The viscous boundary layer at the free surface of a rotating baroclinic fluid: effects due to the temperature dependence of surface tension

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ABSTRACT

A theoretical study is made of the effects due to the dependence on temperature (T) of the coefficient of surface tension (S) on the flow in the viscous boundary layer at the free surface of a baroclinic fluid which rotates rapidly about a vertical axis. Although each component of the flow is altered by an amount of the order of the corresponding ageostrophic component arising when $dS/dT = 0$ multiplied by the dimensionless parameter $C \equiv 2\,\Omega\,(dS/dT)/(g\alpha\varrho_0\nu)$ (where Ω is the rate of rotation of the system, g the acceleration of gravity, and ϱ_0, ν and α denote, respectively, the mean density, kinematic viscosity and thermal coefficient of cubical expansion of the fluid), when the interior flow is strictly geostrophic boundary layer suction remains unaffected.

1. Introduction

The theory of the steady flow in the viscous boundary layer at the free surface of a rotating baroclinic incompressible fluid has been discussed in a previous paper (HIDE, 1964; hereinafter referred to as A). In that paper, surface tension effects were ignored, this being a valid procedure when the free surface is flat and when S, the coefficient of surface tension, is independent of temperature, T, and hence of position. Stresses due to the dependence of S on T call into play additional viscous and Coriolis forces, and if $|dS/dT|$ is sufficiently large, as it may be in certain cases of practical interest in the laboratory, these forces produce significant effects on the boundary layer flow. The purpose of the present paper is to examine some of these effects.

2. The equations of the problem

Consider a region Σ (say) of that part of the bounding surface of the fluid which is free, i.e., subject to no externally applied tangential stresses. Σ is fixed in a frame of reference $\mathcal{S}$ that rotates with uniform angular velocity $\mathbf{\Omega}$ relative to an inertial frame. $|\mathbf{\Omega}|$ is assumed to be so large that, except within the viscous boundary layer, the fluid motion is everywhere geostrophic, satisfying the thermal wind equation (see (7) below, also A (1)).

Following A, we further assume that the acceleration of gravity, $\mathbf{g}$, is parallel to the axis of rotation and that the radius of curvature, $|\mathbf{g}|/\Omega^2$, of the free surface, Σ, is so much greater than any length, L, characteristic of the temperature and dynamic pressure fields in the fluid that Σ can be regarded as flat.

The assumption that ageostrophic motion is restricted to a viscous boundary layer implies that $L \gg \delta$, where

$$\delta \equiv (\nu/|\mathbf{\Omega}|)^{\frac{1}{2}} \tag{1}$$

(see A (4)).

(x, y, z) are the coordinates of a general point in a Cartesian system fixed in $\mathcal{S}$, having its origin in Σ, and aligned so that the positive z axis is normal to Σ and directed into the fluid; the components of $\mathbf{\Omega}$ and $\mathbf{g}$ are then:

$$\mathbf{\Omega} = (0, 0, \Omega), \quad \mathbf{g} = (0, 0, g). \tag{2}$$

Denote by u, v and w the x, y and z components, respectively, of the relative flow velocity vector $\mathbf{u}$, by ϱ_0 the mean density of the fluid and by ϱ the density variations associated with temperature variations in the fluid. If α is the thermal coefficient of cubical expansion and the mean temperature of the fluid is, by definition, zero,

$$\varrho = \varrho_0(1 - \alpha T). \tag{3}$$

The equations governing the boundary layer flow are as given in A (see A (5), A (10) and A (11)). For the problem at hand these equations reduce to:

$$\frac{\partial^4 u}{\partial \sigma^4} + 4(u - \hat{U}) = 0, \tag{4}$$

$$v = \hat{V} - \frac{\operatorname{sgn}(\Omega)}{2}\frac{\partial^2 u}{\partial \sigma^2} \tag{5}$$

and

$$\frac{\partial w}{\partial \sigma} = -\delta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{6}$$

(see A (22) and A (23)), if $\hat{U}$ and $\hat{V}$ are the geostrophic components of u and v near $z=0$ (see A (20)) and $\sigma \equiv z/\delta$ is a dimensionless stretched coordinate (see (1)). The thermal wind equation leads to the following relations:

$$X = -\frac{g}{2\Omega\varrho_0}\left(\frac{\partial \varrho}{\partial y}\right)_{z=0}, \quad Y = \frac{g}{2\Omega\varrho_0}\left(\frac{\partial \varrho}{\partial x}\right)_{z=0}, \tag{7}$$

where $X \equiv \delta\hat{U}/\delta z$ and $Y \equiv \delta\hat{V}/\delta z$ (see A (1) and A (18)).

