On the propagation of gravity waves in a hydrostatic, compressible fluid with vertical wind shear[1,2]

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(Manuscript received December 3; revised version February 12)

ABSTRACT

The speed of propagation of the vertical modes of gravity waves is found in a non-rotating hydrostatic and compressible fluid with a vertical variation of mean temperature and wind. The analysis is based on a set of linearized equations, and it is assumed that the basic flow as well as the perturbations are hydrostatic. We are therefore concerned with one of the possible wave-type solutions to the primitive equations as they are being used in studies of the general circulation of the atmosphere and in other dynamical studies of the atmosphere.

In the case of a constant static stability it is found that the speed of propagation of internal gravity waves is determined by the Richardson number. If the (positive) Richardson number is smaller than $\frac{1}{4}$, the waves will move with a speed determined by the basic flow at the bottom of the fluid. For Richardson numbers larger than $\frac{1}{4}$ several vertical modes moving with different speed will exist. The speed of propagation is evaluated as a function of the vertical wave number and the vertical windshear for given values of the static stability.

The speed of the external gravity waves is evaluated numerically. It is found that the main difference is the existence of a mode which has a numerically large phase speed and which has a maximum amplitude of the vertical velocity at the boundary. The modes of the internal waves move with almost the same speed as before. The amplitudes of vertical velocity and geopotential are computed as functions of pressure for the internal and external gravity waves.

The results of this study which apply to a hydrostatic and compressible atmosphere are compared in the last section with certain general results, obtained in earlier investigations, for a non-hydrostatic, incompressible fluid. A sufficient condition for stability has been found and an upper and lower limit on the magnitude of the complex root has been determined.

1. Introduction

It is well known that the hydrostatic primitive equations generally employed in studies of the large scale dynamics of the atmosphere (the two equations of motion for the horizontal velocity components, the hydrostatic equation, the continuity equation and the thermodynamic equation) have solutions of the following types: Rossby waves, inertia waves, external and internal gravity waves. In the so-called filtering procedures (THOMPSON, 1956) it has been the problem to modify the primitive equations in such a way that solutions of the last three types are excluded from the modified equations and such that the remaining solutions are slightly modified Rossby waves. The modified equations, generally called the balanced equations or the quasi-geostrophic equations, have been used in studies of the dynamics of the atmosphere with some success, but there has in recent years been a return to the unmodified equations.

A partial reason for the use of the primitive equations is the fact that they, at least in a baroclinic atmosphere, permit internal gravity waves as solutions, while the gravity waves in the case of barotropic motion are of the external type. It is known from simplified models (see for example, THOMPSON, 1961) that internal gravity waves may have speeds which are so small

[1] Contribution No. 84, Department of Meteorology and Oceanography.

[2] The present research has been supported by the National Science Foundation under Grant GP-2561.

that they might conceivable interfere with the Rossby waves in such a way that the speed and structure of the pure Rossby waves present in a filtered model may be greatly modified in a non-filtered model permitting the existence of gravity waves. An investigation of this question has been made (WIIN-NIELSEN, 1963) using a two-layer model based upon the primitive equations. It turns out that the inertia and internal gravity waves interfere with the Rossby waves to some extent, especially for small values of the Richardson number.

In view of this result one expects that the speed of either the inertia waves or the internal gravity waves must depend strongly on the quantities which determine the Richardson number, i.e., the vertical windshear and the static stability. It is the purpose of the present paper to investigate this problem by simplifying the general equations for a model with a continuous variation of the horizontal wind and the static stability in the vertical direction to such an extent that only gravity waves are possible solutions. This can be done by neglecting the Coriolis terms in the equations of motion. A further simplification of the mathematical problems is obtained by a linearization of the equations and by restricting the motion to take place in vertical planes.

We are therefore considering the propagation of waves in a long, broad channel of fluid where the basic flow is constant in each isobaric surface but varies with pressure, and where the temperature varies with pressure. The basic flow as well as the perturbations are assumed to be hydrostatic. It is on this point that the present investigation differs significantly from most meteorological investigations of Helmholtz-type waves which have been carried out by a number of authors (see for instance SEKERA (1948), GOSSARD & MUNK (1954) and references therein). We are interested in the gravity waves present in an atmospheric model for the large scale motion, where the hydrostatic assumption can always be made, while most of the previous investigations have been concerned with the explanation of "billow" clouds and short period variations in the atmospheric pressure and wind field.

The present study is, however, not unrelated to meteorological phenomena on a synoptic scale. A recent study by WAGNER (1962) has demonstrated the existence of gravity waves on a scale large enough to be analysed by an admittedly dense network of surface stations. The study contains also a comparison between observed and theoretically estimated speeds of the gravity waves. The theoretical speeds are computed from the formula for gravity waves derived from an atmospheric model consisting of two isentropic layers. This formula resembles the expression for the speed of "shallow water" gravity waves modified by a measure of the static stability. The vertical variation of the wind in the basic current does not enter in this formula. As we shall show in this study, it turns out that the vertical shear of the horizontal wind is very important in the determination of the speed of the internal gravity waves, especially in the case where the wind shear is large. The important non-dimensional number which determines the main characteristics of the speed and structure of the gravity wave is the Richardson number. If the Richardson number is very large, the static stability is of prime importance in determining the speed of the waves, but in the other extreme case, when the Richardson number is very small, the speed of the waves is determined essentially by the wind shear.

There exists a large number of investigations concerned with the stability problem for a stratified fluid with a vertical wind shear. These investigations are carried out under the assumption of incompressibility, while the perturbations are allowed to be non-hydrostatic. The present study supplements these investigations. In the last section of the paper we shall make a comparison with earlier studies and investigate which results we can carry over from the existing theory.

2. The speed of internal gravity waves

We consider a basic state described by a current with an arbitrary direction $U^* = U^*(p)$ and a stratification characterized by a static stability $\sigma^* = \sigma^*(p) = -\alpha^*/\theta^* \partial\theta^*/\partial p$. Quantities with a star as superscript refer to the basic state. α^* in the expression for the static stability measure is the specific volume and θ^* is the potential temperature.

The following linearized perturbation equations apply for disturbances independent of the transversal direction:

$$\left.\begin{aligned}&\frac{\partial u}{\partial t}+U^*\frac{\partial u}{\partial x}+\omega\frac{dU^*}{dp}=-\frac{\partial\phi}{\partial x},\\&\frac{\partial u}{\partial x}+\frac{\partial\omega}{\partial p}=0,\\&\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial p}\right)+U^*\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial p}\right)+\sigma^*\omega=0.\end{aligned}\right\}\qquad(2.1)$$

We have adopted the coordinate system with pressure as vertical coordinate because we have made the hydrostatic assumption. $u(x, p, t)$ is the perturbation velocity in the direction of the basic current U^*, $\omega(x, p, t) = dp/dt$ is the vertical velocity in this sytem and ϕ is the geopotential. In agreement with our desire to study gravity waves in a pure form we have neglected all references to the rotation of the earth, and we shall in this section adopt the boundary conditions that $\omega = 0$ at $p = 0$ and $\omega = 0$ at $p = p_0$, where p_0 is a standard value of the surface pressure. The last condition excludes *external* gravity waves from the system (HOLLMANN, 1956).

The perturbations will be assumed to have the following form

$$u=U(p)\,e^{ik(x-ct)},\quad \omega=\Omega(p)\,e^{ik(x-ct)},$$
$$\phi=\Phi(p)\,e^{ik(x-ct)},\qquad(2.2)$$

in which k is the wave number and c is the wave speed.

