Elementary Rossby waves

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ABSTRACT

The linearized shallow water equations, when applied to a rotating annulus of homogeneous fluid with a free surface, and depth proportional to radius squared, have solutions expressible in elementary functions. The low frequency "second-class" part of the spectrum obeys an equation similar to Haurwitz's formula for Rossby waves on the earth. An approximate analysis shows that the second class oscillations correspond to inertia waves of low frequency in an annulus with a sloping *rigid* top surface. The frequency formula is verified by an experiment in which the waves are created by an oscillating paddle.

1. Background

The free oscillations of a simple rotating planetary atmosphere were analysed by MARGULES in 1893. A few years later HOUGH (1898) independently examined the equivalent problem for a global ocean of uniform depth. Both investigations showed that a special type of low-frequency oscillation can exist. Margules used the phrase "westliche Wellen zweiter Art" for these modes, while Hough called them "oscillations of the second class". Their theoretical existence has also been demonstrated as a general feature of rotating basins of variable depth (LAMB, 1932; VELTKAMP, 1960) and of a rotating stratified planetary atmosphere (ECKART, 1960). They are distinguished from the gravity waves also present in such problems by two properties: (*a*) their frequency is less than twice the angular velocity ω of the coordinate system, and (*b*) their frequency is proportional to ω in the limit $\omega \to 0$. For example, HOUGH (1898, p. 159) showed that in this limit the frequency (σ) of the second class solutions approaches $\sigma = 2\omega s\,[n(n+1)]^{-1}$, where s and $n(\leqslant s)$ are the indices of the associated Legendre functions. ROSSBY (1939) later isolated the second class modes from the gravity waves by considering horizontal non-divergent motion in a coordinate system which crudely approximated the spherical shape of the earth. The physical insight provided by Rossby's analysis, and the possibility which his model provided of analysing actual atmospheric flow patterns, has led to the name "Rossby waves" for the second class oscillations.

HAURWITZ (1940*b*) extended Rossby's model to a sphere and obtained the formula of Hough's cited above. Haurwitz also considered non-divergent flow in a zonal channel, of width W centered at a mean latitude Θ (1940*a*). In this simpler case the frequency σ satisfies the formula:[1]

$$\sigma = \frac{2\omega \cos\Theta}{R_e}\,\frac{\alpha}{\alpha^2 + n^2\pi^2 W^{-2}}, \tag{1}$$

where α is the east-west wave number, R_e is the radius of the earth, and $n = 1, 2, \ldots$ The factor $2\omega \cos\Theta$ represents the latitudinal variation of the Coriolis parameter—a result of assuming horizontal motion on a spherical earth.

The importance of the spherical shape of the earth for day-to-day atmospheric motions has been quantitatively verified in numerical weather prediction, while its significance for ocean currents has been verified in the several steady-state theories of the wind-driven circulation. Considerable interest exists now in the possible role of *transient* Rossby waves in the overall oceanic circulation (see, for example; VERONIS, 1963), stimulated no doubt by SWALLOW's observation (1955) of considerable energy in wavelengths much smaller than the width of an ocean.

[1] Haurwitz and Rossby included a basic current, which we disregard.

Experimental laboratory studies will undoubtedly be necessary to complement analysis of oceanic data in verifying theoretical analyses of possible interactions between currents and transient waves. Unfortunately previous experimental studies of Rossby waves (Fultz and Frenzen, 1955; Fultz and Kaylor, 1959) have not been performed under the most simple conditions. In the present paper I give a theoretical analysis of these waves for circumstances which are simple mathematically and experimentally. Specifically, (a) The Haurwitz formula (1) is a good approximation to the second-class solutions in an exact "shallow-water" analysis of the waves in a rotating annulus with a depth variation $h \propto r^2$; (b) The Haurwitz formula is a good approximation to certain of the *inertia waves* possible in an annulus with a sloping rigid top (i.e. no free surface). The latter point is primarily of academic interest as explaining the appearance of the second class oscillations among the tidal solutions. Finally, a preliminary experimental verification by Dr. Alan Ibbetson of (1) is presented.

2. Rotating annulus with depth proportional to r^2

Previous exact shallow water analyses of second class oscillations in a rotating liquid with a free surface have involved transcendental functions more complicated than trigonometric or exponential functions. It therefore seems worthwhile to consider a special depth profile which has simple trigonometric solutions:

$$gh = \tfrac{1}{2} B\omega^2 r^2; \tag{2}$$

where h is the undisturbed depth of liquid in the basin, ω is the rotation rate, and r is the radius. B is a positive constant, and assumes the value 1 for the case of a flat bottom with a free surface in rotational equilibrium. Since h vanishes at $r=0$, it is necessary to eliminate the origin by considering only the annular region $0 < b \leqslant r \leqslant a$.