The boundary conditions under which (4), (5) and (6) must be solved are:

$$\left.\begin{aligned} &u = \hat{U}, \quad u = \hat{V}, \quad w = W, \quad \frac{\partial u}{\partial \sigma} = X\delta, \\ &\frac{\partial v}{\partial \sigma} = Y\delta \text{ and } \frac{\partial w}{\partial \sigma} = 0 \quad \text{on} \quad \sigma = \infty, \end{aligned}\right\} \tag{8}$$

and

$$w = 0, \ \frac{\partial u}{\partial \sigma} = CY\delta \ \text{ and } \ \frac{\partial v}{\partial \sigma} = -CX\delta \text{o n } \sigma = 0 \tag{9}$$

(see (3)), where W is the value of w at the edge of the boundary layer,

$$C \equiv \frac{2\Omega\, \mathrm{d}S/\mathrm{d}T}{\nu\varrho_0\alpha g}, \tag{10}$$

a pure number, and the other quantities are as defined above. With the exception of the second and third of equations (9) (cf. A (24b)), which state that for the tangential stress to vanish on Σ, $\nu\varrho_0\,\partial u/\partial z + (\mathrm{d}S/\mathrm{d}T)(\partial T/\partial_x)$ and $\nu\varrho_0\,\partial v/\partial z + (\mathrm{d}S/\mathrm{d}T)(\partial T/\partial y)$ must be zero on $\sigma = 0$, these boundary conditions are identical with those expressed by A (24) and A (25).

The surface tension of water decreases almost linearly with temperature from 75.6 dyne cm^{-1} at $T = 0$°C to 58.9 dyne cm^{-1} at $T = 100$°C; the mean value of $\mathrm{d}S/\mathrm{d}T$ between 10°C and 30°C is -0.152 dyne/(cm C°). For $|2\ \Omega| \sim 10$ rad/sec, as $\nu \sim 10^{-2}$ cm^2 sec^{-1}, $\alpha \sim 3 \times 10^{-4}$ (C°)$^{-1}$, and $|g| \sim 10^3$ cm sec^{-2}, in magnitude the parameter $C \sim 100$! The use of detergents to lower the coefficient of surface tension could hardly reduce $|C|$ by more than an order of magnitude.

3. The boundary layer flow

Solutions of (4), (5) and (6) under the boundary conditions (8) and (9) are:

$$\begin{aligned} u = \hat{U} + \tfrac{1}{2}\delta e^{-\sigma}[&\{C(\operatorname{sgn}(\Omega)X - Y) + X \\ &+ \operatorname{sgn}(\Omega)Y\}\cos\sigma + \{C(\operatorname{sgn}(\Omega)X + Y) - X \\ &+ \operatorname{sgn}(\Omega)Y\}\sin\sigma], \end{aligned} \tag{11 a}$$

$$\begin{aligned} v = \hat{V} + \tfrac{1}{2}\delta e^{-\sigma}[&\{C(\operatorname{sgn}(\Omega)Y + X) + Y \\ &- \operatorname{sgn}(\Omega)X\}\cos\sigma + \{C(\operatorname{sgn}(\Omega)Y - X) - Y \\ &- \operatorname{sgn}(\Omega)X\}\sin\sigma], \end{aligned} \tag{11 b}$$

and

$$w = W[1 - e^{-\sigma}(\cos\sigma + C\operatorname{sgn}(\Omega)\sin\sigma)], \tag{11 c}$$

where

$$W = -\tfrac{1}{2}\delta^2 \operatorname{sgn}(\Omega)\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right). \tag{12}$$

These equations constitute a complete description of the boundary layer flow. When surface tension effects are absent (i.e., $C = 0$), equations (11) and (12) reduce to A (26) and A (27) respectively; a detailed discussion of this case is given in A.

We have already seen that typical values of $|C|$ may exceed unity by more than an order of magnitude. According to (11), when $|C|$ is of order unity each component of the boundary layer flow is markedly affected by surface tension and when $|C| \gg 1$ surface tension effects dominate. It is noteworthy, however, that *the normal component of* **u** *at the edge of the boundary layer is unaffected by surface tension* (see (12)).

