Theoretical flow pattern of a vortex in the neighbourhood of a solid boundary

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ABSTRACT

A two-layer model of a vortex in the neighbourhood of a plane solid boundary is proposed. Series solutions in terms of a small parameter are used to determine the flow pattern in the turbulent boundary layer beneath the vortex. It is then found that the axial velocity in the vortex is defined by the flow in the boundary layer. The results are shown to bear a close resemblance to known flows in tornadoes.

1. Introduction

The theoretical study of the boundary layers beneath intense atmospheric vortices such as tornadoes does not appear to have received much attention in the literature. The flow in these boundary layers will certainly be turbulent and even the assumption of laminar flow still leads to considerable difficulty. The crux of the problem lies in the nature of the main vortex. The viscous equations of motion have no solution which can adequately describe a simple circular vortex in which the fluid is at rest on the axis and at rest radially at infinity (other than the Rankine combined vortex which has a discontinuity in the derivative of the velocity). The two vortices which can exist in a viscous fluid and their distortion near a boundary have been discussed by BOEDEWADT (1940) and SCHWIDERSKI & LUGT (1963). The former paper describes the rigid body rotation of the fluid above the boundary whilst the latter considers the case in which the transverse velocity is inversely proportional to radial distance. The unbounded velocities in both cases lead to unrealistic models for the atmosphere.

The large depth of the tornado vortex, the fact that it develops downwards to the solid boundary and the time scale of the motion suggest that the main body of the vortex is not directly affected by the presence of the boundary. However, there will be depth δ through which the boundary has an important influence on the flow. Above this level the motion is determined by a mechanism involving the release of potential energy and latent heat which will not concern us here. In the model we propose, the main body of the tornado is replaced by a non-viscous vortex which maintains at the constant height δ, a fixed transverse velocity profile. In the vortex, a reasonable radial pressure distribution is chosen which then determines the local velocities. An interesting feature of a non-viscous vortex is that any axial velocity component, constant with height, may be associated with it. We shall show that this component is determined by conditions in the boundary layer. We shall make the fundamental assumption that δ/a is small, where a is the effective radius of the tornado and then show that, in the boundary layer, the flow variables possess power-series expansions in terms of this parameter.

In the last section some typical numerical values are inserted in the series which are cut off after the first or second terms.

2. The vortex

The steady-state Navier-Stokes equations of motion and continuity in cylindrical polar coordinates with axial symmetry for an incompressible fluid are:

$$uu_r + wu_z - \frac{1}{r}v^2 = -\frac{1}{\varrho}p_r + \nu\left\{u_{rr} + \left(\frac{u}{r}\right)_r + u_{zz}\right\}, \tag{1}$$

$$uv_r + wv_z + \frac{1}{r}uv = \nu\left\{v_{rr} + \left(\frac{v}{r}\right)_r + v_{zz}\right\}, \tag{2}$$

$$uw_r + ww_z = -\frac{1}{\varrho}p_z + \nu\left\{w_{rr} + \frac{1}{r}w_r + w_{zz}\right\}, \tag{3}$$

$$(ru)_r + (rw)_z = 0, \tag{4}$$

where u, v, w are the velocity components in the cylindrical coordinate system (r,θ,z), p is the pressure, ϱ the density and ν the coefficient of kinematic viscosity.

It is readily seen that in the non-viscous case $(\nu = 0)$

$$p = G(r); \quad u = 0; \quad v = (rG'/\varrho)^{\frac{1}{2}}; \quad w = F(r) \tag{5}$$

is a solution of equations (1) to (4) for arbitrary functions G and F. In general, this vortex will be rotational. It will be assumed that G is a prescribed twice-differentiable function thus ensuring (as we shall see) the continuity of the velocity components in the boundary layer. To ensure convergence throughout we take $G' = o(r^{-\beta})$ as $r \to \infty$ with $\beta > 7$, and to avoid singularities along the axis we assume that $G' \to 0$ as $r \to 0$. Finally we suppose that v has a maximum V_m when $r = a$. We shall consider the boundary layer flow associated with this vortex when an infinite plane boundary is inserted perpendicular to the axis of symmetry.

