On the Boussinesq approximation and its role in the theory of internal waves

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ABSTRACT

Most studies of the dynamics of fluids with weak density gradients use the Boussinesq approximation, which neglects density variations in the inertial terms of the equation of motion. The significance of this approximation is investigated here. As a general matter, it appears that care is called for in the use of the approximation if quantities other than the non-dimensional density difference are small. Also, the existence of certain phenomena of interest and importance may depend on the small terms neglected by the Boussinesq approximation. This is shown by an investigation of solitary waves in a stratified fluid system.

1. Introduction

Most studies of fluids with density gradients use the Boussinesq approximation (BOUSSINESQ, 1903), which neglects density variations in the inertial terms of the equations of motion. The usual argument is that the density difference $\Delta\varrho$ occurs in the acceleration terms as

$$\left(1-\frac{\Delta\varrho}{\bar{\varrho}}\right)\frac{d\mathbf{v}}{dt}, \tag{1}$$

where $\bar{\varrho}$ is the mean density, and, therefore, if $\Delta\varrho/\bar{\varrho} < 1$ the acceleration term may be simplified to $d\mathbf{v}/dt$. But (the argument continues), the term $g\Delta\varrho/\bar{\varrho}$ in the vertical equation of motion should not be neglected because g is large.[2] Below, we show that this argument may lead to inconsistencies if there are other, small non-dimensional quantities besides $\Delta\varrho/\bar{\varrho}$. In addition, we will see that the existence of phenomena of interest and importance may depend on the small terms neglected by the Boussinesq approximation.

2. Small amplitude, long wave-length, internal, gravity waves in a liquid with weak density gradient

Consider an ideal liquid with a weak density gradient in a channel of height h. The bottom, $y'=0$ and top $y'=h$, are rigid planes, the fluid extends to infinity in the horizontal x'-direction, and the motion is two-dimensional.[3] We choose a basic density gradient,

$$\varrho(y') = \varrho_0\,(1-by'). \tag{2}$$

Then, if δ' is the displacement of a particle above its equilibrium height, and if ψ' is the streamfunction, the equations may be put in the form,

$$\begin{aligned}(1-by'+b\delta')\,[\psi'_{x'x't'}+\psi'_{y'y't'}+\psi'_{x'}\,(\psi'_{x'x'y'}\\ +\psi'_{y'y'y'})-\psi'_{y'}\,(\psi'_{x'x'x'}+\psi'_{y'y'x'})]\\ +b\delta'_{x'}\,[\psi'_{x't'}-\psi'_{y'}\,\psi'_{x'x'}+\psi'_{x'}\,\psi'_{x'y'}]\\ -b(1-\delta'_{y'})\,[\psi'_{y't'}-\psi'_{y'}\,\psi'_{y'x'}\\ +\psi'_{x'}\,\psi'_{y'y'}]+gb\delta'_{x'}=0,\end{aligned} \tag{3}$$

$$(\delta'_{y'}-1)\,\psi'_{x'}-\psi'_{y'}\,\delta'_{x'}+\delta'_{t'}=0. \tag{4}$$

In addition to (3) and (4), there are the initial conditions and the homogeneous boundary con-

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[2] More involved arguments have been given by SPIEGEL & VERONIS (1960).

[3] From now on we use primed symbols for dimensional quantities and unprimed symbols for non-dimensional quantities.

ditions that δ' and $\psi'_{x'}$ vanish at the top and bottom of the channel. We assume that initial conditions are given in terms of variables,

$$\frac{x'}{\lambda}, \frac{y'}{h}, \frac{a}{h}, \tag{5}$$

i.e., the characteristic horizontal wave-length is λ, the characteristic vertical length is the depth of the channel, and the amplitude of the disturbance is of the order of a. We suppose that the initial conditions do not introduce any characteristic velocity. (As a special case, the fluid may be initially at rest with a disturbance field of the order of a).

