# Cellular convection with nonisotropic eddys

By DURGA RAY, *Imperial College, London*

(Manuscript received June 1964, revised version April 1, 1965)

ABSTRACT

The effect on the critical Rayleigh number and thus the effects on the ratio of cell width to cell height is found due to the consideration of nonisotropic properties of the eddy viscosities and conductivities in atmospheric cellular convection.

## Introduction

The classical theory of cellular convection in horizontal layers of fluid heated from below has been treated both experimentally and theoretically chiefly by BENARD (1901), RAYLEIGH (1916), JEFFREYS (1926), PELLOW & SOUTHWELL (1940).

In all those calculations, models suitable for the laboratory have been considered. Further, it has been supposed that the kinematic viscosity is either a constant or a function of temperature only and the diffusivity $K$ is a constant.

These theories and the experiments are less realistic from the meteorological point of view because the physical conditions supposed or imposed have not represented the conditions which actually make regular convection patterns possible in the atmosphere.

## Diffusion in the atmosphere

The differential equation which has been the starting point of most mathematical treatments of diffusion from sources is a generalization of the classical equation for conduction of heat in a solid. In the case of a turbulent fluid with the concentration of suspended matter (mass per unit volume) at any point, $(x, y, z)$ denoted by $K$, it is written as follows:

$$\frac{\partial x}{\partial t}=\frac{\partial}{\partial x}\left(K_x\frac{\partial x}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial x}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial x}{\partial z}\right). \quad (1)$$

It will be noted that this equation allows for differences in the eddy diffusivities in the component directions, i.e. for anisotropic diffusion, and also for spatial variations of these diffusivities.

If the $K$'s are constants, independent of $x$, $y$ or $z$, the simplified equation and the type of diffusion implied are usually referred to as Fickian. To some extent Sutton studied this Fickian equation. Comparison of the expressions with experimental data on diffusion in the atmosphere has from the beginning consistently shown that the equivalent values of $K$ vary systemically with the time of travel, with position and with the scale of diffusion process.

This has led to much preoccupation with the solving of equation (1) with the $K$'s variable.

## Non-isotropic $K$

As remarked by PRIESTLEY (1962), in the cellular patterns the ratio of cell width to cell height is independent of the physical properties of the fluid according to the classical theory (6) but this ratio is different in the case of cloud cells.

This is because the braking role played by molecular conductivity and viscosity is taken over by their turbulent counterparts, which are not necessarily isotropic.

In this paper we will mainly investigate the critical Rayleigh number and thus the ratio of cell width to cell height taking the nonisotropic properties of the viscosities and conductivities into consideration. The linearised part of the classical hydrodynamical equation and the BOUSSINSEQ' (1903) approximations are considered in our calculation.

## Basic equations

If the turbulent viscosity and conductivity are independent of space and taking different constant values in different directions, then with the BOUSSINSEQ' approximations the equations of motion in the $i$ direction and the equations of continuity can be written as

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\varrho}\frac{\partial p}{\partial x_i} + \nu_j \frac{\partial^2 u_i}{\partial x_j^2} - \frac{g\varrho}{\varrho_0}\lambda_i \quad (1.1)$$

and
$$\frac{\partial u_i}{\partial x_i} = 0, \quad (1.2)$$

where $j = 1, 2, 3$ corresponds to the $x$, $y$, $z$ directions respectively. $\lambda_i$ is the unit vector along the $x$ axis. The $u$'s are the velocity components $u$, $v$, $w$ along the $x$, $y$, $z$ direction and the viscosities corresponding to those directions are $\nu_1$, $\nu_2$, $\nu_3$ respectively. $p$ the pressure, $\varrho$ the density, $\varrho_0$ the standard density of the fluid.

The equations corresponding to the conduction of heat and state are given as

$$\left(\frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x}\right)\theta = K_j \frac{\partial^2}{\partial x_j^2}\theta \quad (1.3)$$

and
$$g\varrho = g\varrho_0(1 - d\theta), \quad (1.4)$$

where
$$\theta = \mathbf{H} + \theta. \quad (1.5)$$

$\mathbf{H}$ is the undisturbed temperature and $\theta$ the disturbance of temperature.