3. An approximate method

Taking the boundary at $z = 0$, the stretch transformation

$$R = r/a, \quad Z = z/\delta, \tag{6}$$

and the dimensionless variables defined by

$$(u, v, w) = V_m(U, V, W), \quad p = \varrho V_m^2 P$$

are introduced. In terms of the new variables, equations (1)–(4) now become

$$R_e\left\{\varepsilon U U_R + W U_Z - \frac{\varepsilon V^2}{R} + \varepsilon P_R\right\} = \varepsilon^2\left\{U_{RR} + \left(\frac{U}{R}\right)_R\right\} + U_{ZZ}, \tag{7}$$

$$R_e\left\{\varepsilon U V_R + W V_Z + \frac{\varepsilon U V}{R}\right\} = \varepsilon^2\left\{V_{RR} + \left(\frac{V}{R}\right)_R\right\} + V_{ZZ}, \tag{8}$$

$$R_e\{\varepsilon U W_R + W W_Z + P_Z\} = \varepsilon^2\left(W_{RR} + \frac{1}{R}W_R\right) + W_{ZZ}, \tag{9}$$

$$\varepsilon(RU)_R + (RW)_Z = 0, \tag{10}$$

where $\varepsilon = \delta/a$ and R_e is the Reynolds number $V_m\delta/\nu$. With ε small, the relevant flow variables have expansions of the form

$$U = \varepsilon U^{(0)}(R, Z) + \varepsilon^3 U^{(1)}(R, Z) + \dots, \tag{11}$$

$$V = V^{(0)}(R, Z) + \varepsilon^2 V^{(1)}(R, Z) + \dots, \tag{12}$$

$$W = \varepsilon^2 W^{(0)}(R, Z) + \varepsilon^4 W^{(1)}(R, Z) + \dots, \tag{13}$$

$$P = P^{(0)}(R, Z) + \varepsilon^2 P^{(1)}(R, Z) + \dots. \tag{14}$$

It can be shown by reflection arguments that V, W and P are even functions of ε and that U is odd. In dimensionless form, the vortex given by (5) becomes

$$\left.\begin{aligned} &P = G(aR)/\varrho V_m^2 = H(R); \quad U = 0; \\ &V = \left\{\frac{R}{\varrho}\frac{dG(aR)}{dR}\right\}^{\frac{1}{2}} / V_m = (RH')^{\frac{1}{2}}; \\ &W = F(aR)/V_m = T(R). \end{aligned}\right\} \tag{15}$$

In most meteorological situations it is unrealistic to take a laminar boundary layer since turbulence becomes a significant feature. In such circumstances it is usual to follow PRANDTL (1952) and replace the no-slip condition on $Z = 0$ by the requirement that the shearing stress be in the direction of the velocity vector. Further the coefficient of eddy viscosity K is substituted for the coefficient of viscosity ν. In the equations of motion, R_e is now the Reynolds number $V_m\delta/K$. On the upper surface of the boundary layer the velocity components and pressure are continuous. In full, on the assumption that the transverse velocity falls by one-half through the boundary layer, the boundary conditions become

$$V = \tfrac{1}{2}(RH')^{\frac{1}{2}}, \quad W = 0, \quad UV_Z = VU_Z \text{ on } Z = 0; \quad (16)$$

$$U = 0, \quad V = (RH')^{\frac{1}{2}} \text{ on } Z = 1. \quad (17)$$

Inserting the expansions (11)–(14) into (16) and (17) and equating like powers of ε, the terms must satisfy the following conditions:

$$\left.\begin{aligned}
&U^{(n)}(R, 1) = 0, \quad W^{(n)}(R, 0) = 0 \\
&\qquad (n = 0, 1, 2, \ldots), \\
&V^{(0)}(R, 0) = \tfrac{1}{2}(RH')^{\frac{1}{2}}, \\
&\qquad V^{(0)}(R, 1) = (RH')^{\frac{1}{2}}, \\
&V^{(n)}(R, 0) = V^{(n)}(R, 1) = 0 \\
&\qquad (n = 1, 2, \ldots), \\
&P^{(0)}(R, 1) = H(R), \quad P^{(n)}(R, 1) = 0 \\
&\qquad (n = 1, 2, \ldots) \\
&\sum_{m=0}^{n} \{U^{(m)} V_Z^{(n-m)} - V^{(m)} U_Z^{(n-m)}\} \\
&\qquad = 0 \quad \text{at } Z = 0 \quad (n = 0, 1, 2, \ldots).
\end{aligned}\right\} \quad (18)$$