We now non-dimensionalize according to the following scheme:

$$\frac{\psi'}{\varepsilon_1 h^2 \sqrt{gb}} = \psi, \quad \frac{\gamma^{\frac{1}{2}} t' \sqrt{gb}}{\varepsilon_2} = t$$

and

$$\frac{x'}{\lambda} = x, \quad \frac{y'}{h} = y, \quad \frac{\delta'}{\alpha h} = \delta, \quad \frac{a}{h} = \alpha, \quad bh = \beta, \quad \frac{h^2}{\lambda^2} = \gamma. \tag{6}$$

The unknown constants, ε_1, ε_2, are to be determined by the assumptions: (1) that the disturbance has a long wave length, i.e.,

$$\frac{\partial^2}{\partial x^2} < \frac{\partial^2}{\partial y^2}$$

to the first approximation; (2) that the amplitude of the disturbance is small, so that the non-linear terms in the differential equations are small; and (3) that the density gradient is small. These mean that the first approximations to Eqs. (3) and (4) are

$$\psi'_{y'y't'} + gb\delta'_{x'} = 0,$$

$$\delta'_{t'} - \psi'_{x'} = 0.$$

Using Eq. (6), we find that $\varepsilon_2 = 1$ and $\varepsilon_1 = \alpha$. Our non-dimensional quantities are now those in Eq. (6) and

$$\frac{\psi'}{\alpha h^2 \sqrt{gb}} = \psi, \quad \gamma^{\frac{1}{2}} t' \sqrt{gb} = t. \tag{7}$$

Equations (3) and (4) become

$$(1 - \beta y + \alpha\beta\delta)\,[\psi_{yyt} + \gamma\psi_{xxt} + \alpha\psi_x\,(\psi_{yyy} + \gamma\psi_{xxy}) - \alpha\psi_y \times (\psi_{yyx} + \gamma\psi_{xxx})] + \delta_x + \alpha\beta\gamma\delta_x\,[\psi_{xt} - \alpha\psi_y\,\psi_{xx} + \alpha\psi_x\psi_{xy}] - \beta(1 - \alpha_y)\,[\psi_{yt} - \alpha\psi_y\,\psi_{yx} + \alpha\psi_x\,\psi_{yy}] = 0, \tag{8}$$

$$\delta_t - \psi_x\,(1 - \alpha\delta_y) - \alpha\psi_y\,\delta_x = 0. \tag{9}$$

Obviously, according to our assumptions, the boundary and initial conditions also involve only the quantities in Eqs. (6) and (7). The solution may therefore be expanded in terms of small parameters, α, β, γ so that

$$\psi = \psi_{000} + \alpha\psi_{100} + \beta\psi_{010} + \gamma\psi_{001} + \alpha^2\,\psi_{200} + \dots, \tag{10}$$

$$\delta = \delta_{000} + \alpha\delta_{100} + \beta\delta_{010} + \gamma\delta_{001} + \alpha^2\,\delta_{200} + \dots. \tag{11}$$

We see now that if we retain only the first approximation, to obtain

$$\left.\begin{aligned} \frac{\partial^3\,\psi_{000}}{\partial y^2\,\partial t} + \frac{\partial\,\delta_{000}}{\partial x} &= 0, \\ \frac{\partial\,\delta_{000}}{\partial t} - \frac{\partial\psi_{000}}{\partial x} &= 0 \end{aligned}\right\} \tag{12}$$

we are assuming that α, β, γ are all small compared to one. But it is clear that retaining additional terms in (10) and (11), involving α and γ, while at the same time neglecting some or all of the terms involving β, in other words, making the Boussinesq approximation, is fraught with danger. For example, if the amplitude α of the disturbance is about 0.10, and if the non-dimensional density difference β is also about 0.10, it is inconsistent to look for non-linear effects by retaining terms $\psi_{100}, \psi_{200}, \dots$, while neglecting the equally important terms $\psi_{010}, \psi_{020}, \dots$. Further, if the non-linear effects of terms $\psi_{100}, \psi_{200} \dots$ provide, as is common, a tendency for an initial disturbance to steepen and break, it is entirely possible that the $\psi_{010}, \psi_{001}, \dots$ terms could exactly counteract this tendency and permit a steadily progressing disturbance.[1] Finally, in some cases, a solution may not be fully determined if the β-terms in (10) and (11) are neglected. In the next section we illustrate these possibilities by a specific example.

