On the airflow over mountains with gentle slopes[1]

By Z. PETKOVŠEK, *University of Ljubljana, Department of Meteorology*, and
M. RIBARIČ,[2] *Argonne National Laboratory, Argonne, Illinois*

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ABSTRACT

The nonlinear equations for the two-dimensional small scale steady-state flow of a compressible fluid are put in a form appropriate for the treatment of streamlines with gentle slopes. For the limiting case when the slopes of the streamlines are negligibly small, these equations are solved numerically and the obtained results compared with the solutions of the linearized equations to determine the limits within which the linearization is permissible. Few numerical examples are given, to demonstrate the difference between solutions obtained by linearized and nonlinearized models of the atmosphere. It was found that the restriction that the total velocity of the wind be positive plays a decisive role in determining the maximum thickness of the air layer which is able to pass over a mountain of certain height.

I. Introduction

To study the development of weather conditions and flow formations caused by passage of air masses over a mountain, it is necessary to calculate the streamlines above it. In the past, this problem has commanded a considerable amount of interest in connection with mountain waves. A comprehensive survey of the relevant observation data and of the corresponding linear theory is given in the monograph (QUENEY 1960), cf. also Döös (1961, 1962) and SAWYER (1959, 1960). These theoretical results obtained hitherto are due to the linearization of the underlying equations adequate only for the study of mountain waves caused by airflow over a low mountain.

In order to study nonlinear small scale effects of mountain barriers we shall:

(i) develop appropriate theory applicable to mountains with gentle slopes, and
(ii) study the streamline field in a few cases for mountains with extremely gentle slopes.

II. Basic assumptions of our theory

(i) The airflow over the mountain is steady and nonturbulent.
(ii) There is no transfer of energy across or along the streamlines; i.e., the process is adiabatic.
(iii) The flow is two-dimensional; i.e., it depends on the vertical coordinate z and only one horizontal coordinate x.
(iv) The external forces have a two-dimensional potential $\phi = \phi(x,z)$.
(v) The mountain and corresponding streamlines above it have very gentle slopes.
(vi) The air is assumed to be unsaturated; i.e., it is an ideal gas.
(vii) The effect of the earth rotation is neglected.

III. Basic equations of the problem

1. Derivation of the relevant differential equations

In order to obtain differential equations for the field of streamlines above the mountain, we proceed as follows:

We introduce the coordinate system (x,z) as shown in Fig. 1, and identify each streamline by its height ζ_0 at $x = 0$. Now we may define

$$\zeta = \zeta(x, \zeta_0) \tag{1}$$

as the height of the streamline ζ_0 at x, so that

$$\zeta_0 = \zeta(0, \zeta_0).$$

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[2] On leave from the "J. Stefan" Institute, Ljubljana, Yugoslavia.

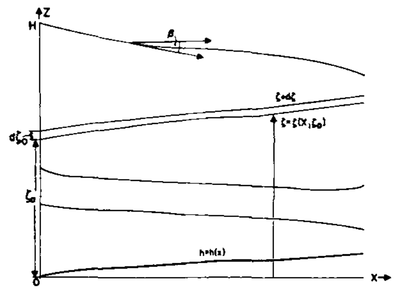

FIG. 1. Streamlines over a mountain, obtained by solving eq. (26) for the initial conditions $\Delta\nu(x, \zeta_0 = H) = 0$ and $-0.15 \leqslant \Delta\pi(x, \zeta_0 = H) \leqslant 0$, with $H = 1.4\ RT_0/g$, $T_0 = 273°\mathrm{K}$, $v_0 = 20$ m/s. All amplitudes $\Delta\zeta$ (29) including the profile of the mountain $\Delta\zeta(\zeta_0 = 0)$ have been drawn magnified by a factor of 2 as compared with the scale for ζ_0.