When (2.2) is introduced in (2.1) we get the following three equations for U, Ω and Φ

$$\left.\begin{aligned}-ik(c-U^*)\,U(p)+\frac{dU^*}{dp}\,\Omega(p)+ik\Phi(p)=0,\\ ikU(p)+\frac{d\Omega}{dp}=0,\\ \sigma^*\Omega-ik(c-U^*)\frac{d\Phi}{dp}=0.\end{aligned}\right\}\qquad(2.3)$$

$U(p)$ can be eliminated from the first equation using the second. We can then solve the resulting equation for $\Phi(p)$, differentiate with respect to pressure and substitute the resulting expression for $d\Phi/dp$ in the last equation. The result is the following equation for $\Omega(p)$:

$$(c-U^*)^2\frac{d^2\Omega}{dp^2}+\left\{(c-U^*)\frac{d^2U^*}{dp^2}+\sigma^*\right\}\Omega=0.\qquad(2.4)$$

We shall treat the case where U^* is a linear function of p. In this case we have $d^2U^*/dp^2=0$, and (2.4) becomes

$$(c-U^*)^2\frac{d^2\Omega}{dp^2}+\sigma^*\Omega=0.\qquad(2.5)$$

It is now convenient to introduce a non-dimensional pressure $p_* = p/p_0$ as a new independent variable. In terms of this variable the boundary conditions become: $\Omega = 0$ for $p_* = 0$ and $p_* = 1$. In agreement with the remarks made above, we express the variation of U^* as follows:

$$U^*=\overline{U^*}+s\left(\tfrac{1}{2}-p_*\right),\quad s=-\frac{dU^*}{dp_*},\qquad(2.6)$$

where $\overline{U^*}$ is a pressure average of U^*. When we introduce these definitions in (2.5) we can write the equation in the form

$$\left\{p^*-\left(\tfrac{1}{2}-c'\right)\right\}^2\frac{d^2\Omega}{dp_*^2}+\frac{\sigma^*p_0^2}{s^2}\,\Omega=0,\qquad(2.7)$$

where $c'=(c-\overline{U^*})/s$.

Equation (2.7) will be solved under the assumption that σ^*, the static stability in the basic state, is a positive constant, $\sigma^* = a > 0$. This representation is by no means the best approximation to the distribution of σ^* in the earth's atmosphere. It has been shown (WIIN-NIELSEN, 1959) that a representation in which σ^* is inversely proportional to the square of pressure is a better approximation to the variation of σ^* in the troposphere of the standard atmosphere. However, for specific values of the constant a one can obtain reasonable approximation to the temperature variation in the standard atmosphere through a rather deep layer. This is shown in Fig. 1 in which the temperature distributions corresponding to different values of the constant a are shown together with the temperature distribution in the troposphere of the standard atmosphere.

The temperature distributions shown in Fig. 1 are obtained as solution to the equation

$$\sigma^*=-\frac{R}{p}\left(\frac{\partial T}{\partial p}-\frac{R}{c_p}\frac{T}{p}\right)=a.\qquad(2.8)$$

If we assume that the temperature for $p = p_0$, $p^* = 1$ is T_0 we can write the solution in the form

$$T(p_*) = \left[T_0 + \frac{a}{R}\frac{p_0^2}{(2-R/c_p)}\right]p_*^{R/c_p} - \frac{a}{R}\frac{p_0^2}{(2-R/c_p)}p_*^2. \tag{2.9}$$

The solid curves in Fig. 1 correspond to $a=1$, 2 and 4, while the dashed curve is the temperature distribution in the troposphere of the standard atmosphere. It is seen that a value of a somewhere between 1 and 2 will give the closest approximation to the dashed curve. Trial calculations show that $a=1.2$ is about the best choice.

The coefficient of Ω in (2.7) is identical with the Richardson number because

$$\mathrm{Ri} = \frac{g\,d(\ln\theta)/dz}{(dU/dz)} = \frac{\alpha\,d(\ln\theta)/dp}{(dU/dp)^2} = \frac{\sigma^* p_0^2}{s^2}. \tag{2.10}$$

With the assumption $\sigma^* = a = \text{const.}$ and the assumption that U^* varies linearly with pressure which means that s is constant we obtain a value of the Richardson number which is independent of pressure.

The determination of the phase speed c' from (2.7) is obtained by transforming the equation in such a way that the speed appears in the lower boundary condition. The following transformation accomplishes this:

$$\zeta = 1 - \frac{p^*}{\frac{1}{2}-c'}. \tag{2.11}$$

(2.7) is by this transformation changed to

$$\xi^2\frac{d^2\Omega}{d\xi^2} + \mathrm{Ri}\,\Omega = 0 \tag{2.12}$$

with the boundary conditions

$$\Omega = 0, \quad \xi = 1, \quad \xi = \frac{c'+\frac{1}{2}}{c'-\frac{1}{2}}. \tag{2.13}$$

Solutions to (2.12) of the form $\Omega \sim \xi^{\varepsilon}$ exists for the following two values of ε

$$\varepsilon = \tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - \mathrm{Ri}}. \tag{2.14}$$

Let us first consider the case where $\mathrm{Ri} > \frac{1}{4}$. In this case we can write the values of ε in the form

$$\varepsilon_{1,2} = \tfrac{1}{2} \pm i\lambda,\ \lambda = \sqrt{\mathrm{Ri} - \tfrac{1}{4}}, \tag{2.15}$$

where ε_1 corresponds to the plus sign.

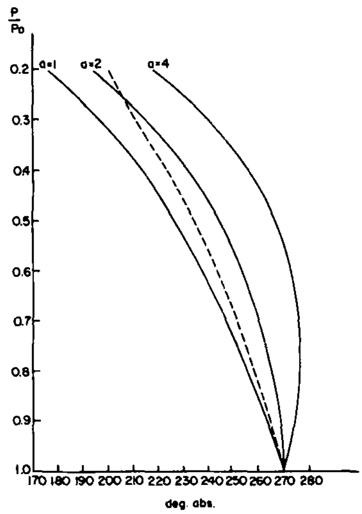

FIG. 1. The dashed curve is the temperature in the standard atmosphere as a function of pressure. The horizontal scale is in absolute degrees and the vertical scale is pressure divided by $p_0 = 100$ cb. The full curves are the temperature distributions for different values of a, a measure of static stability.

The solution to (2.12) can therefore be written

$$\begin{aligned}\Omega(\xi) &= A\,\xi^{\frac{1}{2}+i\lambda} + B\,\xi^{\frac{1}{2}-i\lambda} \\ &= A\sqrt{\xi}\,\{\cos(\lambda\ln\xi) + i\sin(\lambda\ln\xi)\} \\ &\quad + B\sqrt{\xi}\,\{\cos(\lambda\ln\xi) - i\sin(\lambda\ln\xi)\}\end{aligned} \tag{2.16}$$

The boundary condition that $\Omega = 0$ for $\xi = 1$ gives

$$A + B = 0 \tag{2.17}$$

which means that (2.16) can be written in the form

$$\Omega(\xi) = 2iA\sqrt{\xi}\,\sin(\lambda\ln\xi). \tag{2.18}$$

The second boundary condition that $\Omega = 0$ for $\xi = (c'+\frac{1}{2})/(c'-\frac{1}{2})$ gives the equation

$$\sqrt{\frac{c'+\frac{1}{2}}{c'-\frac{1}{2}}}\,\sin\left(\lambda\ln\left(\frac{c'+\frac{1}{2}}{c'-\frac{1}{2}}\right)\right) = 0 \tag{2.19}$$

which is satisfied if

$$\lambda\ln\left(\frac{c'+\frac{1}{2}}{c'-\frac{1}{2}}\right) = m\pi, \quad m = \pm1, \pm2, \ldots, \tag{2.20}$$

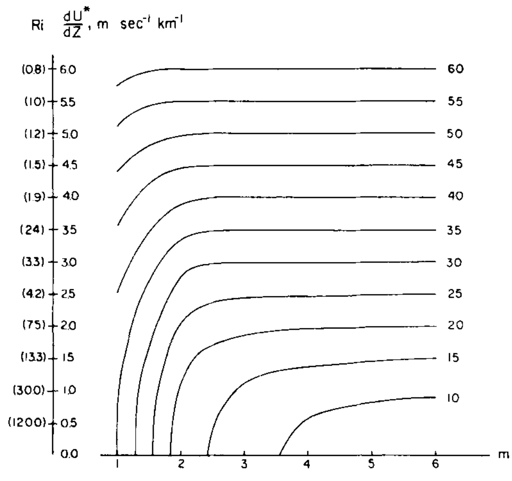

FIG. 2. Isolines for $(c - \overline{U^*})$ for internal gravity waves in a idagram with the vertical wind shear dU^*/dz, measured in the unit m sec^{-1} km^{-1}, as ordinate and the number of modes as abscissa. Values computed for $\sigma^* = 1.2$. The values in parentheses along the ordinate is the Richardson number.