Initially the motion is steady and the velocities are all zero. Therefore from equation (1.3) we get

$$K_3 \frac{\partial^2 \Theta}{\partial z^2} = 0, \quad (1.6)$$

i.e.
$$\Theta = \Theta_0 - \beta z, \quad (1.7)$$

so $\beta$ is the negative temperature gradient.

Therefore neglecting non-linear terms, equation (1.3) can be written as

$$\left(\frac{\partial}{\partial t} - K_j \frac{\partial^2}{\partial x_j^2}\right)\theta = -\left(\frac{\partial}{\partial t} - K_3 \frac{\partial^2}{\partial z^2}\right)\Theta + w\beta. \quad (1.8)$$

Following the usual procedure of eliminating pressure from (1.1) with the use of (1.2) and (1.4) we get,

$$\left(\frac{\partial}{\partial t} - \nu_1 \frac{\partial^2}{\partial x^2} - \nu_2 \frac{\partial^2}{\partial y^2} - \nu_3 \frac{\partial^2}{\partial z^2}\right)\nabla^2 w = g\alpha \nabla_1^2 \theta, \quad (1.9)$$

where $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

From (1.8) and (1.9) eliminating $\theta$, the steady part of the equations is

$$\left(K_1 \frac{\partial^2}{\partial x^2} + K_2 \frac{\partial^2}{\partial y^2} + K_3 \frac{\partial^2}{\partial z^2}\right)$$

$$\times \left(\nu_1 \frac{\partial^2}{\partial x^2} + \nu_2 \frac{\partial^2}{\partial y^2} + \nu_3 \frac{\partial^2}{\partial z^2}\right)\nabla^2 w = g\alpha\beta \nabla_1^2 w. \quad (1.10)$$

Assuming that $K_1 = K_2 = mK_3$

and
$$\nu_1 = \nu_2 = n\nu_3, \quad (1.11)$$

(1.10) may be written as

$$\left(m\nabla_1^2 + \frac{\partial^2}{\partial z^2}\right)\left(n\nabla_1^2 + \frac{\partial^2}{\partial z^2}\right)\nabla^2 w = \frac{g\alpha\beta}{K_3 \nu_3}\nabla_1^2 w, \quad (1.12)$$

where $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

If $w \sim f(xy)\, W(z)$, where $(\nabla_1^2 + a^2/h^2) f = 0$, then we write $\zeta = z/h$, $D = \partial/\partial\zeta$.

Thus the dimensionless form of equation (1.12) can be written down as

$$[(D^2 - ma^2)(D^2 - na^2)(D^2 - a^2) + \Lambda^3 a^6]\, W = 0, \quad (1.13)$$

where $\Lambda^3 a^4$ is the dimensionless Rayleigh number,

$$\frac{g\alpha\beta h^4}{K_3 \nu_3} \quad \text{when} \quad \Lambda^3 = \frac{g\alpha\beta h^4}{K_3 \nu_3 a^4}.$$

## Case of two free surfaces

Taking the origin at midway between the two horizontal surfaces the boundaries are given $\zeta = \pm\frac{1}{2}$.

As the boundaries are free boundary conditions to be satisfied are

$$W = D^2 W = \ldots = D^{2m} W = 0 \text{ (see appendix).}$$

Let
$$W = A \cos p\pi\zeta. \quad (2.1)$$

Substituting from (2.1) in the differential equation (1.13) we get

$$\Lambda^3 = \left(m + \frac{p^2\pi^2}{a^2}\right)\left(n + \frac{p^2\pi^2}{a^2}\right)\left(1 + \frac{p^2\pi^2}{a^2}\right). \quad (2.2)$$

Since $p$ is a positive integer.

Obviously $p = 1$ gives minimum $\Lambda^3$ depending on $m$ and $n$;

$$\therefore R = \Lambda^3 a^4 = \left(ma + \frac{\pi^2}{a}\right)\left(na + \frac{\pi^2}{a}\right)(\pi^2 + a^2). \quad (2.3)$$

The minimum Rayleigh number is given by

$$\frac{\partial R}{\partial a} = 0,$$

i.e. $$4\,mnb^3 + 2\pi^2 b^2(m+n+mn) - 2\pi^6 = 0, \quad (2.4)$$

where $$b = a^2. \quad (2.5)$$

The cubic equation (2.4) has only one positive real root.