Lamb (1932; equation 2, p. 326) gives the differential equation for the free surface perturbation ζ, where θ is the azimuthal angle and the perturbed motion is assumed to vary as $\exp\, i(s\theta + \sigma t)$. (For definiteness, we consider s to be either zero or a positive integer.) If ζ and r are replaced by p and R, where

$$p = r\zeta, \quad R = \ln\,(a/r),$$

this equation takes the form

$$(\sigma^2 - 4\omega^2)\,p + \frac{g}{r}\frac{dh}{dr}\left[-p - \frac{dp}{dR} + \frac{2\omega s}{\sigma}p\right] + \frac{gh}{r^2}\left[\frac{d^2p}{dR^2} + 2\frac{dp}{dR} + (1-s^2)\,p\right] = 0.$$

The special depth law (2) then gives constant coefficients:

$$\frac{d^2p}{dR^2} + \lambda^2 p = 0 \tag{3}$$

where λ^2 is a constant:

$$\lambda^2 = \frac{2s}{f} - 1 - s^2 + \frac{8(f^2-1)}{B}, \tag{4}$$

and $f = \sigma/2\omega$. The boundary conditions that the radial displacement vanish at the outer radius $(r=a,\ R=0)$ and inner radius

$$[r=b,\ R=R_0=\ln\,(a/b)>0]$$

are readily obtained from equation (3) on p. 321 of Lamb. After expressing these in terms of p and R and applying them to the solution

$$p = K_1 \cos \lambda R + K_2 \sin \lambda R$$

of (3), we obtain the frequency relation

$$\left(\frac{s^2}{f^2} - \frac{8}{B}\right)(1-f^2)\sin \lambda R_0 = 0.$$

The possibility $8f^2 = Bs^2$ is readily shown to give modified *Kelvin waves*. In terms of the depth h_a at the outer radius a, these solutions travel in either direction with the frequency

$$\sigma_K = \pm\frac{s}{a}(gh_a)^{\frac{1}{2}}. \tag{5}$$

λ is imaginary and ζ varies as r to the power $(-2\omega s/\sigma)$. The radial displacements vanish identically, as is typical of Kelvin waves. The second possibilitity, $f^2=1$, is not valid unless

$B = 8s^{-2}$, when it is a special case of a Kelvin wave.

The third possibility is satisfied by

$$\lambda R_0 = n\pi; \quad n = 1, 2, \text{ etc.}$$

($n = 0$ is valid only for the special case $B = 8$, when it also coincides with a Kelvin wave.) Insertion of $\lambda = n\pi/R_0$ in (4) gives a cubic equation for the non-dimensional frequency f:

$$f^3 - 2\left[E_n + B\left(\frac{s}{4}\right)^2\right] f + B\left(\frac{s}{4}\right) = 0. \qquad (6)$$

$E_n = \frac{1}{2}[1 + (B/8)(1 + n^2\pi^2 R_0^{-2})]$ contains the dependence of f on n. It can be shown that the discriminant of (6) is always negative, and there are two distinct positive roots and one negative root; $f_1 > f_2 > 0 > f_3$, say.

Consider first the case of s not equal to zero. Differentiation of (6) with respect to s shows that the two positive branches have $\partial f/\partial s = 0$ at $f = s^{-1}$. For the f_1 branch this is a minimum,

$$f_1^2 \geq E_n + (E_n^2 - B/8)^{\frac{1}{2}} \geq E_1 + (E_1^2 - B/8)^{\frac{1}{2}},$$

and for the f_2 branch this is a maximum

$$f_2^2 \leq E_n - (E_n^2 - B/8)^{\frac{1}{2}} \leq E_1 - (E_1^2 - B/8)^{\frac{1}{2}}. \qquad (7)$$

The negative root f_3 has $\partial f/\partial s \leq 0$ for all $s \geq 0$, leading to the result

$$f_3^2 > 2E_n \geq 2E_1.$$

Thus, keeping B and R_0 fixed, but letting s and n range over the positive integers, we conclude that all f_2- values are less than any value of f_1 or $|f_3|$. The minimum Kelvin frequency occurs for $s = 1$, where $f_K^2 = B/8$, and the maximum f_2^2 can also be shown to be smaller than this for finite R_0.

