{\delta z} + C\right], \qquad \text{P}$$

a. by advection of air over a warm or cold surface: *Advection fog, steam fog, sea smoke.*
b. by advection of warm air over a thawing snow surface: *Advection fog.*
c. by radiative heat loss on the ground, snow covered or not: *Radiation fog.*
d. by the gain of solar heat on a wet black surface, for instance the pavement of a road,

the bark of a tree etc. Only patches and traces of fog are produced, often observed shortly after sunrise or at break-up after a rain shower.

e. by the effect of warm (general case) or cold raindrops falling in the air. The surface of each drop constitutes a boundary surface and the fog is the total effect of the temperature gradients at all drops together: *Frontal fog.*

2. Radiative heat loss (*a*) and solar heat gain (*b*) in the topmost layer of the fog, (27),

a. $-\chi \frac{T_v}{T} \frac{Q}{L} > 0$ P

Effective in *advection fog* as well as *radiation fog.*

b. $-\chi \frac{T_v}{T} \frac{Q}{L} < 0.$ D

3. Advective supply (*a*) and withdrawal (*b*) of liquid water content, (16),

a. $-(\bar{u}\delta\bar{r}_w/\delta x + \bar{v}\delta\bar{r}_w/\delta y) > 0$ *Advection fog.* P

b. $-(\bar{u}\delta\bar{r}_w/\delta x + \bar{v}\delta\bar{r}_w/\delta y) < 0.$ D

4. Large-scale vertical motion, (17),

a. upward, $\delta\bar{r}_w/\delta z < (dr_w/dz)_{ad}$ (general case): P
Upslope fog;
downward, $\delta\bar{r}_w/\delta z > (dr_w/dz)_{ad}$. P

b. upward, $\delta\bar{r}_w/\delta z > (dr_w/dz)_{ad}$. D
downward $\delta\bar{r}_w/\delta z < (dr_w/dz)_{ad}$ (general case). D

5. Vertical change of the coefficient of eddy exchange, (17),

a. $\frac{\delta K_r}{\delta z} < 0$, $\delta\bar{r}_w/\delta z < (dr_w/dz)_{ad}$ (general case): P
Fog and *stratus* at the base of inversions.

$\frac{\delta K_r}{\delta z} > 0$, $\delta\bar{r}_w/\delta z > (dr_w/dz)_{ad}$. P

b. $\frac{\delta K_r}{\delta z} < 0$, $\delta\bar{r}_w/\delta z > (dr_w/dz)_{ad}$. D

$\frac{\delta K_r}{\delta z} > 0$, $\delta\bar{r}_w/\delta z < (dr_w/dz)_{ad}$ (general case). D

6. Sublimation on a snow surface, Fig. 5, $r_w < 0$ at the surface. D

7. Fall-out of precipitation, P (15). D

APPENDIX

The differential relations between r_v and h

a. In the psychrometer process, sensible heat is removed from the air and added to the wet bulb for evaporation. The total heat content —sensible plus latent—is retained constant in the air but there is a gain of entropy because of the jump of temperature $T - T_v$ between the air that enters the psychrometer and the wet bulb.

In formation of convective clouds, as well as in the turbulent transfer of eddies, individual volumes of air are subject to continuous changes of pressure and consequently undergo temperature changes. These processes of convection and turbulence are theoretically best approached by the idea of reversible processes, well-known in the form of adiabatic cooling and heating of the air. We shall in the next sections replace the wet-bulb temperature T_v by T_{iv}, the definition of which will be based on a reversible process that is subdivided into three stages, two of which remind us of the adiabatic formation of convective clouds. In the whole, the process is analogous to the psychrometer process and we aim at information about the magnitude of the difference between T_{iv} and T_v.

First stage. We keep the total content of water constant, but bring the vapour pressure to the point of saturation by reducing the pressure of the air volume. During this procedure, the partial pressure of vapour is decreased in proportion to the reduction of total pressure, as well as to the reduction of the partial pressure of the dry-air component. Let us assume that saturation is attained at the temperature T_{sv} and the dry-air pressure p_{sv},

$$c_p(1 + rc_{vp}/c_p)\log T - R(1 + r/\varepsilon)\log p = c_p(1 + rc_{vp}/c_p)\log T_{sv} - R(1 + r/\varepsilon)\log p_{sv}. \quad (32)$$

We recognize T_{sv} and p_{sv} as the temperature and pressure at the level of the base of convective clouds. The specific heat of dry air is signed c_p, and that of vapour c_{vp}. The term ε is identical with m_v/m_d where m_v is the molecular weight of water, and m_d the mean molecular weight of dry air. R is the specific gas constant for dry air. In (32) there are two terms within brackets, differing only a few tenths of a percent from each other since $1/\varepsilon = 1.61$, $c_{vp}/c_p = 1.94$ and $r = 10^{-3}$ to 10^{-2}.