The root required of the equation (2.4) is given by,

$$b = \pi^2\left\{\frac{1}{4mn} - \frac{(m+n+mn)^3}{216m^3n^3} + \sqrt{\frac{1}{16m^2n^2}\left[1 - \frac{(m+n+mn)^3}{27m^2n^2}\right]}\right\}^{\frac{1}{3}} + \pi^2\left\{\frac{1}{4mn} - \frac{(m+n+mn)^3}{216m^3n^3} - \sqrt{\frac{1}{16m^2n^2}\left[1 - \frac{(m+n+mn)^3}{27m^2n^2}\right]}\right\}^{\frac{1}{3}} - \frac{\pi^2}{6mn}(m+n+mn). \quad (2.6)$$

$m = n = 1$ gives $b = \pi/2$, the value obtained by Pellew and Southwell.

## Numerical result

To find the numerical values of $b$ for different values of $m$, $n$ we have to do a little computation. Let

$$A = -\left\{\frac{1}{4mn} - \frac{(m+n+mn)^3}{216m^3n^3}\right\} \quad (3.1)$$

and $$B = -\left\{\frac{1}{16m^2n^2}\left[1 - \frac{(m+n+mn)^3}{27m^2n^2}\right]\right\}.$$

For any positive values of $m$ and $n$, $B^2$ is positive.

Let $$\frac{1}{6mn}(m+n+mn) = C \quad \text{say}, \quad (3.2)$$

then $$b = \pi^2[(-A \pm iB)^{\frac{1}{3}} + (-A \mp iB)^{\frac{1}{3}}] - C\pi^2. \quad (3.3)$$

Let $-A \pm iB = \mathrm{R}e^{i\phi}$,

then $$b = \left(2R^{\frac{1}{3}}\cos\frac{\phi}{3} - C\right)\pi^2, \quad (3.4)$$

where $$R^2 = A^2 + B^2 \quad \text{and} \quad \tan\phi = -\frac{B}{A}. \quad (3.5)$$

Taking the principal value of $\phi$, computation has been done for different values of $m$ and $n$, which are given at the end.

According to KRUGER & FRITZ (1961) in the cellular cloud patterns revealed by satellites the diameter to depth ratio is sometimes around thirty.

From the computational result which is given in Table 1 we see that the diameter to depth ratio i.e. $2\pi/a = (2\pi/\sqrt{b})$ of the convection cells becomes round thirty when

$$m = 500,\ n = 100,$$

or $$m = 1\,000,\ n = 50,$$

$$m = 25\,000,\ n = 1,$$

$m = \infty$, $n = 1$ gives $b = 0$.

## Case of rigid boundaries

The boundary conditions to be satisfied in this case are

$$W = DW = (D^2 - a^2)(D^2 - na^2)W = 0 \quad \text{at} \quad \zeta = \pm\tfrac{1}{2}; \quad (4.1)$$

obviously $W$ will be the superposition of

$$e^{\pm q\zeta}; \quad (4.2)$$

substituting (4.2) in the differential equation (1.13) we get

$$[(q^2 - ma^2)(q^2 - na^2)(q^2 - a^2) + \Lambda^3 a^6] = 0, \quad (4.3)$$

$q_0$, $q_1$, $q_2$ are the three values of $q$ of equation (4.3) and they will be expressed in terms of $m, n, \Lambda$ & $a$.