The maximum value of f_2^2 given by (7) increases monotonically with B (for finite R_0). Its value as $B \to \infty$ gives the following upper bound:

$$f_2^2 \leq (1 + \pi^2 R_0^{-2})^{-1}. \qquad (8)$$

Thus, for $s > 0$, the f_2-values are not only smaller in magnitude than the frequencies of any other mode for fixed B and R_0, but are such that σ_2 is less than 2ω for any value of B. These

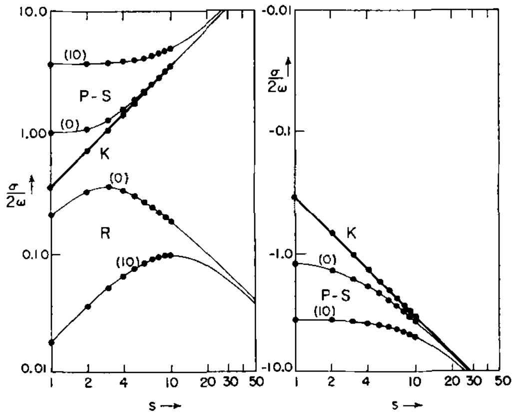

FIG. 1. Spectrum of waves in a rotating annulus of depth $h = \frac{1}{2}\omega^2 r^2/g$. Positive and negative values of $\sigma/2\omega$ are the ordinate on a logarithmic scale, azimuthal wave number s is the abscissa on a logarithmic scale. K, $P-S$ and R refer to the Kelvin, Poincaré–Sverdrup, and Rossby modes. Two sample curves are shown for the $P-S$ and the R modes, one for $n\pi/R_0 = 10$ and one for the limiting case $n\pi/R_0 = 0$. [$R_0 = \ln$ (outer radius $\div$ inner radius).] Individual points are not shown for $s > 10$.

general features are illustrated in Fig. 1 for the case: $B = 1$.[1]

The case $s = 0$ gives $f_1 = \sqrt{2E_n}$, $f_2 = f_K = 0$, and $f_3 = -\sqrt{2E_n}$. This f_2-root and the Kelvin root for $s = 0$ are both absorbed in the well-known result that $\dot r = 0$ and $2\omega r\dot\theta = gd\zeta/dr$ is a steady state solution of the linearized equations for arbitrary $\zeta(r)$ if $\partial/\partial\theta = 0$ and $h = h(r)$.

Approximate expressions for the roots of (6) can be obtained by rewriting it as

$$\varrho^3 - \varrho + \chi = 0, \qquad (9)$$

where $\varrho = f[Bs^2/8 + 2E_n]^{-\frac{1}{2}}$ and $\chi = (Bs/4)[Bs^2/8 + 2E_n]^{-3/2}$. χ^2 is bounded by $(4/27)(1 + \pi^2 R_0^{-2})^{-1}$, so that each of the three roots of (9) has a

[1] Fig. 1 has an interesting resemblance to the similar diagram for the acoustic-gravity oscillations in a non-rotating isothermal atmosphere (ECKART, 1960, Ch. VIII). In both cases, high frequencies (acoustic or Poincaré) are separated from low frequencies (internal gravity waves or Rossby waves) by a boundary-type solution (Lamb-wave or Kelvin-wave). The existence of the low frequencies also depends in both cases on non-uniformity of the medium. The dependence of σ on the vertical (or radial) wave number is also similar for the two high frequency and the two low frequency spectra. An important difference of course is that the Rossby waves move in only one direction.

convergent power series expansion in χ valid for all positive s, n, and B. The leading terms give the approximations.

$$\sigma_1^2 \approx \sigma_3^2 = \left[4\omega^2 + \frac{gh_a}{a^2}\left(1 + s^2 + \frac{n^2\pi^2}{R_0^2}\right)\right][1 + O(\chi)], \tag{10}$$

and

$$\sigma_2 = \left[\frac{4\omega s}{(8/B) + 1 + s^2 + n^2\pi^2 R_0^{-2}}\right][1 + O(\chi^2)]. \tag{11}$$

(These are exact if $s = 0$.)

The σ_1 and σ_3 solutions are similar to the Poincaré–Sverdrup gravity waves (PROUDMAN, 1952) while σ_2 is obviously a second-class (or Rossby) wave, as can be seen by comparing (11) with Haurwitz's formula (1). The σ_2-wave has a retrograde phase velocity (opposite to the direction of rotation), as is true in the tidal solutions for the earth, but this agreement depends upon the outwardly increasing depth in the present example—LAMB's second class solutions (1932, p. 327) for a basin with $h = h_0(1 - r^2a^{-2})$, for example, have forward phase velocities.

