SHORTER CONTRIBUTION

On the scale analysis of traveling planetary waves

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In a well-known study, BURGER (1958) analyzed the importance of the major terms in the vorticity equation, with particular attention to the planetary waves of greatest scale. He concluded that the vorticity equation becomes simply a "balance equation" equivalent to the geostrophic approximation for the largest-scale waves. The purpose of this note is to show that the vorticity equation does retain some useful predictive and diagnostic properties when applied to planetary waves.

In a recent description of the behavior of spherical-harmonic expansions of the 500 mb height (DELAND, 1965) I have shown that they exhibit very clearly a variation of speed of propagation with latitudinal scale that is consistent with the non-divergent vorticity equation (HAURWITZ, 1940), even in the case of the longest wave, wave number 1. For the largest scales, the westward-traveling waves are superimposed on the more familiar "quasi-stationary" waves. The latitudinal, as well as the longitudinal, scale should thus be taken into account in a scale analysis of planetary waves. One way of doing this is to use the spherical harmonic representation.

Consider a single spherical-harmonic wave in the 500-mb surface:

$$Z_n^l = A_n^l P_n^l(\mu) e^{il\lambda},$$

where

A_n^l is the (complex) amplitude,
P_n^l is the associated Legendre function,
$\mu = \sin\varphi$, where φ is latitude,
λ is longitude,
l is the rank(also the longitudinal wavenumber),
n is the degree.

I will follow Burger's procedure, considering the relative amplitude of the major terms in the vorticity equation.

Let us denote the amplitude of a particular harmonic function by [ ], e.g. $[Z_n^l] = A_n^l$. It can be shown that

$$[\nabla^2 Z_n^l] = \frac{n(n+1)}{a^2} A_n^l, \tag{1}$$

where $\nabla^2 \equiv \frac{1}{a^2}\left\{\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial}{\partial\mu}\right] + \frac{1}{1-\mu^2}\frac{\partial^2}{\partial\lambda^2}\right\}$

and a is the radius of the earth; so the amplitude of the (non-spherical-harmonic) geostrophic vorticity is approximately given by

$$[\zeta_n^l] \approx \frac{n(n+1)gA_n^l}{\bar{f}a^2}, \tag{2}$$

where $\bar{f}$ is some intermediate value of the Coriolis parameter $2\Omega \sin\varphi$.

The characteristic length scales are:
in the longitudinal direction

$$S_\lambda \sim \frac{a}{l} \tag{3}$$

and in the latitudinal direction

$$S_\varphi \sim \cdot \frac{a}{\partial(n-l)}. \tag{4}$$

The last follows from the fact that there are n–l zero-crossings of P_n^l between the poles.

From (3) it follows that the β-term in the vorticity equation has amplitude

$$[v\beta] \sim \frac{gA_n^l l}{a^2}, \tag{5}$$

where $\beta \equiv \delta f/\delta y$.

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From (2) and (3), the local derivative of vorticity has amplitude

$$\left[\frac{\partial \zeta}{\partial t}\right] \sim c\left[\frac{\partial \zeta}{\partial x}\right] \sim c\frac{n\,(n+1)\,gA_n^l\,l}{\bar{f}a^3} \tag{6}$$

which, from (5)

$$\sim \frac{cn\,(n+1)}{\bar{f}a}[v\beta], \tag{7}$$

where c is the speed of propagation of the waves.

Using observed speeds (Deland, 1965) of 60 and 20 degrees of longitude per day for the waves of lowest degree (broadest latitudinal scale, relation [4]) for l equal to 1 and 2 respectively, approximate magnitudes for the factor $cn(n+1)/\bar{f}a$ are found to be:

for $l = 1 (n = 2)$, .7,

and for $l = 2 (n = 3)$, .5.

Since average zonal wind speeds are of approximately equal magnitude to the wave-speeds, the rate of zonal advection of relative vorticity is also of similar magnitude to the local change.

The local rate of change and the zonal advection of relative vorticity are thus not much less than the "beta-term" in the vorticity equation. The difference between this result and Burger's arises mainly from the inclusion of the latitudinal scale, and to a smaller extent, for wave-number 1, from the relatively large wave-speed used in the above analysis.

It follows that the vorticity equation does not necessarily display a quasi-stationary character, even for the largest scales. It should therefore be useful in understanding the behavior of the planetary waves, especially their travelling components.

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