Second stage. We add $(r_{iv} - r)$ units of cloud droplets per unit mass of the dry-air component, the droplets having the temperature T_{sv}.

Third stage. We restore the original partial pressure p of the dry air of the volume. Because of the liquid water added in the second stage, the vapour content now remains fixed under saturation conditions. The final temperature T_{iv} thus differs from the original T,

$$\begin{aligned} c_p(1 + rc/c_p)\log T_{sv} + Lr/T_{sv} - R\log p_{sv} \\ + (r_{iv} - r)c\log T_{sv} \\ = c_p(1 + r_{iv}c/c_p)\log T_{iv} + Lr_{iv}/T_{iv} - R\log p. \end{aligned} \quad (33)$$

The first terms enclosed in brackets in the left-hand and right-hand parts of (33) are some per cent greater than the corresponding terms of (32) because c/c_p is 4.17 while c_{vp}/c_p is 1.94; c is the specific heat of liquid water. L is the latent heat of evaporation of water.

There is no change of entropy in the first and third stages but in the second there is an increase because of the liquid water added. Thus, the amount of entropy of the original volume is given by the right-hand part of (33) *minus* the entropy of the liquid water added, or

$$\begin{aligned} S = (c_p + rc)\log T_{iv} + Lr_{iv}/T_{iv} - R\log p \\ + (r_{iv} - r)c\log T_{iv}/T_{sv}. \end{aligned} \quad (34)$$

S is the entropy per unit mass of the dry-air component.

NORMAND (see BRUNT, 1944) simplifies the matter and puts $T_{iv} = T_{sv} = T_v$. His well-known *Proposition II* reads, "The entropy of air is equal to the entropy of the same air saturated at the wet-bulb temperature, *minus* the entropy of the additional liquid water required so to saturate it" (Brunt, 1944). However, the wet-bulb temperature is strictly fixed by the following formula which leads to Normand's Proposition I (see equation (6)),

$$c_p(1 + rc_{vp}/c_p)(T - T_v) = L(r_v - r). \quad (35)$$

In order to find the degree of approximation which is involved in Normand's Proposition II, let us firstly delete the above-mentioned terms within brackets of (35), (32) and (33),

$$c_p(T - T_v) = L(r_v - r), \quad (36)$$

$$c_p\log T - R\log p = c_p\log T_{sv} - R\log p_{sv}, \quad (37)$$

$$\begin{aligned} c_p\log T_{sv} - R\log p_{sv} = c_p\log T_{iv} - R\log p \\ + Lr_{iv}/T_{iv} - Lr/T_{sv}. \end{aligned} \quad (38)$$

Secondly, we disregard the terms of the second and higher powers in the logarithm functions of the left-hand side of (37) and the right-hand side of (38) which are equal,

$$c_p(T - T_{iv}) = L(r_{iv} - rT_{iv}/T_{sv}). \quad (39)$$

After elimination of the original temperature T in (36) and (39), we get

$$c_p(T_{iv} - T_v) + L(r_{iv} - r_v) = Lr(T_{iv}/T_{sv} - 1). \quad (40)$$

Since

$$\left.\begin{aligned} r_{iv} - r_v &= \left(\frac{\partial r_v}{\partial T_v}\right)_p (T_{iv} - T_v), \\ r_{iv} - r &= \left(\frac{dr_v}{dT_v}\right)_{ad} (T_{iv} - T_{sv}) \end{aligned}\right\} \quad (41)$$

we get

$$c_p(T_{iv} - T_v) = \frac{L(r_{iv} - r)}{1 + \frac{L}{c_p}\left(\frac{\partial r_v}{\partial T_v}\right)_p} \frac{r}{T_{sv}\left(\frac{dr_v}{dT_v}\right)_{ad}}, \quad (42)$$

where

$(\partial r_v/\partial T_v)_p$ is the variation of the saturation mixing-ratio at constant pressure,

$(dr_v/dT_v)_{ad}$ is the corresponding variation along the moist adiabat, see (66) and (67).