3. Rossby waves and inertia waves

One result of section 2 was that the simple second class oscillations studied there have a frequency less than 2ω. This is not unique to the depth law $h \propto r^2$; it is true of all second-class oscillations (VELTKAMP, 1960).[1] The same limitation exists for the *inertia waves* which can exist in a rotating homogeneous incompressible fluid under conditions where gravity has no effect (BJERKNES, *et al.*, 1933; GREENSPAN, 1964). It is therefore likely that the second class waves are simply a special type of inertia wave—a thought which has probably occurred to many before. I present here a brief argument that this is indeed the case. The mathematical analysis is only approximate but it has the virtue of being based on a simple physical model which is quite different from that employed by Rossby. It is significant that the shallow water hydrostatic approximation characteristic of previous "exact" analyses is not introduced *a priori* but is a result of the analysis.

Choose $(2\omega)^{-1}$ for the time scale and take the z-axis parallel to the axis of rotation of a homogeneous incompressible fluid. Let (α, β, γ) represent the wave number vector and let u, v, w be the velocity components in the x, y, and z directions. Plane inertia waves of the form $\exp i(\alpha x + \beta y + \gamma z + ft)$ satisfy the frequency relation

$$\frac{\sigma}{2\omega} = f = \delta\gamma(\alpha^2 + \beta^2 + \gamma^2)^{-\frac{1}{2}}, \quad \delta = \pm 1, \tag{12}$$

where f is the non-dimensional frequency. For definiteness we consider α, β, and γ as positive, and add together four such waves of equal amplitude and frequency, but with different vector wave numbers: (α, β, γ), $(\alpha, -\beta, \gamma)$, $(\alpha, \beta, -\gamma)$ and $(\alpha, -\beta, -\gamma)$. The resulting velocities in the y- and z-directions, apart from a factor $\exp i(\alpha x + ft)$, are:

$$v = \cos\gamma z \sin\beta y,$$

$$w = -\frac{\delta\alpha(\alpha^2 + \beta^2 + \gamma^2)^{\frac{1}{2}}}{\alpha^2 + \gamma^2} \times \sin\gamma z\left(\sin\beta y + \frac{\beta f}{\alpha}\cos\beta y\right). \tag{13}$$

These satisfy the boundary condition of zero normal velocity at the three plane surfaces $z = 0$, $y = 0$ and $y = D$ if $\beta = n\pi/D$, $n = 1, 2, \ldots$. The solutions represent waves traveling in either direction along the x-axis in a straight channel with a lateral width D (see Fig. 2).

Imagine now that a sloping top boundary is introduced:

$$z_{\mathrm{TOP}} = 1 + \varepsilon y/D. \tag{14}$$

(It is convenient to use the height of this upper surface at $y = 0$ as the length scale for x, y, z.) (13) alone will not satisfy the condition of zero normal velocity at z_{TOP}, although this could be

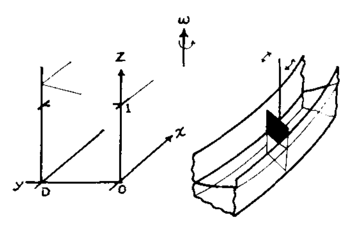

FIG. 2. Coordinate system used to show that Rossby waves are related to inertia waves. Right: Sketch of paddle generator for Rossby waves in an annulus. In practice, leakage around the paddle must be minimized for efficient generation of the waves.

[1] The approximation $W \ll R_e$ is implicit in (1).

accomplished by viewing (13) as defining only a single term in a sum over n and letting (12) define γ_n for a fixed but unknown value of f. (Reflection conditions at plane surfaces have been analysed in detail by O. PHILLIPS, 1963.)