(2.20) leads to the following value of c':

$$c' = \tfrac{1}{2} \coth \left(\frac{m\pi}{2\lambda}\right) \tag{2.21}$$

which in turn gives the following values for c:

$$c = \overline{U^*} + s \cdot \tfrac{1}{2} \coth \left(\frac{m\pi}{2\lambda}\right). \tag{2.22}$$

(2.22) gives the complete solution in the case $Ri > \frac{1}{4}$. If $Ri < \frac{1}{4}$ it is easily seen from (2.12)–(2.14) that the only possible wave solution is $c' = -\frac{1}{2}$ or $c = \overline{U^*} - \frac{1}{2}s = U_0^*$, where U_0^* is the basic velocity at the "ground" ($p^* = 1$).

In the special case when $Ri = \frac{1}{4}$ we get only one solution from (2.14), $\Omega_1 = \sqrt{\xi}$. It is, however, easily seen that a second particular solution is given by $\Omega_2 = \sqrt{\xi} \ln \xi$ resulting in the complete solution

$$\Omega = A\sqrt{\xi} + B\sqrt{\xi} \ln \xi. \tag{2.23}$$

The boundary condition $\Omega = 0$ for $\xi = 1$ gives $A = 0$, and the other boundary condition gives

$$\sqrt{\frac{c'+\frac{1}{2}}{c'-\frac{1}{2}}} \ln \frac{c'+\frac{1}{2}}{c'-\frac{1}{2}} = 0 \tag{2.24}$$

giving the solution $c = U_0^*$.

The method of solution used above breaks down in the case where $s = 0$, i.e. when we have no vertical windshear or in other words when the Richardson number is infinite. In this case we have to return to the original equation (2.5), where we can now consider U^* as a constant. The equation (2.5) can in this case be written as

$$\frac{d^2\Omega}{dp_*^2} + q^2\Omega = 0, \quad q^2 = \frac{\sigma^* p_0^2}{(c - U^*)^2} \tag{2.25}$$

with the general solution

$$\Omega = A \sin (qp_*) + B \cos (qp_*). \tag{2.26}$$

One boundary condition ($\Omega = 0$ for $p_* = 0$) gives $B = 0$, while the other results in the following equation for the determination of the phase speed

$$q^2 = \frac{\sigma^* p_0^2}{(c - U^*)^2} = m^2\pi^2, \quad m = \pm 1, \pm 2, \ldots \tag{2.27}$$

from which we determine c to be:

$$c = U^* + \frac{p_0 \sqrt{\sigma^*}}{m\pi}. \tag{2.28}$$

The results obtained in this section may be summarized in the following way. For a hydrostatic and compressible fluid with a constant vertical wind shear and a constant value of the stability parameter, we find only stable waves if $Ri > \frac{1}{4}$ and the simple wave-solutions $c = U_0^*$ if $Ri \leq \frac{1}{4}$. Formally, this result is identical to the result obtained by TAYLOR (1931), but it should be realized that he considered non-hydrostatic, incompressible disturbances. A further difference is that our vertical profile is linear with respect to pressure while he considered a linear profile with respect to height.

3. Discussion of the results

The expressions (2.22) and (2.28) give the speed of the different modes of the internal gravity waves. The first formula applies for a finite Richardson number, while the second holds in the special case of an infinite Richardson number.

Certain general conclusions can be made directly from the solutions. We note first of all

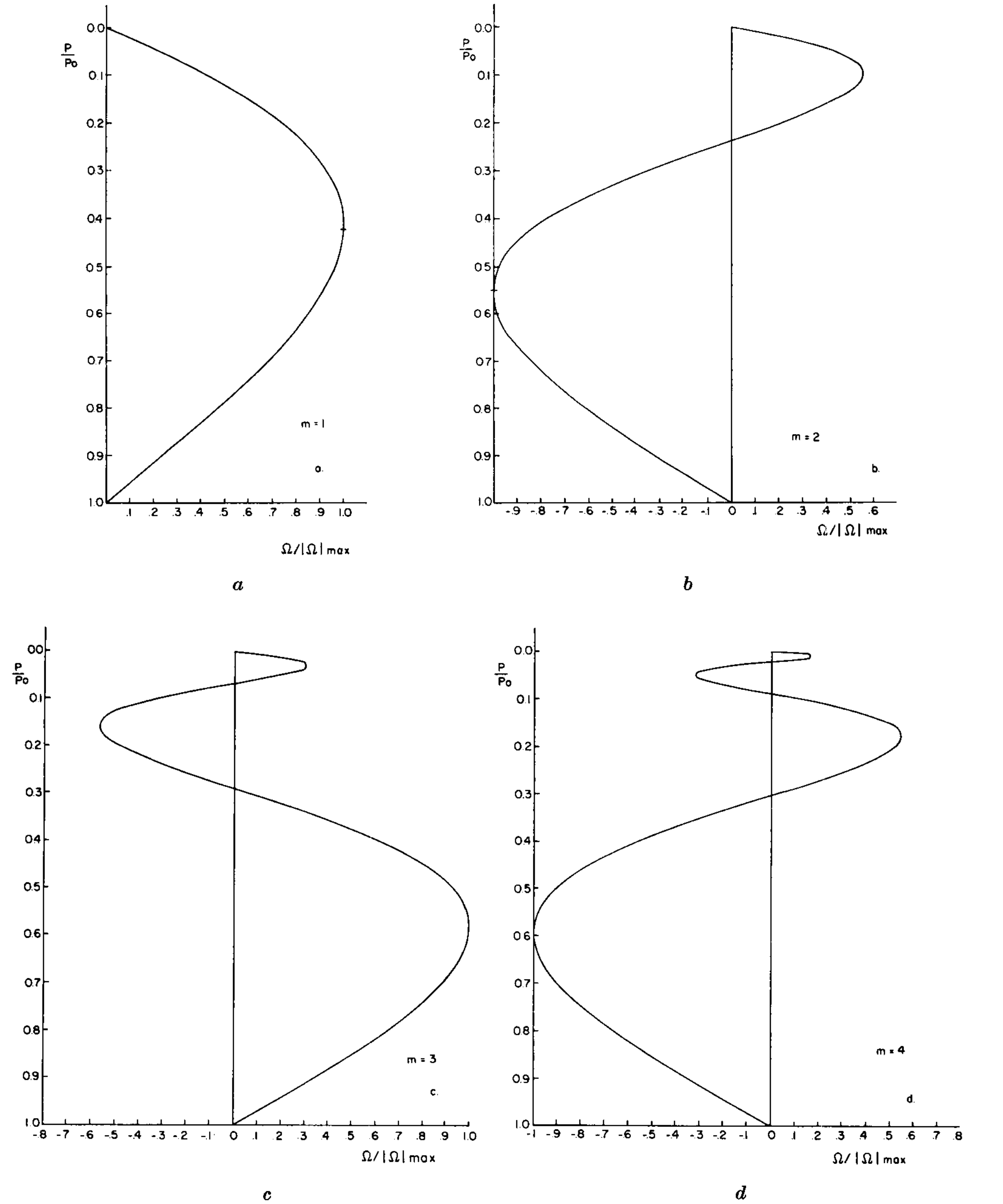

FIG. 3. Normalized amplitude of the vertical velocity for the different modes of internal gravity waves as a function of pressure. The ordinate is p/p_0, the abscissa $\Omega/|\Omega|$ max.

that c' changes sign when m changes sign. This means that the progressive waves ($m > 0$) move with the same numerical speed relative to the vertically averaged current $\overline{U^*}$ as the retrogressive waves ($m < 0$) for the same absolute value of m. For numerical estimates it is therefore sufficient to consider positive values of m.