The magnitude of $(r_{iv} - r)$ is almost that of $(r_v - r)$. Thus, we put (42) together with (36) and get the approximate relationship

$$\frac{T_{iv} - T_v}{T - T_v} \sim \frac{1}{1 + \frac{L}{c_p}\left(\frac{\partial r_v}{\partial T_v}\right)_p} \frac{r_v}{T_v\left(\frac{dr_v}{dT_v}\right)_{ad}}. \quad (43)$$

TABLE 8

$$f(T_v, p) = \frac{1}{1+\frac{L}{c_p}\left(\frac{\partial r_v}{\partial T_v}\right)_p} \frac{r_v}{T_v\left(\frac{dr_v}{dT_v}\right)_{ad}}.$$

T_v	$p = 1000$ mb	700 mb
+30°	0.023	0.019
+20	0.028	0.023
+10	0.034	0.029
0	0.040	0.035
−10	0.045	0.041
−20	0.048	0.045
−30	0.049	0.047
−40	0.048	0.046

In Table 8, the right-hand side of (43) is evaluated for the conditions of the 700 and 1000 mb levels. We see that the difference between T_{iv} and T_v is 2–5% of the difference between the dry-bulb and the wet-bulb temperatures.

b. By disregarding the difference between T_v and T_{iv}, NORMAND undoubtedly simplifies the theory relating to the entropy of air within reasonable limits of approximation. We follow his line of approach and drop the index i of T_{iv} and r_{iv}. In addition, we disregard the small amount of entropy that is included in the liquid water content of the air. After these approximations, (34) reads

$$S = c_p \log T_v + Lr_v/T_v - R \log p \qquad (44)$$

which means that with a small degree of approximation we regard *the entropy of air as a function of two variables only, i.e. the wet-bulb temperature and the partial pressure of dry air.*

A reversible change of total heat content is related to the simultaneous change of entropy (34 or 44) by the equation

$$dh = TdS.$$

Thus, (44) leads to a relation between reversible variations of total heat content and variations of wet-bulb temperature and partial pressure of dry air,

$$dh = c_p T dT_v/T_v + Td(Lr_v/T_v) - RT dp/p. \qquad (45)$$

The saturation mixing-ratio r_v is fixed by the wet-bulb temperature and the partial pressure of dry air. The heat of evaporation L depends upon the wet-bulb temperature only. Since r_v varies inversely with pressure,

$$\left[\frac{\partial}{\partial p}\left(\frac{Lr_v}{T_v}\right)\right]_{T_v} = -\frac{Lr_v}{pT_v}. \qquad (46)$$

On the other hand, the partial pressure of dry air is coupled to the height above the ground level by the hydrostatic equation,

$$g dz = -RT dp/p. \qquad (47)$$

In consequence, equation (45) may be written in the following form,

$$dh = \left[c_p + T_v \frac{\partial}{\partial T_v}\left(\frac{Lr_v}{T_v}\right)\right]\frac{T}{T_v} dT_v + \left[1 + \frac{Lr_v}{RT_v}\right] g dz. \qquad (48)$$

By means of this differential equation we obtain some derivatives which were fundamental in the main part of the present paper.

$$\frac{dh}{dt} = \left[c_p + T_v \frac{\partial}{\partial T_v}\left(\frac{Lr_v}{T_v}\right)\right]\frac{T}{T_v}\frac{dT_v}{dt} + \left(1 + \frac{Lr_v}{RT_v}\right) g w, \qquad (49)$$

$$\frac{\delta h}{\delta z} = \left[c_p + T_v \frac{\partial}{\partial T_v}\left(\frac{Lr_v}{T_v}\right)\right]\frac{T}{T_v}\frac{\delta T_v}{\delta z} + \left(1 + \frac{Lr_v}{RT_v}\right) g, \qquad (50)$$

$$\begin{aligned}\frac{\delta^2 h}{\delta z^2} = &\left[c_p + T_v \frac{\partial}{\partial T_v}\left(\frac{Lr_v}{T_v}\right)\right]\frac{T}{T_v}\frac{\delta^2 T_v}{\delta z^2} \\ &+ \frac{\partial}{\partial T_v}\left[c_p + T_v \frac{\partial}{\partial T_v}\left(\frac{Lr_v}{T_v}\right)\right]\frac{T}{T_v}\left(\frac{\delta T_v}{\delta z}\right)^2 \\ &+ 2\frac{g}{R}\frac{\partial}{\partial T_v}\left(\frac{Lr_v}{T_v}\right)\frac{\delta T_v}{\delta z} + \frac{Lr_v}{RT_v}\frac{g^2}{RT_v} \\ &+ \left[c_p + T_v \frac{\partial}{\partial T_v}\left(\frac{Lr_v}{T_v}\right)\right]\frac{\delta T_v}{\delta z}\frac{\delta}{\delta z}\left(\frac{T}{T_v}\right). \end{aligned} \qquad (51)$$

The derivatives of r_v are easily derived on the basis of the differential of the first order

$$dr_v = \frac{\partial r_v}{\partial T_v} dT_v - r_v \frac{dp}{p}, \qquad (52)$$

the partial derivative (46) and the hydrostatic equation (47).