Introduction of the expansions (11)–(14) into (7)–(10) leads to the following systems of partial differential equations which, to zero order in ε, are

$$V_{ZZ}^{(0)} = 0, \quad (19)$$

$$P_Z^{(0)} = 0; \quad (20)$$

to the first order

$$R_e\left[P_R^{(0)} - \frac{1}{R}\{V^{(0)}\}^2\right] = U_{ZZ}^{(0)}; \quad (21)$$

to the second order

$$R_e\left\{U^{(0)}V_R^{(0)} + W^{(0)}V_Z^{(0)} + \frac{1}{R}U^{(0)}V^{(0)}\right\}$$

$$= V_{RR}^{(0)} + \frac{1}{R}V_R^{(0)} - \frac{1}{R^2}V^{(0)} + V_{ZZ}^{(1)}, \quad (22)$$

$$R_e P_Z^{(1)} = W_{ZZ}^{(0)}, \quad (23)$$

$$\{RU^{(0)}\}_R + RW_Z^{(0)} = 0; \quad (24)$$

to the third order

$$R_e\left\{U^{(0)}U_R^{(0)} + W^{(0)}U_Z^{(0)} - \frac{2}{R}V^{(0)}V^{(1)} + P_R^{(1)}\right\}$$

$$= U_{RR}^{(0)} + \frac{1}{R}U_R^{(0)} - \frac{1}{R^2}U^{(0)} + U_{ZZ}^{(1)} \quad (25)$$

etc. The scheme continues in a similar manner for succeeding terms.

From (19), (20) and (18) it is a simple matter to show that

$$V^{(0)} = \tfrac{1}{2}(RH')^{\frac{1}{2}}(Z+1), \quad (26)$$

$$P^{(0)} = H. \quad (27)$$

The latter expression merely confirms the usual boundary layer assumption that the pressure, to the lowest order, is unaffected by the presence of the boundary. Integrations, performed in the sequence (21), (24), (22), (23) subject to the end-conditions (18), now give the first one or two terms in the expansions, namely

$$U^{(0)} = \frac{R_e H'}{96}(1-Z)(2Z^3 + 10Z^2 - 26Z - 13), \quad (28)$$

$$W^{(0)} = \frac{R_e(RH')'}{960R} Z(4Z^4 + 20Z^3 - 120Z^2 + 65Z + 130), \quad (29)$$

$$\begin{aligned}
V^{(1)} = {} & \frac{R_e^2 Z(1-Z)(RH')^{\frac{1}{2}}}{79{,}690\, R}[H'\{26Z^5 + 208Z^4 - 422Z^3 - 1857Z^2 - 37Z + 4058\} \\
& + 3RH''\{2Z^5 + 16Z^4 + 2Z^3 - 208Z^2 - 208Z + 247\}] \\
& + \frac{1}{12}Z(1-Z)(Z+4)\left[\{(RH')^{\frac{1}{2}}\}'' + \left\{\frac{1}{R}(RH')^{\frac{1}{2}}\right\}'\right], \quad (30)
\end{aligned}$$

$$P^{(1)} = \frac{(RH')'}{96R}(Z-1)(2Z^3 + 10Z^2 - 26Z - 13). \quad (31)$$

Since all terms in the series are polynomials in Z, there are no mathematical difficulties in calculating further terms, although they rapidly become complicated. To the lowest order, the vertical velocity profile in the main vortex is given by

$$T(R) \approx \varepsilon^2 W^{(0)}(R, 1) = \frac{33 R_e \varepsilon^2 (RH')}{320 R} \quad (Z \geqslant 1) \qquad (32)$$

from (29).

From the equation of continuity (10), a stream function ψ for secondary flow generated by the vortex can be introduced through

$$\varepsilon R U = \psi_Z, \quad R W = -\psi_R. \qquad (33)$$

It is easy to see that we can write

$$\psi = \varepsilon^2 \psi^{(0)} + \varepsilon^4 \psi^{(1)} + \dots \qquad (34)$$

and evaluate each $\psi^{(n)}$ separately in terms of $U^{(n)}$ and $W^{(n)}$. For example, from (28), (29) and (32)

$$\psi^{(0)} = \begin{cases} -\dfrac{R_e}{960} RH'Z(4Z^4 + 20Z^3 - 120Z^2 \\ \qquad + 65Z + 130) \quad (0 \leqslant Z \leqslant 1), \\ -\dfrac{33}{320} R_e RH' \quad (Z > 1). \end{cases} \qquad (35)$$

In passing, we remark that the case of laminar flow can be evaluated in a very similar manner.