To understand the last result it will be instructive to evaluate the components (F_x, F_y) of the horizontal vector **F**, which measures the difference between the total volume flow (per unit length normal to **F** in the (x, y) plane) in the boundary layer and the (hypothetical) geostrophic volume flow associated with the pressure field in the boundary layer. Thus

$$F_x \equiv \delta \int_0^\infty (u - \hat{U})\,d\sigma, \quad F_y \equiv \delta \int_0^\infty (u - \hat{V})\,d\sigma, \quad (13)$$

where the range of integration covers the whole vertical extent of the boundary layer (see A (4), A (29)). Substitute (11) in (13) and find:

$$\begin{aligned} F_x &= \tfrac{1}{2}\delta^2 \operatorname{sgn}(\Omega)\,(CX + Y), \\ F_y &= \tfrac{1}{2}\delta^2 \operatorname{sgn}(\Omega)\,(CY - X). \end{aligned} \quad (14)$$

As W, the value of w at the edge of the boundary layer, must, by (6) and (13), satisfy

$$W = -\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right), \quad (15)$$

it follows from (14) that W is independent of C (see (12)) because $\partial X/\partial x + \partial Y/\partial y = 0$ (cf. (7)). Hence, the result that boundary layer suction remains unaltered when the temperature dependence of surface tension is taken into account is a direct consequence of the assumption that the interior flow is strictly geostrophic.

If we make use of (1), (3), (7) and (10), (14) may be re-written as follows:

$$\left.\begin{aligned} F_x &= \frac{g\nu}{4\Omega^2 \varrho_0}\left(\frac{\partial \varrho}{\partial x}\right)_{z=0} - \frac{\mathrm{d}S/\mathrm{d}T}{2\Omega\varrho_0}\left(\frac{\partial T}{\partial y}\right)_{z=0}, \\ F_y &= \frac{g\nu}{4\Omega^2 \varrho_0}\left(\frac{\partial \varrho}{\partial y}\right)_{z=0} + \frac{\mathrm{d}S/\mathrm{d}T}{2\Omega\varrho_0}\left(\frac{\partial T}{\partial x}\right)_{z=0}. \end{aligned}\right\} \quad (14')$$

The contribution involving ν corresponds to the ageostrophic flow down the pressure gradient. The other contribution to $\mathbf{F}$ corresponds to the additional geostrophic flow required to balance the stress arising as a result of the temperature dependence of surface tension.

4. Conclusion

In conclusion, consider the range of validity of the foregoing analysis.

The *assumption that the interior flow is strictly geostrophic* requires that both the Rossby number,

$$R \equiv Q/L|\Omega| \quad (16)$$

(where Q is a typical relative flow speed) and the Ekman number,

$$E \equiv \nu/L^2|\Omega| = \delta^2/L^2 \quad (17)$$

(see (1)), should vanish. When both E and R are much less than unity, but at least one of these parameters is non-zero, the flow is then quasi-geostrophic.

In treating the boundary layer flow only viscous and surface tension effects were taken into account. Hence, in addition to the assumptions that $E < 1$ and $R < 1$, the equations leading to (4) and (5) contain implicitly the further assumption that non-linear *inertial forces* are negligible in comparison with viscous forces. A direct calculation, based on (11), of the order of magnitude of the neglected terms shows that the last assumption is valid only when

$$R \max(1, C^2E) < E^{\frac{1}{2}} \max(1, |C|).$$

Hence, the conditions under which equations (11) constitute an accurate description of the boundary layer flow are as follows:

$$R < 1, \quad E < 1 \quad (18a)$$

and

$$\begin{aligned} &R < E^{\frac{1}{2}} \text{ when } |C| < 1, \\ &R < |C|E^{\frac{1}{2}} \text{ when } E^{-\frac{1}{2}} > |C| > 1, \\ &R < (|C|E^{\frac{1}{2}})^{-1} \text{ when } |C| > E^{-\frac{1}{2}}. \end{aligned} \quad (18b)$$

These conditions are realized in certain laboratory studies.

When (18a) is satisfied but (18b) is not, although the interior flow is still quasi-geostrophic, non-linear inertial forces, rather than viscous forces, determine the boundary layer flow. This case is mathematically more difficult to treat than the one analysed above, and lies beyond the scope of the present paper.

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