TABLE 1. *The ratio of cell width to cell height for different values of m and n in the case of free boundaries.*

| $m$ or $n$ | $m$ or $n$ | $b$ | $2\pi/\sqrt{b}$ |
|---|---|---|---|
| 1 | 1 | 4.928 | 2.8 |
| 1 | 50 | 0.9388 | 6.49 |
| 1 | 100 | 0.674 | 7.6 |
| 1 | 500 | 0.3072 | 11.33 |
| 1 | 10,000 | 0.06954 | 23.8 |
| 1 | 25,000 | 0.0438 | 30.02 |
| 1 | 27,500 | 0.0419 | 30.68 |
| 1 | 30,000 | 0.0402 | 31.33 |
| 1 | 50,000 | 0.0312 | 35.5 |
| 1 | 100,000 | 0.0220 | 42.4 |
| 10 | 50 | 0.4027 | 9.91 |
| 10 | 100 | 0.2887 | 11.7 |
| 10 | 500 | 0.1314 | 17.35 |
| 10 | 1,000 | 0.09326 | 20.6 |
| 10 | 5,000 | 0.0419 | 30.7 |
| 50 | 100 | 0.1357 | 17.7 |
| 50 | 500 | 0.0614 | 25.03 |
| 50 | 1,000 | 0.0435 | 30.2 |
| 50 | 5,000 | 0.0195 | 44.8 |
| 50 | 10,000 | 0.0138 | 53.7 |
| 100 | 100 | 0.0968 | 20.2 |
| 100 | 500 | 0.0437 | 30.06 |
| 100 | 1,000 | 0.0309 | 35.7 |
| 0.1 | 100,000 | 0.0220 | 42.4 |
| 0.1 | 1 | 8.43 | 2.17 |
| 0.1 | 0.1 | 19.74 | 1.5 |
| 0.1 | 0.01 | 28.87 | 1.17 |
| 0.01 | 0.01 | 67.36 | 0.75 |

Thus the even solutions of $W$ can be written as

$$W = A_0 \cos q_0 \zeta + A_1 \cosh q_1 \zeta + A_2 \cosh q_2 \zeta. \quad (4.4)$$

Substituting the expression of $W$ from (4.4) in (4.1) and eliminating $A_0$, $A_1$, $A_2$ from them we get a transcendental equation relating $a$ and ($\Lambda = 3\sqrt{R/a^4}$) and it must be solved numerically by trial and error.

The method is to determine $q_0$, $q_1$, $q_2$ in terms of $\Lambda$ for a given $a$ and given $m$, $n$ and then by trial and error to determine those values of $\Lambda$ which satisfy the transcendental equation.

The same procedure can be done in the case of odd solution of $W$.

## Conclusions

It is seen from the table that as $m$ and $n$ increase, i.e. when the components of eddy transfer coefficients in the vertical direction differ much from their components in the horizontal directions the ratio of cell width to cell height also increases.

From the table we see that the result, diameter to depth ratio, of thirty can be achieved by several combinations of unequal values of $m$ and $n$.

In general a horizontal transfer coefficient of any kind that is large compared with the vertical coefficient will flatten the cells, and the amount of the flattening is indicated in Table 1.

By putting particular values of $m$ and $n$ in the transcendental equation obtained in the case of rigid boundaries, computation could be done to find the value of $m$, $n$ for different values of "$a$" from which the minimum value of $\Lambda$ ($m$, $n$, $a$) and the corresponding cell shape number "$a$" can be obtained, in the case of rigid boundaries.

We would expect the difference between the case of rigid and free boundaries to be of the same kind as in the classical isotropic case. It has been demonstrated that very flat cells correspond to very large horizontal transfer coefficients for free boundaries and the same will be the case for rigid boundaries.

These calculations would be very laborious and are not considered worthwhile.

## Appendix

On pages 435 and 436, boundary conditions are written for free and rigid boundaries. Derivation of the boundary conditions are given below.

On a rigid surface no slip occurs. On a free surface no tangential stresses act.

The condition that no slip occurs on this surface implies that not only $w$, but also the horizontal components of the velocities $u$ and $v$ vanish.

Thus $u = 0$, $v = 0$ in addition to $w = 0$ on a rigid surface. Since this condition must be satisfied for all $x$ and $y$ on the surface, it follows from the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

that $$\frac{\partial w}{\partial z} = 0 \text{ on a rigid surface.} \quad (1)$$

The conditions on a free surface are

$$P_{xz} = P_{yz} = 0,$$

where $P_{xz}$, $P_{yz}$ are the stresses on the $x$, $y$ plane. This is equivalent to the vanishing of the components $p_{xz}$, $p_{yz}$ of the viscous stress tensor

$$p_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \quad \text{and} \quad p_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right).$$

Since $w$ vanishes (for all $x$ and $y$) on the boundary surface it follows that

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \quad \text{on a free surface.}$$

Equation of continuity is differentiated again which gives

$$\frac{\partial^2 w}{\partial z^2} = 0 \quad \text{on a free surface,} \tag{2}$$

i.e. $\quad D^2 w = 0 \quad$ on a free surface.