Let the height h of the mountain be given by the equation

$$h = h(x). \tag{2}$$

Now let us consider the air traveling through the infinitely thin tube defined by the streamlines ζ_0 and $\zeta_0 + d\zeta_0$, $d\zeta_0 > 0$; i.e., by

$$\left.\begin{aligned} z &\geqslant \zeta = \zeta(x, \zeta_0) \quad \text{and} \\ z &\leqslant \zeta + d\zeta = \zeta + \frac{\partial}{\partial \zeta_0}\zeta(x, \zeta_0)\, d\zeta_0. \end{aligned}\right\} \tag{3}$$

Since we assume that the process is adiabatic, frictionless, steady-state, and the air an ideal gas, the state of the air in the tube and the position of the streamline $\zeta(x, \zeta_0)$ are determined by:

(a) *The Bernoulli equation*

$$\tfrac{1}{2} v_0^2 + \phi_0 + c_p T_0 = \tfrac{1}{2} v^2 + \phi + c_p T, \tag{4}$$

where

$$v = v(x, \zeta_0), \quad T = T(x, \zeta_0), \quad \phi = \phi(x, \zeta_0)$$

and

$$v_0 = v(0, \zeta_0), \quad T_0 = T(0, \zeta_0), \quad \phi_0 = \phi(0, \zeta_0)$$

are initial velocity, temperature and potential energy of the air at $x = 0$.

(b) *The equations of state*

$$\left.\begin{aligned} &T/T_0 = \pi^{1-1/\varkappa}, \quad p = R\varrho T, \quad \text{where} \\ &\pi = \pi(x, \zeta_0) = p/p_0 = p(x, \zeta_0)/p(0, \zeta_0) \\ &\qquad \text{and} \quad \varrho = \varrho(x, \zeta_0) \end{aligned}\right\} \tag{5}$$

with $\varkappa = c_p/c_v = 1.404.$

(c) *The continuity equation*

$$\sigma_0 \varrho_0 v_0 = \sigma \varrho v, \tag{6}$$

where $\sigma = \sigma(x, \zeta_0)$ is the area of the opening of the stream tube (3). So that

$$\left.\begin{aligned} \sigma_0 &= \sigma(0, \zeta_0) = d\zeta_0 \\ \text{and} \qquad \sigma(x, \zeta) &= \cos\beta \frac{\partial \zeta(x, \zeta_0)}{\partial \zeta_0}\, d\zeta_0, \end{aligned}\right\} \tag{7}$$

where

$$\mathrm{tg}\,\beta(x, \zeta_0) = \frac{\partial \zeta(x, \zeta_0)}{\partial x}. \tag{8}$$

(d) *The vertical component of the Euler equation of motion*

$$\frac{d}{dt}(v \sin\beta) = -\varrho^{-1}\frac{\partial p}{\partial z} - \frac{\partial \phi}{\partial z}, \tag{9}$$

where

$$\frac{d}{dt} = v\cos\beta \frac{\partial}{\partial x} + v \sin\beta \frac{\partial}{\partial z} \tag{10}$$

and

$$\frac{\partial}{\partial z} = \left(\frac{\partial \zeta}{\partial \zeta_0}\right)^{-1} \frac{\partial}{\partial \zeta_0}. \tag{11}$$

Eliminating T and ϱ from the above equations, we get the basic equations of our problem:

$$\frac{\partial \zeta}{\partial \zeta_0} = \nu^{-1} \pi^{-1/\varkappa} \cos^{-1}\beta \tag{12}$$

and

$$\frac{\partial \pi}{\partial \zeta_0} = -\left[\frac{\partial \phi}{\partial z} + \frac{d}{dt}(v \sin\beta)\right] \Big/ (\nu R T_0 \cos\beta) - \pi \frac{d \ln p_0}{d\zeta_0} \tag{13}$$

with

$$\nu = \frac{v}{v_0} = \left[1 - 2v_0^{-2}(\phi - \phi_0) + 2v_0^{-2} c_p T_0 (1 - \pi^{1-1/\varkappa})\right]^{\frac{1}{2}}, \tag{14}$$

where β and d/dt are given by (8) and (10), respectively.

2. THE BOUNDARY CONDITIONS

Two boundary conditions, namely

$$\pi(0, \zeta_0) = 1, \quad \zeta_0 = \zeta(0, \zeta_0) \quad \text{for} \quad \zeta_0 \geqslant 0 \tag{15}$$

and

$$\zeta(x, 0) = h(x) \quad \text{for} \quad x \geqslant 0, \tag{16}$$

are uniquely determined by the nature of the problem. The third boundary condition, which determines the behavior of $\pi(x,\zeta_0)$ and $\zeta(x,\zeta_0)$ on some streamline $\zeta_0 = H$, depends on the meteorological situation; and as yet it is not known what kind one encounters most often in nature. In the examples, we are going to examine two cases for the boundary condition:

(i) We assume that some streamline, say $\zeta_0 = H$, is a straight line, i.e.,

$$\zeta(x,H) = H, \quad \text{for} \quad x \geqslant 0. \tag{17}$$

(ii) We assume $v_0(0,\zeta_0) \equiv 0$ for $\zeta_0 > H$; then $v(x,\zeta_0) \equiv 0$ for $\zeta_0 > H$; and from (4), (5) we infer that $\phi(x,\zeta)$ and $\pi(x,\zeta)$ are for $\zeta_0 > H$ related by

$$\phi - \phi_0 = c_p T_0(1 - \pi^{1-1/\varkappa}).$$

Since ζ and π are continuous functions of ζ_0 we get the boundary condition

$$\phi(x,\zeta(x,H)) - \phi(0,H) = c_p T_0(1 - \pi^{1-1/\varkappa}(x,H)), \quad \text{for} \quad x \geqslant 0. \tag{18}$$

It follows from the physical situation that boundary conditions (15), (16) together with either one of (17), (18) should uniquely determine the solution of the partial differential equations (12), (13).

IV. Methods of solution of the basic differential equations

When the mountain has very gentle slopes, i.e., when

$$\frac{d}{dx}h(x) = \beta(x,0) < 1, \quad x \geqslant 0,$$

one expects also that $\beta(x,\zeta_0) < 1$ for all $0 \leqslant \zeta_0 \leqslant H$ and $x \geqslant 0$. If such is the case we may hope that by putting $\beta \equiv 0$ in eqs. (12), (13) we obtain equations the solutions of which, say $\zeta_h^{(0)}$, $\pi_h^{(0)}$, are useful approximations to the exact solutions of the equations (12), (13) with $\beta(x,0) < 1$.

Equations (12), (13) with $\beta \equiv 0$ form a system of first-order ordinary differential equations and can easily be solved numerically, since one independent variable, x, appears only as a parameter in ϕ and in the boundary conditions on the lower ($\zeta_0 = 0$) and upper ($\zeta_0 = H$) boundary streamlines. So we have a boundary value problem for ordinary differential equations. However, if ϕ and the upper boundary condition are independent of x, cf. as is (17) or (18), then we can convert the boundary value problem into a numerically easier initial value problem as follows:

(i) We note that in this case

$$\left.\begin{aligned} \zeta_{h_1}^{(0)}(x,\zeta_0) &= \zeta_h^{(0)}(\psi(x),\zeta_0) \quad \text{and} \\ \pi_{h_1}^{(0)}(x,\zeta_0) &= \pi_h^{(0)}(\psi(x),\zeta_0), \end{aligned}\right\} \tag{19}$$

(where $\psi(x) \geqslant 0$, $x \geqslant 0$, and $\psi(0) = 0$) satisfy the same differential equations and the same upper boundary condition as $\zeta_h^{(0)}$, $\pi_h^{(0)}$; on the lower boundary streamline $\zeta_{h_1}^{(0)}$ satisfies the boundary condition.

$$\zeta_{h_1}^{(0)}(x,0) = h_1(x) = h(\psi(x)). \tag{20}$$

(ii) We introduce an additional auxiliary upper boundary condition $\pi(x,H) = \pi_H(x)$, which together with the original upper boundary condition defines an initial value problem on the upper boundary streamline $\zeta_0 = H$. Let $\pi_{h_2}^{(0)}(x,\zeta_0)$, $\zeta_{h_2}^{(0)}(x,\zeta_0)$ be a solution of this auxiliary initial value problem, and let

$$h_2(x) = \zeta_{h_2}^{(0)}(x,0). \tag{21}$$

Now, if $\pi_H(x)$ is chosen so that $\max_{0 \leqslant x} h_2(x) \geqslant \max_{0 \leqslant x} h(x)$ we may define a function $\psi_1(x) \geqslant 0$ such that

$$h(x) = h_2(\psi_1(x)); \tag{22}$$

and we obtain the solution of the original boundary problem as

$$\pi_h^{(0)}(x,\zeta_0) = \pi_{h_2}^{(0)}(\psi_1(x),\zeta_0)$$

$$\text{and} \quad \zeta_h^{(0)}(x,\zeta_0) = \zeta_{h_2}^{(0)}(\psi_1(x),\zeta_0). \tag{23}$$

Let us note that Fig. 1 is obtained using such procedure.