4. Boundary layer thickness δ

In any physically real situation the retarding couple acting on a free vortex due to the presence of the solid boundary would ultimately affect the whole depth of the vortex in any steady-state investigation. We felt however that in a tornado, for example, the constant regeneration of the vorticity in the vortex, due to some mechanism involving the latent instability of the air, would imply a typical depth δ of the boundary layer at which the retarding couple due to the solid boundary becomes equal to the accelerating couple in the main body of the tornado. Now in our simplified two-layer model, the upper vortex in the non-viscous layer contains zero couple by definition. In this case, δ is determined by the condition that the total couple over the disc at the top of the boundary layer is zero, that is

$$M = 2\pi \varrho \nu \int_0^\infty r^2 v_z(r, \delta)\, dr = 0, \qquad (36)$$

or, after non-dimensionalising and using the first two terms of the expansion for V

$$\int_0^\infty R^2 \{ V_Z^{(0)}(R, 1) + \varepsilon^2 V_Z^{(1)}(R, 1) \}\, dR = 0. \qquad (37)$$

It is implicit in (37) that ε must be "small" since we have cut off the expansion after two terms.

We note that the transverse velocity $v \sim 1/r$ for large r will not be admissible since the couple given by (36) will be infinite. In consequence, the Rankine combined vortex, for example, cannot be used as a basis for the free vortex. This is not surprising since an infinite couple would be required to generate such a vortex in a viscous fluid.

5. The pressure distribution

Having established the main points of the theory, it remains to choose a realistic pressure distribution for a vortex such as a tornado. We shall consider a simple exponential distribution

$$p = G(r) = p_0 \{1 - \alpha \exp(-r^2/a^2)\}, \qquad (38)$$

where p_0 is normal atmospheric pressure and α a constant which fixes the pressure drop in the core of the vortex. This is the normal vortex referred to by JAMES (1950) and confirmed for such meteorological features as the tropical cyclone by SELLICK (1958).

Using (15) we get

$$H(R) = \frac{p_0}{\varrho V_m^2} \{1 - \alpha \exp(-R^2)\}, \qquad (39)$$

and from (5)

$$v = \left(\frac{2 p_0 \alpha}{\varrho}\right)^{\frac{1}{2}} \frac{r}{a} \exp(-r^2/2a^2).$$

The maximum transverse velocity V_m occurs when $r = a$ and is given by

$$V_m = (2 p_0 \alpha / \varrho e)^{\frac{1}{2}}. \qquad (40)$$

Thus in the vortex

$$V = \frac{v}{V_m} = R \exp\{(1-R^2)/2\}.$$

With the pressure field fully specified, it is an easy matter to determine explicitly the lowest order terms from (26), (28), (29) and (35). They are

$$V^{(0)} = \tfrac{1}{2}(Z+1)R \exp\{\tfrac{1}{2}(1-R^2)\}, \tag{41}$$

$$U^{(0)} = \frac{R_e}{96}(1-Z)(2Z^3+10Z^2-26Z-13)R \exp(1-R^2), \tag{42}$$

$$W^{(0)} = \frac{R_e}{480} Z(4Z^4+20Z^3-120Z^2+65Z+130)(1-R^2)\exp(1-R^2), \tag{43}$$

$$\psi^{(0)} = \begin{cases} -\dfrac{R_e}{960} Z(4Z^4+20Z^3-120Z^2+65Z+130)R^2 \exp(1-R^2), & 0 \leqslant Z \leqslant 1, \\ -\dfrac{33R_e}{320} R^2 \exp(1-R^2) & Z \geqslant 1, \end{cases} \tag{44}$$