Let us denote the value of c' corresponding to $m > 0$ by $c'(+)$, while the value corresponding to $(-m)$ will be denoted $c'(-)$. Since coth (x)

is greater than 1 when $x > 0$ we find for $m > 0$ that $c'(+) > \frac{1}{2}$, or equivalently $c(+) > \overline{U^*} + \frac{1}{2}s$. The value $(\overline{U^*} + \frac{1}{2}s)$ is the value of U^* for $p_* = 0$ or, in other words, the maximum value of $U^* = U^*_{max} = \overline{U^*} + \frac{1}{2}s$. We find consequently that the speed of propagation of the progressive waves is larger than the maximum speed of the basic current.

Considering next the case corresponding to $-m$ we see that $c'(-) < -\frac{1}{2}$ or $c(-) < \overline{U^*} - \frac{1}{2}s = U^*_{min}$ which means that the retrogressive waves move with a speed smaller than the minimum value of the basic current.

For a given value of $m > 0$ we find that c' increases with increasing values of λ, i.e. with increasing values of the Richardson number. On the other hand, when the Richardson number becomes small, (but still larger than $\frac{1}{4}$) we find that $c'(+)$ approaches $\frac{1}{2}$, or $c(+) \to \overline{U^*} + \frac{1}{2}s = U^*_{max}$. Similar considerations show that $c(-)$ becomes numerically larger for increasing values of Ri, while $c(-) \to \overline{U^*} - \frac{1}{2}s = U^*_{min}$ when Ri becomes small.

The conclusions, summarized in the preceding paragraphs, are illustrated in Fig. 2 where isolines for $(c - \overline{U^*})$ computed from (2.22) with $\sigma^* = 1.2$ are drawn in a diagram with m as abscissa and the vertical wind shear as ordinate. Continuous lines are drawn although it is realized that m can have only integer values. The values of $c - \overline{U^*}$ for $dU^*/dz = 0$ were obtained from (2.28). On the ordinate we have furthermore given the values of the Richardson number corresponding to dU/dZ.

Fig. 3*a*, *b*, *c* and *d* show the amplitude of the vertical velocity as a function of pressure starting with a positive slope ($d\Omega/dp > 0$ for $p = 0$) and normalized in such a way that $|\Omega|_{max} = 1$. It is seen that the position of the extremum moves toward higher pressure as m increases. For $m = 1$ the extremum occurs at $p = 420$ mb, for $m = 2$ at 550 mb, for $m = 3$ at $p = 570$ mb and for $m = 4$ at 590 mb. The amplitude in the geopotential is shown in Fig. 4*a*, *b*, *c* and *d* for the same values of m. The extreme amplitude is found at the lower boundary. The curves in Fig. 3 were computed from (2.18) in which we have substituted the transformation (2.11) and the solution (2.21) for c. The curves in Fig. 4 were computed from the first equation in (2.3) after substitution of the known solutions for $\Omega(p)$ and $U(p)$.

The curves in Fig. 3 and 4 are valid for positive values of m, in which case $(c - \overline{U^*})$ is positive. The structure of the waves when $m < 0$ and $c - \overline{U^*} < 0$ is easily visualized because it is possible to show that the curves showing the variation of the amplitudes in the vertical velocity and the geopotential for a given negative value of m are symmetric to the curves for the corresponding positive value of m around the line $p^* = \frac{1}{2}$.

In order to demonstrate the preceding statement we introduce the new indpendent coordinate

$$p' = p^* - \tfrac{1}{2}. \tag{3.1}$$

In terms of p' we may write (2.11) in the form

$$\xi = \frac{c' + p'}{c' - \frac{1}{2}}. \tag{3.2}$$

For a given value of $m > 0$ we have the corresponding values $c'(+)$ and

$$\xi(+) = \frac{c'(+)p'}{c'(+) - \frac{1}{2}}. \tag{3.3}$$

When we next consider the value $(-m)$ we know that $c'(-) = -c'(+)$ and we find that

$$\xi(-) = \frac{c'(-) + p'}{c'(-) - \frac{1}{2}} = \frac{c'(+) - p'}{c'(+) + \frac{1}{2}}. \tag{3.4}$$

From (3.3) and (3.4) it is seen that $\xi(-)$ at p' is proportional to $\xi(+)$ at $(-p')$. The symmetry relation is now evident by noting that the basic differential equation is invariant with respect to a change of sign of ξ and with respect to a replacement of ξ by a constant times ξ.

4. The speed of external gravity waves

The external gravity waves were excluded from the solutions obtained in Sections 2 and 3 because of the simplified lower boundary condition $\Omega = 0$ for $p = p_0$. In this section we shall remove this restriction and apply a different boundary condition. The basic equations are naturally the same as in Section 2, and the formal solution is unchanged. The difference appears after (2.16) when we try to determine the integration constants from the boundary