Using $\pi_h^{(0)}$ and $\zeta_h^{(0)}$, higher approximations to π and ζ can be obtained; e.g.,

(i) From $\zeta_h^{(0)}$ one can compute a second approximation, say $\beta^{(1)}$, for β; the first one is $\beta^{(0)} \equiv 0$. Then substituting $\beta^{(1)}$ for β in eqs. (12), (13) one again obtains a system of ordinary first-order differential equations, solutions $\zeta_h^{(1)}$, $\pi_h^{(1)}$ of which are higher approximations to the exact solutions ζ, π. Now repeating this procedure one obtains higher approximations to π, ζ. These approximations $\pi_h^{(i)}$, $\zeta_h^{(i)}$ need not converge to π, ζ, but it seems that they are at least asymptotic approximations of the order

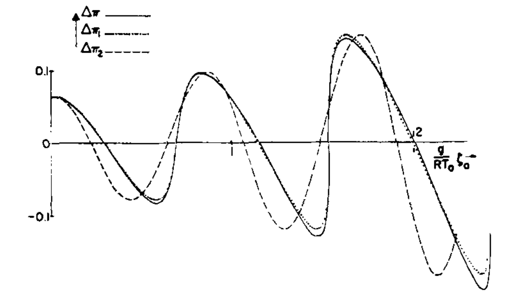

FIG. 2. Relative change of the pressure $\Delta\pi(\zeta_0)$ versus the initial height ζ_0 of the streamline when the initial temperature $T_0 = 273°\mathrm{K}$, velocity $v_0 = 20$ m/s and the lower boundary conditions are $\Delta\pi(\zeta_0 = 0) = 0.0625$, $\Delta\nu(\zeta_0 = 0) = 0$. Values of $\Delta\pi$ are solutions of the nonlinear eq. (26), those of the approximations $\Delta\pi_1$, $\Delta\pi_2$ are solutions of the corresponding linearized equations.

$$\left(\max_{0\leqslant x}\left|\frac{d}{dx}h(x)\right|\right)^i$$

to the exact solution π, ζ, by construction.

(ii) Another possibility is to introduce new independent variables $\Delta\pi = \pi - \pi_h^{(0)}$, $\Delta\zeta = \zeta - \zeta_h^{(0)}$. Now linearizing equations (12), (13) with respect to $\Delta\pi$, $\Delta\zeta$ we would obtain linear partial differential equations yielding first approximations for $\Delta\pi$, $\Delta\zeta$. Higher approximations to $\Delta\pi$, $\Delta\zeta$ could be obtained by the so-called Perturbation–Iteration scheme recently used for the numerical treatment of a similar problem, cf. KRISHNAMURTI (1964).

V. Example

Let us consider the case then

$$\left.\begin{aligned} T_0 &= T(0,\zeta_0) = \text{const},\\ v_0 &= v_0\,(0,\zeta_0) = \text{const},\\ \phi(x,z) &= gz - \phi(x), \quad \phi(0) = 0,\\ \text{and}\qquad \beta &= 0, \end{aligned}\right\} \tag{24}$$

where the potential $\phi(x)$ can be due to the horizontal pressure gradient, while we neglect earth rotation. Then it follows from (13), (14), (15), (24) that

$$\frac{d\ln p_0}{d\zeta_0} = -\frac{g}{RT_0}, \tag{25}$$

and, inserting $\beta = 0$ into eqs. (12), (13), (14), one obtains after a short manipulation, the following system

$$\left.\begin{aligned} \frac{d\pi}{d\zeta} &= \pi\nu - 1,\\ \frac{d\nu}{d\zeta} &= q(1-\pi^{1-1/\varkappa}), \quad q = RT_0 v_0^{-2}. \end{aligned}\right\} \tag{26}$$

And we have

$$\zeta_0 = RT_0/g\int_0^{\xi}\nu\,d\xi, \tag{27}$$

$$-\frac{c_p}{g}\frac{dT}{dz} = 1-\pi\nu \tag{28}$$

and
$$\Delta\zeta = \frac{\varkappa}{q(\varkappa-1)}\frac{d\nu}{d\xi} - \frac{1}{2q}(\nu^2-1), \tag{29}$$

where
$$\Delta\zeta = \frac{g}{RT_0}(\zeta-\zeta_0) - \frac{1}{RT_0}\phi(x). \tag{30}$$

So the upper boundary condition (17) now reads

$$\Delta\zeta = -\frac{1}{RT_0}\phi(x) \quad \text{at} \quad \zeta_0 = H \tag{31}$$

and (18) simplifies into

$$\nu = 1 \quad \text{at} \quad \zeta_0 = H. \tag{32}$$

Let us note that only if $\nu > 0$ and $\pi > 0$ do the solutions represent a physically possible situation. As long as