3. Internal solitary waves

Let us now investigate the special case of a steady disturbance in a stratified fluid, propa-

[1] Similar comments may be made about the hydrostatic approximation in the theory of water waves (LONG, 1964 b).

gating at a constant, non-dimensional speed c. We may use Eqs. (8) and (9) or, more conveniently, notice that the primitive equations in a coordinate system moving with the disturbance may be integrated once (LONG, 1953) to yield

$$\nabla^2 \delta' - \frac{1}{2}\frac{d \ln \varrho c^2}{dy_0'}\left[(\nabla\delta')^2 - 2\frac{\partial \delta'}{\partial y'}\right] - \frac{g}{c^2 \varrho}\frac{d\varrho}{dy_0'}\delta' = 0. \tag{13}$$

In the derivation of (13) it was assumed that the disturbance vanishes at $x = \infty$ or $x = -\infty$. The quantity c is now the uniform horizontal velocity in the undisturbed region; y_0' is the height, in the undisturbed region, of the fixed streamline through the point (x', y').

The mathematics is a bit simpler if we direct attention to flow with an exponential basic density gradient,

$$\varrho = \varrho_0 e^{-by_0}. \tag{14}$$

In terms of the quantities in (6), Eq. (13) becomes

$$\frac{\partial^2 \delta}{\partial y^2} + \gamma \frac{\partial^2 \delta}{\partial x^2} + \frac{\beta}{2}\left[\alpha\left(\frac{\partial \delta}{\partial y}\right)^2 + \alpha\gamma\left(\frac{\partial \delta}{\partial x}\right)^2 - 2\frac{\partial \delta}{\partial y}\right] + \sigma^2 \delta = 0, \tag{15}$$

where
$$\sigma^2 = \frac{gbh^2}{c^2}. \tag{16}$$

Let us now assume various expressions for γ in terms of α, β and search for solutions of (15) in the form,

$$\delta = \delta_{00} + \alpha\delta_{10} + \beta\delta_{01} + \alpha\beta_{11} + \dots, \tag{17}$$

$$\sigma^2 = \sigma_{00}^2 + \alpha\sigma_{10}^2 + \beta\sigma_{01}^2 + \alpha\beta\sigma_{11}^2 + \dots. \tag{18}$$

(A) $\gamma = \beta$.

The first approximation is

$$\frac{\partial^2 \delta_{00}}{\partial y^2} + \sigma_{00}^2 \delta_{00} = 0. \tag{19}$$

Since δ_{00} must vanish at the top and bottom of the channel, the solution is

$$\delta_{00} = f(x) \sin n\pi y, \tag{20}$$

$$\sigma_{00}^2 = n^2 \pi^2, \tag{21}$$

where $f(x)$ is arbitrary. A second approximation is

$$\frac{\partial^2 \delta_{01}}{\partial y^2} + n^2\pi^2 \delta_{01} = -\sigma_{01}^2 \delta_{00} - \frac{\partial^2 \delta_{00}}{\partial x^2} + \frac{\partial \delta_{00}}{\partial y} = -(f'' + \sigma_{01}^2 f)\sin n\pi y + n\pi f \cos n\pi y \tag{22}$$

with solution

$$\delta_{01} = f_1 \sin n\pi y + f_2 \cos n\pi y + \frac{y \cos n\pi y}{2n\pi} \times (f'' + \sigma_{01}^2 f) + \frac{f}{2} y \sin n\pi y, \tag{23}$$

where f_1 and f_2 are arbitrary. But $f_2 = 0$, since δ_{01} must vanish at $y = 0$. Then the condition that $\delta_{01} = 0$ at $y = 1$ yields

$$f'' + \sigma_{01}^2 f = 0. \tag{24}$$

This has no solution for f that vanishes at either $x = \infty$ or $x = -\infty$. There is no steady-state disturbance for $\gamma = \beta$ that vanishes at infinity.

(B) $\gamma = \alpha$.