Equation (1.9) with the use of (1.11) on page 435 can be written as,

$$(D^2 - na^2)(D^2 - a^2)\, w = \frac{g\alpha}{\nu_3} a^2 z \tag{3}$$

where $\quad \nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

and $\quad w \sim f(x,y)\, W(z)$

and $\quad \theta \sim f(x,y)\, \mathbf{z}$

and $\quad \left(\nabla_1^2 + \frac{a^2}{h^2}\right) f = 0,$

$$\zeta = \frac{z}{h}, \quad D = \frac{\partial}{\partial \zeta}.$$

Operating by the operator $\nabla_1^2$ steady part of equation (1.8) using (1.11) can be written as

$$(D^2 - ma^2)\, \mathbf{z} = -\beta a^2 W. \tag{4}$$

On the bounding surfaces $W$ and $\mathbf{z}$ are zero. From (1) and (2)

$DW = 0 \quad$ if the boundary surfaces are rigid and

$D^2 W = 0 \quad$ if the boundary surfaces are free.

Both boundary surfaces are free, in which case,

$$\left.\begin{aligned} W = (D^2 - a^2)(D^2 - na^2)\, W = 0, \quad \text{and} \\ D^2 W = 0 \quad \text{on the boundaries.} \end{aligned}\right\} \tag{5}$$

Both boundary surfaces are rigid,

$$\left.\begin{aligned} W = (D^2 - a^2)(D^2 - na^2)\, W = 0, \\ DW = 0 \quad \text{on the boundaries.} \end{aligned}\right\} \tag{6}$$

(5) shows that in the case of free boundaries

$$W = D^2 W = D^4 W = 0 \quad \text{on the boundaries.}$$

On page 435 equation (1.13) satisfied by $W$ shows that on the boundaries

$$D^6 W = 0. \tag{7}$$

From (1.13) on page 435 differentiated twice with respect to $z$, we next conclude that $D^8 W = 0$ is also zero on the boundaries.

By further differentiations of equation (1.13) we conclude successively, that all the even derivatives of $W$ vanish on the boundaries.

Thus $\quad D^{(2m)} W = 0 \quad$ on the boundaries. $\quad$ (8)

Thus for any constant value of $m$ and $n$, boundary conditions are

$$W = D^2 W = D^4 W = \ldots = D^{(2m)} W = 0 \tag{9}$$

when the boundaries are free.

From (6)

$$W = DW = (D^2 - a^2)(D^2 - na^2)\, W = 0 \tag{10}$$

when the boundaries are rigid, $n = 1$, (10) gives

$$W = DW = (D^2 - a^2)^2 W = 0, \quad \text{on the rigid boundaries.} \tag{11}$$

## REFERENCES

BENARD, H., 1901, Les tourbillons cellulaires dans une nappe liquide transportant de la chaleur par convection en regime permanent. *Ann. Chim. Phys.*, **23**, pp. 62–144.

BOUSSINEQ, J., 1903, *Théorie Analytique de la chaleur 2*, p. 172. Gauthier-Villars, Paris, 1903.

JEFFEREYS, H., 1926, The stability of a layer of fluid heated from below. *Phil. Mag.* (7), **2**, pp. 833–844.

KRUGER and FRITZ, 1961, C. V. Press.

PELLOW, A., and SOUTHWELL, R. V., 1940, On maintained convective motion in a fluid heated from below. *Proc. Roy. Soc. A*, **176**, pp. 312–343.

PRIESTLEY, C. H. B., 1962, Width–height ratio of large convection cells. *Tellus*, **14**, No. 1, pp. 123–124.

RAYLEIGH, 1916 *Phil. Mag.*, **32**, pp. 529–546. (Reprinted in collected papers, 6, 432–36).