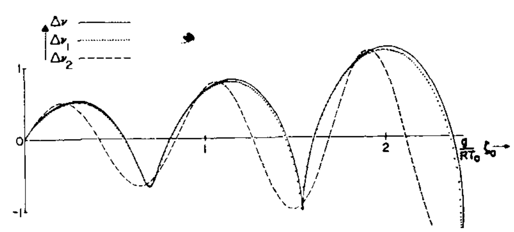

FIG. 3. Relative change $\Delta\nu(\zeta_0)$ of the wind velocity v versus the initial height ζ_0 of the streamline, when $T_0 = 273°\mathrm{K}$, $v_0 = 20$ m/s, $\Delta\pi(\zeta_0 = 0) = 0.0625$, $\Delta\nu(\zeta_0 = 0) = 0$. Values of $\Delta\nu$ are solutions of the nonlinear eq. (26), and those of the approximations $\Delta\nu_1$, $\Delta\nu_2$ are solutions of the corresponding linearized equations.

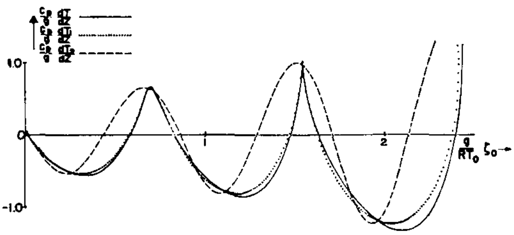

FIG. 4. Relative lapse of temperature $c_p/g\,dT/dz$ (eq. (28)) versus ζ_0. Values of $c_p/g\,dT/dz$ are calculated from exact values $\Delta\pi$, $\Delta\nu$ and those of $c_p/g\,dT_1/dz$, $c_p/g\,dT_2/dz$ from the approximate values $\Delta\pi_1$, $\Delta\nu_1$, and $\Delta\pi_2$, $\Delta\nu_2$, respectively.

$$-\frac{c_p}{g}\frac{dT}{dz}<1$$

the stratification of the flow is stable against vertical displacements. So it follows from (28) that for small displacements the stratification is stable since then $\pi\sim 1$ and $\nu\sim 1$. Moreover, it follows from (28) that stratification becomes unstable (i.e. $-c_p/g\,dT/dz\geqslant 1$) only when $\pi=0$ or $\nu=0$, which is physically impossible.

Let us now introduce new dependent variables

$$\Delta\pi=1-\pi,\quad \Delta\nu=\nu-1 \tag{33}$$

and linearize the equations (26) with respect to $\Delta\pi$ and $\Delta\nu$; we get

$$\left.\begin{aligned}\frac{d\Delta\nu}{d\xi}&\approx q\frac{\varkappa-1}{\varkappa}\Delta\pi,\\ \frac{d\Delta\pi}{d\xi}&\approx\Delta\pi-\Delta\nu.\end{aligned}\right\} \tag{34}$$

So we get that for small $\Delta\nu$, $\Delta\pi$ the function

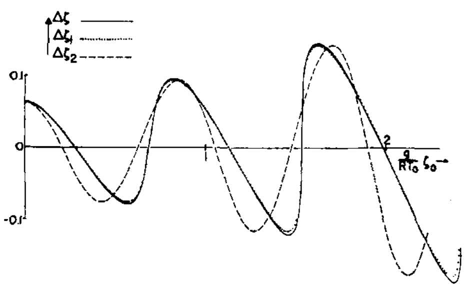

FIG. 5. Relative vertical displacements $\Delta\zeta$, $\Delta\zeta_1$, $\Delta\zeta_2$ versus ζ_0 as calculated by eq. (30) from the values of the corresponding $\Delta\pi$, $\Delta\nu$, $\Delta\pi_1$, $\Delta\nu_1$, $\Delta\pi_2$, $\Delta\nu_2$.