Again the first approximation is

$$\delta_{00} = f \sin n\pi y, \tag{25}$$

$$\sigma_{00}^2 = n^2 \pi^2 \tag{26}$$

and a second approximation is

$$\frac{\partial^2 \delta_{10}}{\partial y^2} + n^2\pi^2 \delta_{10} = -\frac{\partial^2 \delta_{00}}{\partial x^2} - \sigma_{10}^2 \delta_{00} = -\sin n\pi y \left(f'' + \sigma_{10}^2 f\right). \tag{27}$$

A solution satisfying $\delta_{10} = 0$ at $y = 0$ is

$$\delta_{10} = g_1(x) \sin n\pi y + \frac{y \cos n\pi y}{2n\pi}(f'' + \sigma_{10}^2 f). \tag{28}$$

The requirement that $\delta_{10} = 0$ at $y = 1$ yields

$$f'' + \sigma_{10}^2 f = 0. \tag{29}$$

Again there is no steady-state disturbance for $\gamma = \alpha$ that vanishes at infinity.

(C) $\gamma = \alpha\beta$

We again have

$$\delta_{00} = f \sin n\pi y, \tag{30}$$

$$\sigma_{00}^2 = n^2 \pi^2. \tag{31}$$

The differential equation for δ_{10} is the same as Eq. (27), except that the first term on the right is missing. Therefore,

$$\delta_{10} = g_1(x) \sin n\pi y + \frac{\sigma_{10}^2 fy \cos n\pi y}{2n\pi}. \tag{32}$$

Since $\delta_{10} = 0$ at $y = 1$, we see that

$$\sigma_{10}^2 = 0, \quad \delta_{10} = g_1 \sin n\pi y. \tag{33}$$

The equation for δ_{01} is

$$\frac{\partial^2 \delta_{01}}{\partial y^2} + n^2\pi^2 \delta_{01} = -\sigma_{01}^2 \delta_{00} + \frac{\partial \delta_{00}}{\partial y}$$

$$= -\sigma_{01}^2 f \sin n\pi y + n\pi f \cos n\pi y. \tag{34}$$

The solution satisfying $\delta_{01} = 0$ at $y = 0$ is

$$\delta_{01} = f_1 \sin n\pi y + \frac{\sigma_{01}^2 fy \cos n\pi y}{2n\pi} + \frac{f}{2} y \sin n\pi y. \tag{35}$$

Since $\delta_{01} = 0$ at $y = 1$, we see that

$$\sigma_{01}^2 = 0, \quad \delta_{01} = f_1 \sin n\pi y + \frac{f}{2} y \sin n\pi y. \tag{36}$$

Finally, the equation for δ_{11} is

$$\frac{\partial^2 \delta_{11}}{\partial y^2} + n^2\pi^2 \delta_{11} = -\frac{\partial^2 \delta_{00}}{\partial x^2} - \frac{1}{2}\left(\frac{\partial \delta_{00}}{\partial y}\right)^2 + \frac{\partial \delta_{10}}{\partial y}$$

$$-\sigma_{11}^2 \delta_{00} = -(f'' + \sigma_{11}^2 f) \sin n\pi y + n\pi g_1$$

$$\times \cos n\pi y - \frac{1}{4} f^2 n^2 \pi^2 (1 + \cos 2n\pi y) \tag{37}$$

so that

$$\delta_{11} = f_2 \sin n\pi y$$

$$+ \frac{f^2}{6} \cos n\pi y + \frac{(f'' + \sigma_{11}^2 f)\, y \cos n\pi y}{2n\pi}$$

$$+ \frac{g_1 y \sin n\pi y}{2} - \frac{1}{4} f^2 + \frac{f^2}{12} \cos 2n\pi y. \tag{38}$$

Since $\delta_{11} = 0$ at $y = 1$,

$$[(-1)^n - 1]\frac{f^2}{6} + \frac{(-1)^n}{2n\pi}(f'' + \sigma_{11}^2 f) = 0. \tag{39}$$