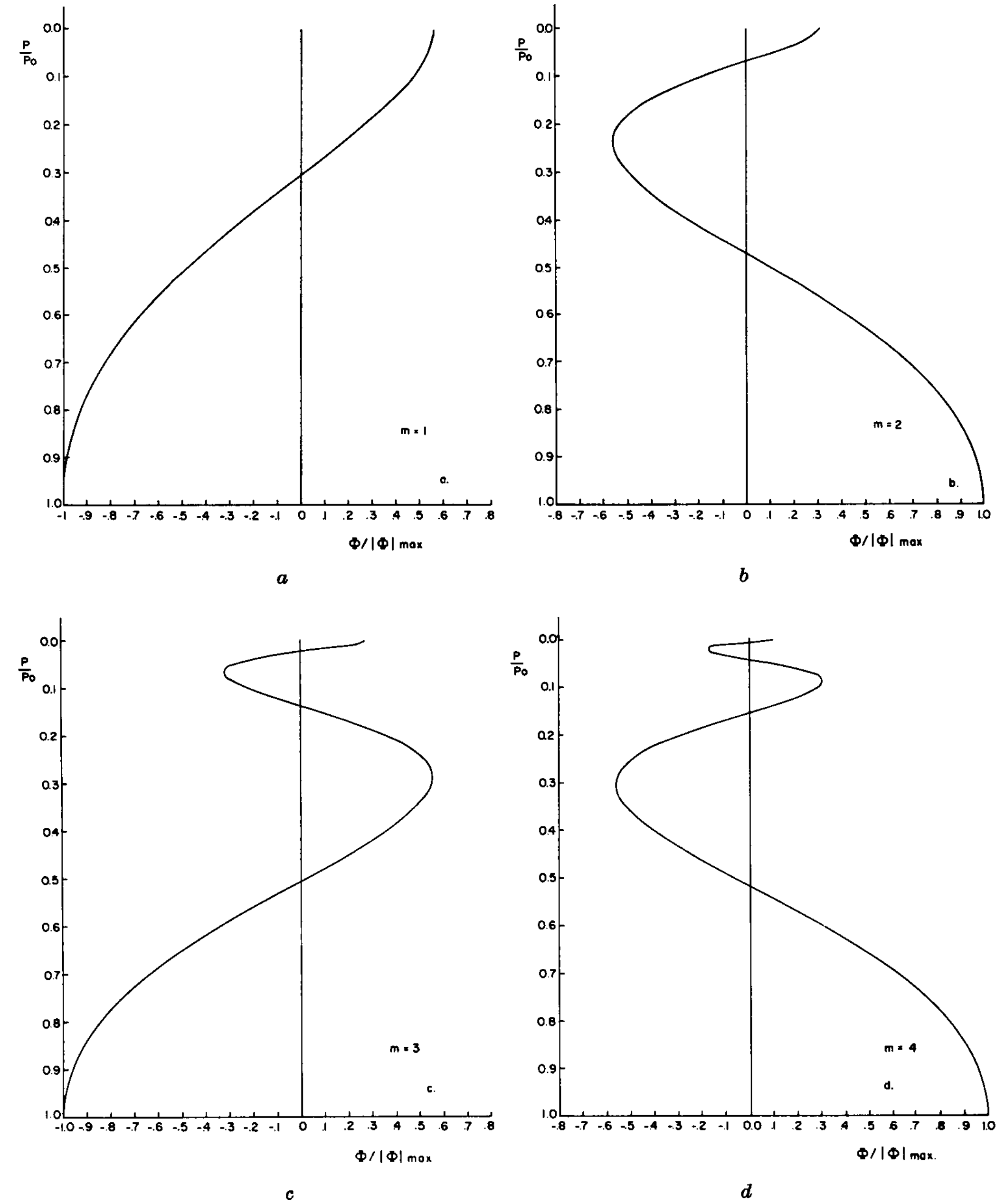

Fig. 4. Normalized amplitude of the geopotential for the different modes of internal gravity waves as a function of pressure. The ordinate is p/p_0, the abscissa $\Phi/|\Phi$ max.

conditions and arrive at the formula for the phase speed.

The true boundary condition for a flat surface is naturally

$$w=0 \quad \text{for} \quad z=0. \tag{4.1}$$

We shall approximate the condition (4.1) by assuming that

$$w=0 \quad \text{for} \quad p=p_0, \tag{4.2}$$

where p_0 is a standard pressure equal to 100 cb.

(4.2) may then be expressed as a relation between the vertical velocity amplitude $\Omega_0 = \Omega(p_0)$ and the amplitude of the geopotential at the same pressure $\Phi_0 = \Phi(p_0)$ because (4.2) can be written in the following form:

$$gw = \left(\frac{d\Phi}{dt}\right)_0 = \left(\frac{\partial \phi}{\partial t}\right)_0 + U_0^* \left(\frac{\partial \phi}{\partial x}\right)_0 + \omega_0 \left(\frac{\partial \phi}{\partial p}\right)_0 = 0. \tag{4.3}$$

For the type of disturbances treated in this paper (see (2.2)) (4.3) reduces to

$$\Omega_0 = -ik\varrho_0(c - U_0^*)\,\Phi_0, \quad p = p_0. \tag{4.4}$$

Using the definition of c' which appears just after (2.7) we may write (4.4) in the form

$$\Omega_0 = -ik\,\varrho_0\,s\,(c' + \tfrac{1}{2})\,\Phi_0, \quad \xi = \frac{c' + \frac{1}{2}}{c' - \frac{1}{2}}. \tag{4.5}$$

The condition at the upper boundary is still that $\Omega = 0$ for $\xi = 1$ which leads to the expression (2.18) for Ω. In order to apply (4.5) we must find the amplitude of the geopotential. From the first equation in (2.3) we find that

$$\Phi = -\frac{1}{ik}\left[(c - U_0^* - s(1 - p_*))\frac{1}{p_0}\frac{d\Omega}{dp_*} - \frac{s}{p_0}\Omega\right] \tag{4.6}$$

which leads to the following expression when we transform to the independent variable ξ:

$$\Phi = -\frac{1}{ik}\frac{s}{p_0}\left[\xi\frac{d\Omega}{d\xi} - \Omega\right]. \tag{4.7}$$

We evaluate Φ using the solution (2.18) for $\Omega(\xi)$ giving

$$\Phi = -\frac{2s}{kp_0}A\sqrt{\xi}\,[\lambda \cos(\lambda \ln \xi) - \tfrac{1}{2}\sin(\lambda \ln \xi)]. \tag{4.8}$$

Let us in order to apply the boundary condition (4.5) use the following notation

$$\xi_0 = \frac{c' + \frac{1}{2}}{c' - \frac{1}{2}}, \quad c' = \frac{1}{2}\frac{\xi_0 + 1}{\xi_0 - 1}. \tag{4.9}$$

With this notation we can write (4.5) in the form

$$F(\xi_0) = 0, \tag{4.10}$$

where

$$F(\xi) = \left[\left(1 + \frac{1}{2}\frac{\varrho_0}{p_0}s^2\right)\xi - 1\right]\tan(\lambda \ln \xi) - \lambda\frac{\varrho_0}{p_0}s^2\xi. \tag{4.11}$$

The phase speed can be determined when we know the roots satisfying (4.10). Once the roots are determined we can find c' from the second expression in (4.9), and the phase-speed c is evaluated from the definition of c'.

The procedure described above is valid in all cases except when $s = 0$, i.e. $U^* = \text{constant}$.

The solution in this special case is given by (2.27) with the quantity q defined by (2.25). When we apply the new boundary conditions to this solution we find again $B = 0$ while the lower boundary conditions applied at $p^* = 1$ gives the equation

$$G(q) = q \tan q - \sigma^*\varrho_0 p_0 = 0 \tag{4.12}$$

from which we have to determine q and then c from the definition of q given in (2.25). The roots to (4.10) and (4.12) were found using standard trial and error methods.

We can now compare the solutions to (4.10) and (4.12) with the phase speeds determined from (2.22) and (2.28). The main difference is that the boundary conditions applied in this section allow a wave solution whose speed is numerically much larger than the speeds determined in the previous sections. In addition, the possibility exists that the speed and structure of the different modes might have been changed significantly because of the new lower boundary condition.

Table 1 contains the positive values of $(c - \overline{U^*})$ for the first seven modes for different values of the vertical shear of the horizontal wind. Two numbers are given in each position. The upper one corresponds to the speeds determined with the simplified lower boundary condition, while the lower numbers in parentheses give the speeds determined from (4.10) and (4.12). The speeds corresponding to $m = 0$ exist only in the latter case. If no figure appears in the lower position it means that it is equal to the figure in the upper position.