Consider instead the more heuristic procedure of choosing γ in (13) so that the upper boundary condition is satisfied as well as possible by the single v and w field given there. Let V represent the normal component of (13) at the sloping top surface. It can be written as

$$(D^2+\varepsilon^2)^{\frac{1}{2}}V = -(I^2+J^2)^{\frac{1}{2}} \sin(\beta y+\tau),$$

where I and J are functions of y:

$$I=\delta M \sin\psi(y)+\varepsilon\cos\psi(y);\quad J=N\sin\psi(y);$$

$$\psi(y)=\gamma(1+\varepsilon y/D);\quad \tan\tau=J/I;$$

$$M=D\alpha(\alpha^2+\beta^2+\gamma^2)^{\frac{1}{2}}\div(\alpha^2+\gamma^2);\ N=D\beta\gamma\div(\alpha^2+\gamma^2)$$

The specific technique will be to minimize the mean value of I^2+J^2 over the interval $0\leqslant y\leqslant D$ by an appropriate selection of γ. In doing this we will consider only small values of ε, i.e. the percentage change in z_{TOP} is taken to be small. (The restriction that $D\sim O(\varepsilon^0)$ is also implied. This permits V, f, and γ to be expanded as a power series in ε, viz. $\gamma=\gamma_0+\varepsilon\gamma_1+\dots$. Since $v=O(1)$ in (13), it is clear that this technique will be fruitful only to the extent we are able to obtain $V<v\,dz/dy\sim\varepsilon/D$.

By requiring $V_0=0$ we find first that both $I_0=\delta M_0\sin\gamma_0$ and $J_0=N_0\sin\gamma_0$ must be zero. This is satisfied only by $\gamma_0=l\pi$, where $l=0,1,2,\dots$. (These are of course the exact solutions for $\varepsilon=0$.) The next contribution to V is given by

$$I_1^2+J_1^2=[M_0(\gamma_1+\gamma_0 y/D)+\delta]^2+N_0^2(\gamma_1+\gamma_0 y/D)^2. \tag{15}$$

Integration of this with respect to (y/D) from 0 to 1 gives $\overline{I_1^2+J_1^2}$, which is a function of γ_0, γ_1, δ, and the fixed parameters α and β. It has a minimum with respect to γ_1 at

$$\gamma_1=-\frac{\gamma_0}{2}-\frac{\delta M_0}{M_0^2+N_0^2}.$$

Insertion of this value of γ_1 into the average value of (15) gives

$$\overline{(I_1^2+J_1^2)}_{\min}=\gamma_0^2\left[\frac{1}{12}+\frac{\beta^2}{D^2(\alpha^2+\beta^2)^2}\right]$$

This in turn is minimized with respect to γ_0 by choosing $\gamma_0=0$ from the set of values $\gamma_0=l\pi$ allowed by the earlier condition $V_0=0$. V_1 is now identically zero, as is N_0. γ_1 reduces to

$$\gamma_1=-\frac{\delta}{M_0}=-\frac{\delta\alpha}{D(\alpha^2+\beta^2)^{\frac{1}{2}}},$$

and the resulting frequency is of order ε:

$$\frac{\sigma}{2\omega}=f=\frac{-\varepsilon\alpha}{D(\alpha^2+\beta^2)}+O(\varepsilon^2). \tag{16}$$

The agreement in form of this with the Haurwitz formula (1) is clear, while the vanishing of I_1 and J_1 allowed by $\gamma_0=0$ means that $V\sim\varepsilon^2$. We therefore conclude that there is an intimate relation between the second class (Rossby) waves and inertia waves. The inertia waves in question are evidently those of lowest modality in the direction parallel to $\boldsymbol{\omega}$. Consideration of the next term in the ε expansion shows that

$$DV=-\varepsilon^2(I_2^2+J_2^2)^{\frac{1}{2}}\sin(\beta y+r)$$

and the best choice of γ_2 turns out to be

$$-(2M_0)^{-1}.$$

$\overline{I_2^2+J_2^2}$ however is then equal to

$$\tfrac{1}{12}+\beta^2[D(\alpha^2+\beta^2)]^{-2}>0.$$

V therefore can no longer be kept zero to higher orders of ε by selection of γ alone, and improvement of (16) requires combination of several fields (13) with different values of $n(\beta)$.

The relation (16) can also be derived by using a theorem due to GREENSPAN (1964), in the spirit of Rayleigh's principle. The frequency of the inertia modes in a closed rigid container is related to the eigen function velocities corresponding to that frequency by

$$\frac{\sigma}{2\omega}=\frac{Im\int(u^*v-v^*u)}{\int|u|^2+|v|^2+|w|^2} \tag{17}$$

where the integration extends over the complete volume. The theorem also applies to the x-

periodic solutions appropriate to the system sketched in Fig. 2. If a steady solution of u, v, and w with $\gamma=0$ and $\beta=n\pi/D$ satisfying the "unperturbed" upper boundary condition (i.e. $\varepsilon=0$) is substituted into (17) as an approximation to the true eigenfunctions, and the integration is carried out over the "perturbed" volume defined by (14), equation (16) results.