Thus,

$$\frac{dr_v}{dt} = \frac{\partial r_v}{\partial T_v}\frac{dT_v}{dt} + r_v\frac{g}{RT}w, \tag{53}$$

$$\frac{\delta r_v}{\delta z} = \frac{\partial r_v}{\partial T_v}\frac{\delta T_v}{\delta z} + r_v\frac{g}{RT}, \tag{54}$$

$$\frac{\delta^2 r_v}{\delta z^2} = \frac{\partial^2 r_v}{\partial T_v^2}\left(\frac{\delta T_v}{\delta z}\right)^2 + \frac{\partial r_v}{\partial T_v}\left[\frac{\delta^2 T_v}{\delta z^2} + 2\frac{g}{RT}\frac{\delta T_v}{\delta z}\right] + r_v\left[\frac{g}{RT} - \frac{1}{T_v}\frac{\delta T_v}{\delta z} - \frac{T_v}{T}\frac{\delta}{\delta z}\left(\frac{T}{T_v}\right)\right]\frac{g}{RT}. \tag{55}$$

We substitute the derivatives of total heat content for those of temperature by means of (49), (50) and (51),

$$L\frac{dr_v}{dt} = \chi\frac{T_v}{T}\frac{dh}{dt} - [\chi RT_v - (1-\chi)Lr_v]\frac{g}{RT}w, \tag{56}$$

$$L\frac{\delta r_v}{\delta z} = \chi\frac{T_v}{T}\frac{\delta h}{\delta z} - [\chi RT_v - (1-\chi)Lr_v]\frac{g}{RT}, \tag{57}$$

$$L\frac{\delta^2 r_v}{\delta z^2} = \chi\frac{T_v}{T}\frac{\delta^2 h}{\delta z^2} + L\left[A\left(\frac{\delta T_v}{\delta z}\right)^2 + B\left(\frac{\delta T_v}{\delta z}\right) + C\right] - L\frac{\delta r_v}{\delta z}\frac{T_v}{T}\frac{\delta}{\delta z}\left(\frac{T}{T_v}\right). \tag{58}$$

The following abbreviations are introduced in (56) to (58) for the substitution of terms which almost only depend upon T_v and p,

$$\chi = \frac{L\dfrac{\partial r_v}{\partial T_v}}{c_p + T_v\dfrac{\partial}{\partial T_v}\left(\dfrac{Lr_v}{T_v}\right)}, \tag{59}$$

$$A = \left[c_p + T_v\frac{\partial}{\partial T_v}\left(\frac{Lr_v}{T_v}\right)\right]\frac{\partial}{\partial T_v}\left(\frac{\chi}{L}\right), \tag{60}$$

$$B = \frac{1}{L}\left[2c_p\chi - \frac{Lr_v}{T_v}\right]\frac{g}{RT}, \tag{61}$$

$$C = r_v(1-\chi)\left(\frac{g}{RT}\right)^2. \tag{62}$$

A, B and C are coefficients of a quadratic polynomial in $\delta T_v/\delta z$. The same polynomial may also be written in terms of $\delta h/\delta z$. The transformation is done by means of (50). We sign the new coefficients A_h, B_h and C_h. These are related to the former as follows,

$$A_h\left[c_p + T_v\frac{\partial}{\partial T_v}\left(\frac{Lr_v}{T_v}\right)\right]^2\left(\frac{T}{T_v}\right)^2 = A, \tag{63}$$

$$B_h\left[c_p + T_v\frac{\partial}{\partial T_v}\left(\frac{Lr_v}{T_v}\right)\right]\frac{T}{T_v} = -2A\left(\frac{dT_v}{dz}\right)_{ad} - B, \tag{64}$$

$$C_h = A\left(\frac{dT_v}{dz}\right)_{ad}^2 + B\left(\frac{dT_v}{dz}\right)_{ad} + C. \tag{65}$$

c. If we put $\delta h/\delta z = 0$, the derivative (50) leads us to the well-known moist-adiabatic lapse rate of temperature,

$$\left(\frac{dT_v}{dz}\right)_{ad} = -\frac{RT_v + Lr_v}{c_p + T_v\dfrac{\partial}{\partial T_v}\left(\dfrac{Lr_v}{T_v}\right)}\frac{g}{RT}. \tag{66}$$