The vertical velocity profile in the vortex now becomes, to the lowest order,

$$T(R) \approx \frac{33R_e}{160}\varepsilon^2(1-R^2)\exp(1-R^2). \tag{45}$$

We observe from (42) that the air always flows towards the axis of the vortex in the boundary layer. Further, equation (45) shows that the air is moving up the core of the vortex and is being drawn into the boundary layer outside the core. The first-order term in V (equation (30)) is given by

$$V^{(1)} = \frac{R_e^2 Z(1-ZR \exp\{\tfrac{3}{2}(1-R^2)\}}{79{,}690} \left[\{32Z^5+256Z^4-416Z^3-2481Z^2-661Z+4799\} - 6R^2\{2Z^5+16Z^4+2Z^3-208Z^2-208Z+247\}\right] + \tfrac{1}{12}Z(1-Z)(Z+4)R(R^2-4)\exp\{\tfrac{1}{2}(1-R^2)\}. \tag{46}$$

The requirement that the moment of the drag is zero at the top of the boundary layer leads to the condition $\varepsilon R_e \approx 7.0$ after the integrals in (37) have been evaluated. Thus

$$\delta \approx 2.65\left(\frac{aK}{V_m}\right)^{\frac{1}{2}}. \tag{47}$$

Using $\varepsilon R_e = 7$, we then find that the dimensionless velocities are given by

$$U \approx \varepsilon U^{(0)} = 7.3\times 10^{-2}(1-Z)(2Z^3+10Z^2-26Z-13)R\exp(1-R^2), \tag{48}$$

$$W \approx \varepsilon^2 W^{(0)} = 3.8\times 10^{-2}\left(\frac{K}{aV_m}\right)^{\frac{1}{2}} Z(4Z^4+20Z^3-120Z^2+65Z+130)(1-R^2)\exp(1-R^2), \tag{49}$$

$$V \approx V^{(0)} + \varepsilon^2 V^{(1)}. \tag{50}$$

For this perturbation problem we shall restrict ε to be less than 1/3, which means that R_e must exceed 21. With these values, V, given by (50), is substantially a function of εR_e alone since the last term of (46) is numerically much smaller than the first.

A graph showing the distribution of V throughout the boundary layer at $R=1$ is shown in Fig. 1. Fig. 2 shows the distribution of $W(aV_m/K)^{\frac{1}{2}}$ at $Z=1$ and Fig. 3 that of U at $R=1/\sqrt{2}$, the value of R at which the maximum radial speed occurs. Some typical streamlines, $\psi \approx \varepsilon^2\psi^{(0)} = \text{constant}$, are indicated in Fig. 4.

6. Discussion

We may now compare the theoretical results with observations on intense atmospheric vortices in which the effective width is much less than the total depth, e.g. tornadoes. BROOKS (1951), summarising previous observations, gave mean values of diameters and maximum values of wind speeds recorded. He concluded that an average diameter was about 220 m, implying a value of $a = 60$ m in our work (since this is the radius of maximum transverse wind speed and not the radius of damage), and that maximum wind speeds varied between 50 m/sec and 150 m/sec. Since the latter were probably due to

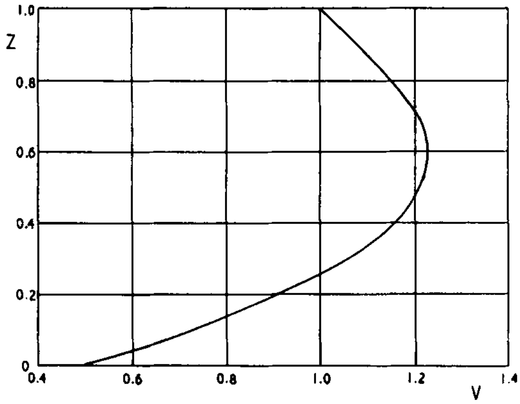

FIG. 1. The dimensionless transverse velocity component V vs. Z at $R = 1$.

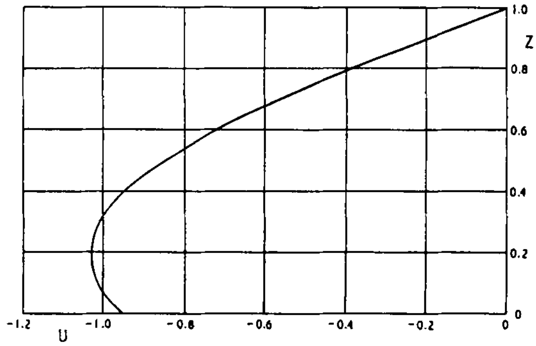

FIG. 3. The dimensionless radial velocity U vs. Z at $R = 1/\sqrt{2}$, the value of R at which maximum radial speeds occur.