It is first of all seen from Table I that the more correct boundary condition applied in this sec-

TABLE 1. *Positive values of* $(c - \overline{U^*})$ *in m sec*$^{-1}$ *as a function of the vertical windshear in m sec*$^{-1}$ *km*$^{-1}$ *and of the number of modes.*

The upper figure in each position gives the value determined from the simplified lower boundary condition while the lower figure in parentheses gives the value determined from the boundary condition described in Section 4. The values corresponding to $m = 0$ exist only in the latter case. If the figure in the lower position is missing it indicates equality to the figure in the upper position.

dU^*/dz m	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
0	(322.4)	(322.6)	(322.9)	(323.2)	(323.6)	(324.0)	(324.6)	(325.1)	(325.8)	(326.5)	(327.4)	(328.3)	(329.2)
1	35.0	35.0	35.7	36.7	38.1	39.9	42.0	44.6	47.4	50.6	54.1	57.9	62.0
	(34.4)	(34.4)	(35.4)	(36.3)	(37.6)	(39.5)	(41.4)	(44.1)	(46.9)	(50.2)	(53.7)	(57.6)	(61.7)
2	17.5	17.9	19.2	21.4	24.3	27.8	31.7	36.0	40.6	45.3	50.1	55.0	60.0
	(17.4)	(17.8)	(19.1)	(21.3)	(24.2)	(27.7)	(31.6)	(35.9)	(40.5)				
3	11.7	12.3	14.3	17.3	21.2	25.6	30.3	35.1	40.0	45.0	50.0	55.0	60.0
4	8.8	9.6	12.0	16.0	20.4	25.1	30.0	35.0	40.0	45.0	50.0	55.0	60.0
5	7.0	8.1	11.1	15.4	20.1	25.0	30.0	35.0	40.0	45.0	50.0	55.0	60.0
6	5.8	7.1	10.7	15.1	20.0	25.0	30.0	35.0	40.0	45.0	50.0	55.0	60.0

tion results in only small changes in the speed of the higher modes. As a matter of fact the differences are negligible for $m \geqslant 3$. For $m = 1$ and $m = 2$ we find differences but they are in all cases less than 1 m sec^{-1}. The only difference of real importance is therefore in the existence of the wave-solutions corresponding to $m = 0$ in Table 1. These external gravity waves exist because of the relaxation of the lower boundary condition. The speeds are of the same order of magnitude as the speeds of the external gravity waves in a homogeneous fluid with a free surface. It is well known that the speed of such waves is described by the formula

$$c - U_0^* = \pm\sqrt{gH}, \tag{4.13}$$

where H is the mean depth of the fluid. To evaluate this speed we may write

$$gH = p_0/\varrho_0, \tag{4.14}$$

where p_0 is the pressure at the bottom of the fluid and ϱ_0 is the constant density. Using $p_0 = 100$ cb and $\varrho_0 = 10^{-3}$ t m^{-3} we find $c - U_0^* = \pm 316$ m sec^{-1}.

The structure of the waves can be found ((2.18) and (4.8)) by substitution of

$$\xi = 1 - \frac{p_*}{\frac{1}{2} - c'} \tag{4.15}$$

and making use of the numerical values of c' which were computed for the construction of Table 1.

The almost straight line in Fig. $5a$ gives $\Omega = \Omega(p_*)$ for $m = 0$, $dU^*/dz = 2$ m sec^{-1} km^{-1} and $\sigma^* = 1.2$. It is seen that the maximum value of Ω for the external gravity wave is found at the lower boundary. This distribution is analogous to the distribution of the vertical velocity as a function of height in a homogeneous fluid with a free surface where it is found that the vertical velocity depends linearly on height, is zero at the lower boundary and maximum at the free surface. Fig. $6a$ shows the geopotential amplitude $\Phi = \Phi(p_*)$ for the same parameters as in Fig. $5a$. As expected it turns out that $\Phi(p_*)$ is almost but not quite independent of pressure.

The next problem to investigate is the importance of the lower boundary condition for the distribution of the amplitude of the vertical velocity and the geopotential in the higher modes. Fig. $5b$, c and d show the normalized amplitude of the vertical velocity as a function of pressure for modes 1, 2, and 3. These figures should be compared with Fig. $3a$, b and c. There is a great similarity between the two sets of curves. The maximum amplitude is found at a somewhat lower pressure with the improved boundary condition, and the amplitude of the vertical velocity does not go to zero at the lower boundary. The value of Ω for $p_* = 1$ is less than 10% of the maximum value for all three modes. Fig. $6b$, c and d show the normalized amplitude of the geopotential as a function of pressure. These figures should be compared with Fig. $4a$, b and c. We find again a very great similarity

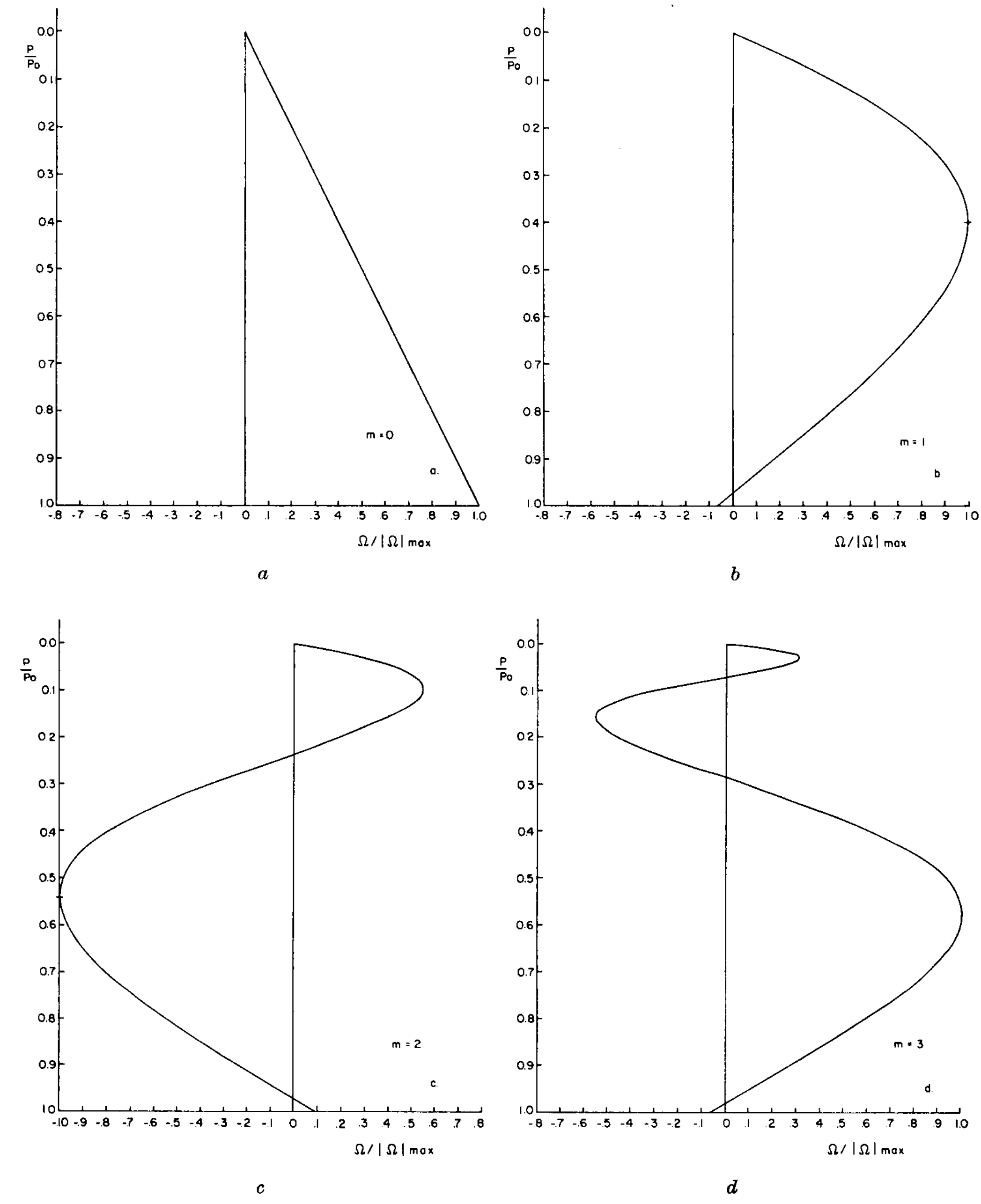

FIG. 5. Normalized amplitude of the vertical velocity for the different modes of external gravity waves as a function of pressure. Coordinates as in Fig. 3.

between the two sets of curves; the only difference is that the maximum amplitude is no longer at the lower boundary, but is found at a slightly lower pressure. An evaluation of $\Omega(p_*)$ and $\Phi(p_*)$ for negative values of $(c-\overline{U^*})$ produces curves which are almost symmetric to the curves in Fig. 3*a*, *b*, *c*, *d* and Fig. 4*a*, *b*, *c*, *d* around the line $p^* = \frac{1}{2}$. The values of Ω for $p^* = 1$ are of the same order of magnitude as those found in Fig. 5*b*, *c* and *d*.