The change of saturation mixing-ratio along the moist adiabat follows if we put $\delta h/\delta z = 0$ in (57),

$$\left(\frac{dr_v}{dz}\right)_{ad} = -\frac{1}{L}[\chi RT_v - (1-\chi)Lr_v]\frac{g}{RT}. \tag{67}$$

A closed volume of air, which is lifted adiabatically and retains its total content of water, changes its liquid water content accordingly,

$$\left(\frac{dr}{dz}\right)_{ad} = \left(\frac{dr_w}{dz}\right)_{ad} + \left(\frac{dr_v}{dz}\right)_{ad} = 0. \tag{68}$$

Thus,

$$\left(\frac{dr_w}{dz}\right)_{ad} = \frac{1}{L}[\chi RT_v - (1-\chi)Lr_v]\frac{g}{RT}. \tag{69}$$

The derivative $(dr_w/dz)_{ad}$ is tabulated in Table 1.

d. We got in (14) the term (31) because of the factor $\bar{T}/\bar{T}_v$ in formula (48) for the total heat content. Two derivatives are involved in (31). The first one reminds of (54), which we rewrite,

$$\frac{\delta \bar{r}_v}{\delta z} = \frac{\partial r_v}{\partial T_v}\left[\frac{\delta \bar{T}_v}{\delta z} - \left(\frac{dT_v}{dz}\right)_{r_v}\right], \tag{70}$$

TABLE 9

$$f(T_v, p) = \frac{L}{c_p T_v} \frac{\partial r_v}{\partial T_v}.$$

T_v	$p = 1000$ mb	700 mb
+30°C	13.3×10^{-3}	19.7×10^{-3}
+20	7.9	11.5
+10	4.6	6.6
0	2.6	3.7
−10	1.4	2.0
−20	0.7	1.0
−30	0.3	0.5
−40	0.1	0.2

where

$$\left(\frac{dT_v}{dz}\right)_{r_v} = -\frac{r_v}{\frac{\partial r_v}{\partial T_v}} \frac{g}{RT} \tag{71}$$

is the wet-bulb temperature gradient which corresponds to constant saturation mixing-ratio of vapour and dry air along the vertical.

The second derivative found in (31) is easily transformed into a derivative of r_w if we make use of (37) and (38) and recall that we in (44) and the following sections introduced $T_{sv} = T_{tv} = T_v$. We get

$$\log T_v/T = \frac{Lr_w}{c_p T_v}, \quad r_w \leqslant 0. \tag{72}$$

Thus, (31) reads

$$K_r \frac{L}{c_p T_v} \frac{\partial r_v}{\partial T_v} \left[\frac{\delta \bar{T}_v}{\delta z} - \left(\frac{dT_v}{dz}\right)_{r_v}\right] \frac{\delta \bar{r}_w}{\delta z}. \tag{73}$$

When $r_w > 0$, (31) is zero and (73) is not valid.

The reader finds in Table 9 values of the first part of (73)—K_r is excluded, and in Table 10 he finds data illustrating the derivative (71). For comparison, values of the moist adiabat are added in the latter table. In the case of an adiabatic distribution of temperature ($\delta \bar{h}/\delta z = 0$) as well as humidity ($\delta \bar{r}/\delta z = 0$), information on $\delta \bar{r}_w/\delta z$ is found in Table 1. From the figures in the Tables 1, 9 and 10 we see that (31) under these conditions is about $-10^{-15} \times K_r$ sec^{-1}.

TABLE 10

$f_1(T_v, p) = (dT_v/dz)_{r_v}$. (See (71).)

$f_2(T_v, p) = (dT_v/dz)_{ad}$. (See (66).)

	$p = 1000$ mb		$p = 700$ mb	
T_v	f_1	f_2	f_1	f_2
+30°C	-0.188×10^{-4}	-0.366×10^{-4}	-0.184×10^{-4}	-0.317×10^{-4}
+20	−0.184	−0.444	−0.182	−0.384
+10	−0.178	−0.546	−0.177	−0.476
0	−0.171	−0.662	−0.171	−0.592
−10	−0.164	−0.775	−0.164	−0.716
−20	−0.156	−0.863	−0.156	−0.824
−30	−0.149	−0.921	−0.149	−0.899
−40	−0.141	−0.952	−0.142	−0.942

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