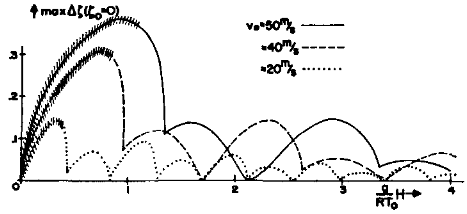

FIG. 6. Max $\Delta\zeta(\zeta_0=0)$ versus thickness H of the air column when the upper boundary condition is $\Delta\zeta(H)=0$, and $\Delta\pi(H)$ variable, for three different initial velocities v_0 of the wind: (i) $v_0=20$ m/s and $-0.0024\leqslant\Delta\pi(H)\leqslant 0.0105$; (ii) $v_0=40$ m/s and $-0.0096\leqslant\Delta\pi(H)\leqslant 0.06$; and (iii) $v_0=50$ m/s and $-0.016\leqslant\Delta\pi(H)\leqslant 0.112$. Taking for the second upper boundary condition $\Delta\pi(H)$ values outside the indicated limits would essentially increase max $\Delta\zeta(0)$ only within the segments of the curves denoted by crosshatching.

$$\Delta\nu_1(\xi)=Ae^{\xi/2}\sin\left(\sqrt{\frac{\varkappa-1}{\varkappa}q-\frac{1}{4}}\,(\xi-B)\right), \tag{35}$$

where A and B are integration constants and

$$\zeta_0=\frac{RT_0}{g}\int_0^{\xi}(1+\Delta\nu_1(\xi))\,d\xi, \tag{36}$$

is an approximation to $\Delta\nu$. How good it is can be seen from Figs. 2, 3, 4, and 5. Now if we replace ξ in (35) by $g/RT_0\,\zeta$ we obtain the true linear approximation

$$\Delta\nu_2(\zeta)=\Delta\nu_1\left(\frac{g}{RT_0}\xi\right) \tag{37}$$

to $\Delta\nu$. From Figs. 2, 3, 4, and 5 one sees that approximation $\Delta\nu_1(\zeta)$ and corresponding $\Delta\pi_1$,

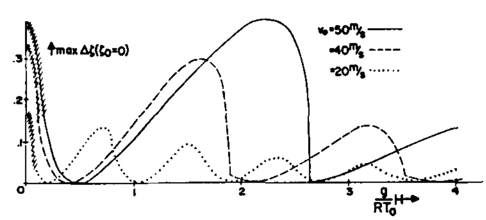

FIG. 7. Max $\Delta\zeta(0)$ versus thickness H of the air column when the upper boundary condition is $\Delta\nu(H)=0$ and $\Delta\pi(H)$ variable, for three different initial velocities v_0 of the wind: (i) $v_0=20$ m/s and $-0.195\leqslant\Delta\pi(H)\leqslant 0.135$; (ii) $v_0=40$ m/s and $-0.69\leqslant\Delta\pi(H)\leqslant 0.29$; and (iii) $v_0=50$ m/s and $-1.2\leqslant\pi\leqslant 0.4$. Taking for $\Delta\pi(H)$ values outside the above-indicated limits would essentially increase max $\Delta\zeta(0)$ only in the segments of the waves denoted by crosshatching.

$\Delta\zeta_1$, dT_1/dz are even good for a very large vertical displacement, where straight linear approximation $\Delta\nu_2$ and corresponding $\Delta\pi_2$, $\Delta\zeta_2$, dT_2/dz are less satisfactory. This is due to the fact that

(i) $\Delta\nu$ has an extremum when $\Delta\pi = 0$, cf. (26), and

(ii) the magnitude of $\Delta\nu = (v - v_0)/v_0$ increases faster as the vertical displacements increase than the magnitude of pressure changes $\Delta\pi = (p - p_0)/p_0$; so that the condition $\Delta\nu > -1$ mainly determines the height H of the air column which can still pass over a given obstacle having a certain maximum height $\max h(x)$. As can be seen from Fig. 3, for the lower boundary conditions $\Delta\nu(\zeta_0 = 0) = 0$, $\Delta\pi(\zeta_0 = 0) = 0.0625$ we have the restriction $H < H_{\max} \sim 2.\ 2RT_0/g \sim 18.5$ km, since at $gR/T_0\zeta_0 \sim 2.\ 2$ velocity v gets negative which is physically impossible. In order to demonstrate quantitatively how the condition $v \geq 0$ determines the maximum reduced amplitude $\Delta\zeta(\zeta_0 = 0)$ (30) of the lower boundary streamline we plotted in Figs. 6 and 7, max $\Delta\zeta(\zeta_0 = 0)$ versus the height H of the air column for two upper boundary conditions $\Delta\zeta(\zeta_0 = H) = 0$ and $\Delta\pi(\zeta_0 = H) = 0$.

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