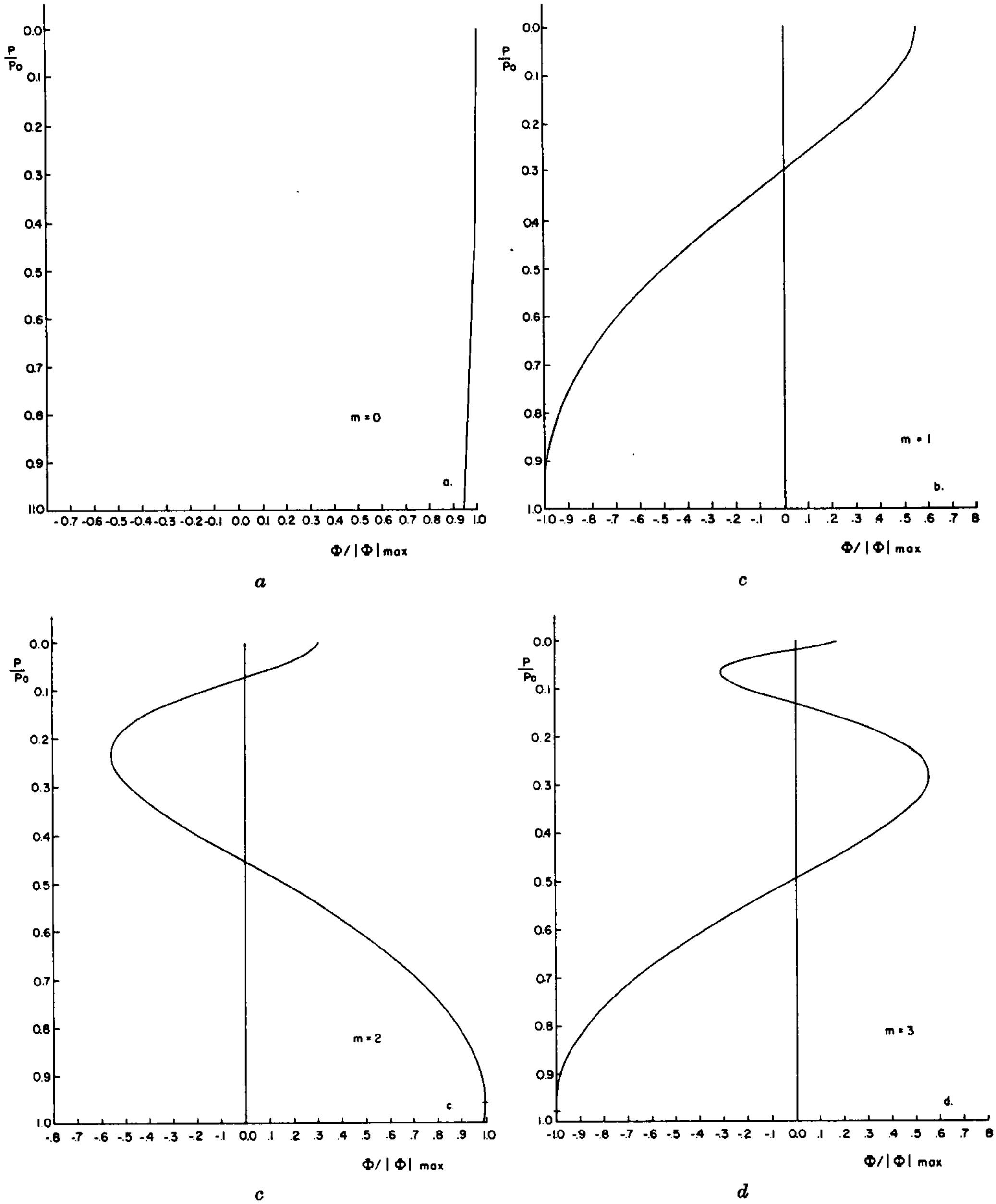

Fig. 6. Normalized amplitude of the geopotential for the different modes of external gravity waves as a function of pressure. Coordinates as in Fig. 4.

5. A sufficient condition for stability

The problem of stability of stratified shear flow has received considerable attention over the past years. The perturbations have in most cases been considered to be incompressible and nonhydrostatic. The only exception is a paper by Kuo (1963). G. I. Taylor (1931) investigated the problem and was able to show that neutral disturbances can exist in a fluid over a flat plate if the Richardson number (Ri) is larger than $\frac{1}{4}$, and if the fluid extends to infinity with a linear velocity profile and an exponentially decreasing density. If Ri < ¼ no simple sinusoidal waves can exist. The stability problem has later been extended to other cases with different velocity and density variations, continuous and discontinuous, by many investigators either as an initial value problem (ELIASSEN, HØILAND & RIIS, 1953) or as a linear perturbation problem (DRAZIN, 1958). Recently, MILES (1961) and HOWARD (1961) have shown under rather general conditions that a sufficient condition for stability in a parallel, stratified, inviscid fluid is that the local Richardson number should exceed ¼ everywhere. The proof of the theorem holds for non-hydrostatic and incompressible perturbations. Howard (loc. cit.) has also given an upper bound to the values of the complex wave velocity and the growth rate of infinitesimal disturbances if they exist.

It is the purpose of this section to show that Miles' theorem also holds for hydrostatic, compressible fluids, and that we can obtain the same upper bound on the complex wave velocity as obtained by Howard (loc. cit.). In order to do this we shall apply a method very similar to Howard's. We return to the basic differential equation (2.4), and we do not make any restrictions on the basic wind profile and the stability measure. The perturbation will be assumed to be unstable. This means that the imaginary part of the wave velocity, is positive. If we define the function E as $E = U^* - c = (U^* - c_r) - ic_i$ we know that we can select a definite branch of $E^{\frac{1}{2}}$. When E is introduced in the basic differential equation (2.4) we get

$$E^2 \frac{d^2\Omega}{dp^2} + \left(\sigma - E\frac{d^2E}{dp^2}\right)\Omega = 0 \qquad (5.1)$$

with the boundary conditions $\Omega = 0$ for $p = 0$ and $p = p_0$. The dependent variable Ω is replaced by a new dependent variable G defined by

$$\Omega = E^{\frac{1}{2}} G. \qquad (5.2)$$

(5.1) is then transformed to the following equation

$$\frac{d}{dp}\left(E\frac{dG}{dp}\right) + \left[\frac{1}{E}\left(\sigma - \frac{1}{4}\left(\frac{dE}{dp}\right)^2 - \frac{1}{2}\frac{d^2E}{dp^2}\right]G = 0 \qquad (5.4)$$

with the boundary conditions $G = 0$ for $p = 0$ and $p = p_0$.