The above analysis shows that certain of the steady inertia waves allowed in a container whose top and bottom boundaries are planes normal to the axis of rotation (namely the solutions with $\gamma=0$) are converted into non-steady solutions if one of these boundaries is inclined slightly. [This is consistent with the proof by GREENSPAN (1964) that steady inertia waves are not possible in any container where contours of equal depth are not closed curves.] These oscillations have a frequency formula similar to that for second class oscillations and have a vertical wave length larger than the depth of the container. This latter circumstance explains why these inertia waves are retained when the "shallow water" hydrostatic assumption is made in a barotropic rotating fluid.

4. Experimental verification

Perhaps the simplest generating mechanism for Rossby waves is the arrangement sketched as part of Fig. 2. A rotating annulus of water with a sloping free surface and horizontal bottom contains an oscillating paddle. The axis of the paddle is vertical and fixed in place with respect to the annulus. Small slow oscillations of the paddle about an orientation transverse to the annulus can be expected to excite oscillatory motions possessing relative vorticity about the vertical axis. This vorticity would be created by the stretching of vortex lines accompanying horizontal displacements against the radial gradient of the undisturbed depth.

The paddle will excite primarily the transverse wave number $n=1$. However, the azimuthal wave number α associated with a prescribed frequency of the paddle has two values according to (16). The proper selection of these is made by considering the group velocity $d\sigma/d\alpha$:

$$\frac{d\sigma}{d\alpha} = -2\omega\frac{\varepsilon}{D}\left(\frac{\beta^2-\alpha^2}{(\alpha^2+\beta^2)^2}\right).$$

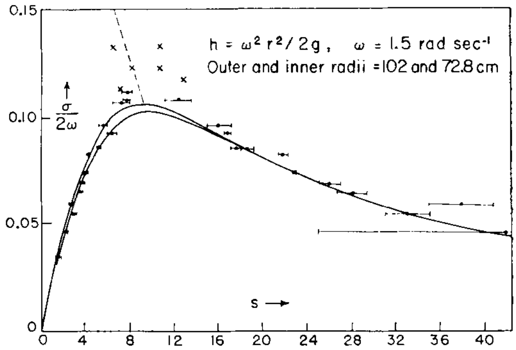

FIG. 3. Experimental verification of Rossby waves in a rotating annulus. Abscissae is the azimuthal wave number (= 360 ÷ wavelength in degrees), ordinate is $\sigma/2\omega$. Observed values are shown by crosses for some single observations, and by dots with horizontal lines to indicate the mean and variance from repeated photographs of a single experiment. Wavelengths were measured from streak photographs of aluminum powder on the surface of the water. (The paddle amplitude was $\pm$ 5 cm. Wave number 40 corresponds to an azimuthal wavelength of only 11 cm at the inner radius. The small amplitude assumption of the linear theory undoubtedly fails under these conditions and is responsible for the large variance at high wave numbers.)

The lower solid curve is the exact shallow water solution from the cubic equation (6), while the upper solid curve is from equation (16). The dashed line is the real part of the wavenumber from (16) when $\sigma > \omega\varepsilon/\pi$. *Wave numbers less than 10 were always observed on the up-rotation side of the paddle, while wave numbers greater than 10 occurred only on the down-rotation side.*

If the x-axis is chosen in the azimuthal direction along the annulus in the direction of rotation, we have

$$\varepsilon/D = dz/dy = -dz/dr \sim -\omega^2\bar{r}/g,$$

where $\bar{r}$ is the mean radius of the annulus. The group velocity is then positive (energy transport in the negative x-direction) for $\alpha^2 < \beta^2$, and is negative (energy transport in the positive x-direction) for $\alpha^2 > \beta^2$. Long waves will therefore appear on the up-rotation side of the paddle, while shorter waves will appear on the down-rotation side. On both sides, individual waves will move opposite to the direction of rotation since $f > 0$, but energy will be transmitted away from the paddle in both directions.

The two wave trains will meet on the opposite side of the annulus unless they are damped. Under the experimental conditions (water,

$\bar{r} \sim 1$ meter, $\omega \sim 1$ rad sec^{-1}) this damping occurs through viscous Ekman layers at the bottom, and the two trains are absorbed rapidly enough that they do not interfere with each other. Fig. 3 shows an example of experimental results obtained by Dr. Alan Ibbetson of the Woods Hole Oceanographic Institution. Further results, including quasi-non-linear interactions within these wave trains, will be reported in a later paper.

Acknowledgments

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