gusts we chose $V_m = 75$ m/sec as a typical value. We see from equation (40) that this implies $\alpha \approx 1/10$ or a 10 % drop in pressure between the environment and the centre of the tornado, a value which again bears good comparison with values of the pressure fall found by Brooks and confirmed by BATTAN (1961). Since

$$v = V_m V, \quad u = V_m U$$

the corresponding values of the transverse and radial wind speeds at the given radii in the boundary layer may be deduced from Figs. 1 and 3. Another feature noted by Brooks was that the radial speed was often greater than the tangential speed. We see this in our model at, for example, $R = 1/\sqrt{2}$, $Z = 0.2$ where

$$v = 0.96\ V_m, \quad u = 1.03\ V_m.$$

So far we have not been required to estimate K since the condition of zero drag at the top of the boundary layer gave us U and V directly to the order required. To evaluate the vertical velocity and the boundary layer thickness we do however require a numerical value for K. Now all previous evaluations of K have to our knowledge been obtained in relatively steady conditions of overall straight flow, e.g. DOBSON (1914), and have yielded values of K of the order of 5 m^2/sec for moderate values of the

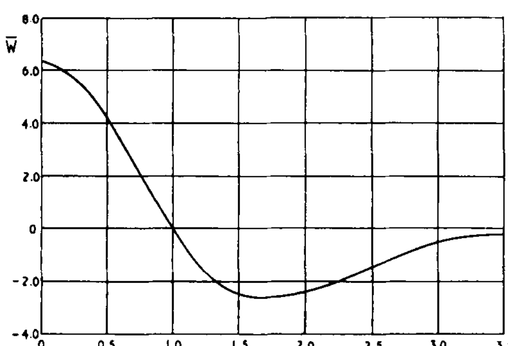

FIG. 2. The dimensionless variable $\overline{W} = (K/aV_m)^{\frac{1}{2}} W$, where W is the dimensionless vertical velocity, vs. R at the top of the boundary layer $Z = 1$. This shows downdraught outside the vortex core with updraught inside.

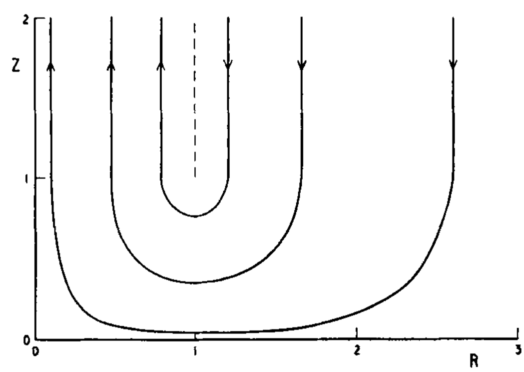

FIG. 4. Some typical streamlines of the secondary flow produced by the vortex motion normal to a flat surface.

wind speed [BRUNT (1952)], the value increasing with the speed. From estimates of the vertical velocity mentioned by Brooks a value of K of 20 m^2/sec may be deduced. This leads from (47) to a depth of the boundary layer of about 10 m which would seem to be of the right order in the circumstances. We note finally that this implies $\varepsilon = 1/6$, a value encouraging confidence in the method we have used which assumed initially that ε was small.

REFERENCES

BATTAN, L., 1961, *The Nature of Violent Storms*, p. 79. Anchor Books, New York.

BOEDEWADT, U. T., 1940, Die Drehströmung über festerem Grunde, *ZAMM*, **20**, pp. 241–253.

BROOKS, E. M., 1951, Tornadoes and related phenomena. In *Compendium of Meteorology*, pp. 673–680. Amer. Met. Soc., Boston.

BRUNT, D., 1952, *Physical and Dynamical Meteorology*, p. 255. Univ. Press, Cambridge.

JAMES, R. W., 1950, On the theory of large-scale vortex motion in the atmosphere. *Quart. J. Roy. Met. Soc.*, **76**, pp. 255–276.

PRANDTL, L., 1952, *Fluid Dynamics*, p. 359. Blackie, London.

SCHWIDERSKI, E., and LUGT, H., 1963, Vortex flows normal to a flat surface. *Tellus*, **15**, pp. 241–245.

SELLICK, N. P., 1958, Modification of the structure of a tropical cyclone. *Quart. J. Roy. Met. Soc.*, **84**, pp. 259–269.