If we have an unstable wave solution, Ω (and therefore also G) will be complex. Suppose that

$$G = G_r + iG_i, \quad G^* = G_r - iG_i, \qquad (5.4)$$

where G^* is the complex conjugate to G. We multiply (5.3) by G^* and integrate from 0 to p_0 with respect to pressure. Using the notation $|G|^2 = G \cdot G^*$ and $|dG/dp|^2 = (dG/dp)\cdot(dG^*/dp)$ where $|G|^2$ and $|dG/dp|^2$ are real numbers we obtain after integration by parts in the first term and the use of the boundary conditions:

$$\int_0^{p_0}\left[E\left|\frac{dG}{dp}\right|^2 + \frac{1}{2}\frac{d^2E}{dp^2}|G|^2 + E^*\left\{\frac{1}{4}\left(\frac{dE}{dp}\right)^2 - \sigma\right\} \times \left|\frac{G}{E}\right|^2\right]dp = 0. \qquad (5.5)$$

We shall next divide this integral in its real and imaginary parts. Since c s independent of p it follows that dE/dp and d^2E/dp^2 are real numbers. The imaginary part is

$$ic_i\int_0^{p_0}\left[\left|\frac{dG}{dp}\right|^2 + \left(\sigma - \frac{1}{4}\left(\frac{dU^*}{dp}\right)^2\right)\left|\frac{G}{E}\right|^2\right]dp = 0. \qquad (5.6)$$

Since $c_i > 0$ by assumption, it follows that the integral in (5.6) must vanish. This is clearly impossible if

$$\sigma - \frac{1}{4}\left(\frac{dU^*}{dp}\right)^2 > 0 \quad \text{everywhere.} \qquad (5.7)$$

It follows therefore that a sufficient condition for stability is that (5.7) holds everywhere in the fluid. If we define the Richardson number to be ∞ if $dU/dp = 0$, we can also write the sufficient condition for stability in the form

$$Ri > \tfrac{1}{4}. \qquad (5.8)$$

The criterion derived above does not show the existence of unstable internal gravity waves, because it is only a necessary condition for the existence of these waves. It has not been possible so far to demonstrate the existence of such unstable modes for the system we are considering. The same statement holds for the non-hydrostatic incompressible case (MILES, 1961).

Assuming that unstable modes exist, we are, however, able to give an upper and lower limit for the magnitude of the real and imaginary part of the wave velocity. Such limits have been given by HOWARD (1961) for the non-hydrostatic, incompressible case. We shall, as a matter of fact, use his technique to establish the limits in our case.

We consider again the differential equation (5.1) and assume the existence of an unstable mode ($c_i > 0$). The dependent variable Ω is replaced by a new dependent variable F defined by the relation

$$\Omega = E \cdot F. \tag{5.9}$$

The equation (5.1) is transformed to the following equation in the new dependent variable F

$$\frac{d}{dp}\left(E^2 \frac{dF}{dp}\right) + \sigma F = 0. \tag{5.10}$$

(5.10) is multiplied by the complex conjugate F^* of F and integrated from 0 to p_0. After integration by parts and using the boundary conditions $F = F^* = 0$ for $p = 0$ and $p = p_0$ we obtain

$$\int_0^{p_0} \left\{ E^2 \left|\frac{dF}{dp}\right|^2 - \sigma |F|^2 \right\} dp = 0. \tag{5.11}$$

From the definition of E we have

$$E^2 = [(U^* - c_r)^2 - c_i^2] - i2c_i(U^* - c_r). \tag{5.12}$$

The integral (5.11) is separated into its real and imaginary parts:

$$\int_0^{p_0} \left[\left\{ (U^* - c_r)^2 - c_i^2 \right\} \left|\frac{dF}{dp}\right|^2 - \sigma |F|^2 \right] dp = 0, \tag{5.13}$$

$$-2\,i\,c_i \int_0^{p_3} (U^* - c_r) \left|\frac{dF}{dp}\right|^2 dp = 0. \tag{5.14}$$

Under the assumption that $c_i > 0$ it follows from (5.14) that

$$a = U^*_{\min} < c_r < U^*_{\max} = b. \tag{5.15}$$

The relation (5.15) gives an upper and lower limit to the real part of the phase velocity for an unstable wave. We shall next proceed to get information about the position of $c = c_r + ic_i$ in the complex plane. Let $|dF/dp|^2 = Q > 0$. For $c_i \neq 0$ we obtain from (5.14)

$$\int_0^{p_0} U^* Q \, dp = c_r \int_0^{p_0} Q \, dp. \tag{5.16}$$

From (5.13) we obtain

$$\int_0^{p_0} [(U^{*2} - 2c_r U^* + c_r^2 - c_i^2) Q] \, dp = \int_0^{p_0} \sigma |F|^2 dp \tag{5.17}$$

but this relation can also be written in the following form after substitution from (5.16)

$$\int_0^{p_0} U^{*2} Q \, dp = (c_r^2 + c_i^2) \int_0^{p_0} Q \, dp + \int_0^{p_0} \sigma |F|^2 dp. \tag{5.18}$$

Let us next consider the integral

$$\int_0^{p_0} (U^* - a)(U^* - b) Q \, dp, \tag{5.19}$$

where a and b are defined in (5.15). According to the definition of a and b it is obvious that the integral (5.19) is negative. It follows therefore that

$$0 > \int_0^{p_0} U^{*2} Q \, dp - (a+b) \int_0^{p_0} U^* Q \, dp + ab \int_5^{p_0} Q \, dp. \tag{5.20}$$

The right-hand side of (5.20) can be transformed using (5.16) and (5.18), and we get:

$$0 > (c_r^2 + c_i^2) \int_0^{p_0} Q \, dp - (a+b) c_r \int_0^{p_0} Q \, dp + ab \int_0^{p_0} Q \, dp + \int_0^{p_0} \sigma |F|^2 dp \tag{5.21}$$

or

$$0 > \left\{ \left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 - \left(\frac{b-a}{2}\right)^2 \right\} \int_0^{p_0} Q \, dp + \int_1^{p_0} \sigma |F|^2 dp. \tag{5.22}$$

Since the two integrals in (5.22) are both positive it follows from (5.22) that

$$\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 > \left(\frac{b-a}{2}\right)^2 \qquad (5.23)$$

which says that the complex wave velocity must lie inside the semicircle in the upper half plane with $(\frac{1}{2}(a+b), 0)$ as center and $b-a = U_{\max} - U_{\min}$ as diameter. It follows that

$$0 < c_i < \tfrac{1}{2}(U_{\max} - U_{\min}) \qquad (5.24)$$

which gives the upper limit to the imaginary part of the wave speed.

6. Conclusions

The speed and structure of internal and external gravity waves have been found in a non-rotating atmosphere where the horizontal wind and the temperature are simple functions of pressure. The first part of the study is limited by the fact that the basic current is a linear function of pressure, and that the vertical variation of temperature corresponds to a static stability $\sigma^* = -\alpha^*/\theta^* \, \partial\theta^*/\partial p$ independent of pressure.

The main difference between the two solutions obtained with different lower boundary conditions is the existence of a fast moving zero mode in the case where $w=0$ at the boundary $p=p_0$. This mode has a maximum value of the vertical velocity and the geopotential at one of the boundaries. The higher modes move with essentially the same speed and have approximately the same structure for the two sets of boundary conditions. The vertical velocity at the lower boundary is less than 10% of the maximum vertical velocity.

The speed of the higher modes is determined by the Richardson number. It is found that the higher modes move faster if the Richardson number decreases. The eastward progressing waves will approach the speed of the basic current at the upper boundary when the Richardson number becomes small, while the other wave solution will approach the speed at the lower boundary.

The results derived in Section 5 show that the main results from the theory for incompressible, non-hydrostatic disturbances can be carried over to the compressible, hydrostatic case.

7. Acknowledgements

The author wishes to thank Mrs. Charmian Jefferies for programming all calculations performed in this study.

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