If n is even, the first term vanishes completely, and the remaining differential equation has no solution vanishing at $x = \infty$. *For odd n*, however, we get

$$f'' + \sigma_{11}^2 f + \frac{2}{3} n\pi f^2 = 0, \quad n = 1, 3, \ldots \tag{40}$$

with solution

$$f = -\frac{9\sigma_{11}^2}{4n\pi} \operatorname{sech}^2\left(\frac{x}{2} i\sigma_{11}\right), \quad n = 1, 3, \ldots \tag{41}$$

vanishing at $|x| = \infty$ if σ_{11}^2 is negative. Notice that if we regard α in (6) as the non-dimensional maximum amplitude of the disturbance, then

$$-\frac{9\sigma_{11}^2}{4n\pi} = 1 \quad n = 1, 3, \ldots. \tag{42}$$

To the present order then,

$$\frac{\delta'}{h} = \alpha \operatorname{sech}^2 \frac{x'}{3h} (n\pi\alpha\beta)^{\frac{1}{2}} \sin n\pi \frac{y'}{h}, \quad n = 1, 3, \ldots, \tag{43}$$

$$\sigma^2 = n^2\pi^2 - \frac{4n\pi}{9}\alpha\beta, \quad n = 1, 3, \ldots. \tag{44}$$

(D) $\gamma = \alpha^2 \beta$.

As in (C), we have

$$\sigma_{00}^2 = n^2\pi^2, \quad \delta_{00} = f \sin n\pi y, \tag{45}$$

$$\sigma_{10}^2 = 0, \quad \delta_{10} = g_1 \sin n\pi y, \tag{46}$$

$$\sigma_{01}^2 = 0, \quad \delta_{01} = f_1 \sin n\pi y + \frac{f}{2y} \sin n\pi y. \tag{47}$$

In addition, the solution for δ_{11} in (38) and (39) is applicable here if we omit the terms involving f''. Equation (39) then reveals that $f = 0$ if n is odd, and that $\sigma_{11}^2 = 0$ is n if even. Hence

$$\sigma_{11}^2=0,\quad \delta_{11}=f_2\sin n\pi y+\frac{g_1 y\sin n\pi y}{2}+\frac{f^2}{12}(2\cos n\pi y-3+\cos 2n\pi y)\quad n=2,4,\dots.\tag{48}$$

Continuing, we obtain

$$\sigma_{20}^2=0,\quad \delta_{20}=g_2\sin n\pi y\quad n=2,4,\dots\tag{49}$$

and

$$\frac{\partial^2\delta_{21}}{\partial y^2}+n^2\pi^2\delta_{21}=-\sigma_{21}^2\delta_{00}-\frac{\partial^2\delta_{00}}{\partial x^2}+\frac{\partial\delta_{20}}{\partial y}-\frac{\partial\delta_{00}}{\partial y}\frac{\partial\delta_{10}}{\partial y}=-(\sigma_{21}^2 f+f'')\sin n\pi y+g_2 n\pi\cos n\pi y-\frac{fg_1 n^2\pi^2}{2}\times(1+\cos 2n\pi y).\tag{50}$$

The solution satisfying $\delta_{21}=0$ at $y=0$ is

$$\delta_{21}=f_3\sin n\pi y+\frac{fg_1}{3}\cos n\pi y+\frac{y\cos n\pi y}{2n\pi}\times(f''+\sigma_{21}^2 f)+\frac{g_2 y}{2}\sin n\pi y-\frac{fg_1}{2}+\frac{fg_1}{6}\cos 2n\pi y\quad n=2,4,\dots.\tag{51}$$

Applying the condition that $\delta_{21}=0$ at $y=1$, we get

$$f''+\sigma_{21}^2 f=0.\tag{52}$$

There is no steady-state disturbance for $\gamma=\alpha^2\beta$ that vanishes at infinity.

(E) $\gamma=\alpha\beta^2$.

As in (D) we have

$$\sigma_{00}^2=n^2\pi^2,\quad \delta_{00}=f\sin n\pi y,\tag{53}$$

$$\sigma_{10}^2=0,\quad \delta_{10}=g_1\sin n\pi y,\tag{54}$$

$$\sigma_{20}^2=0,\quad \delta_{20}=g_2\sin n\pi y,\tag{55}$$

$$\sigma_{01}^2=0,\quad \delta_{01}=f_1\sin n\pi y+\frac{f}{2}y\sin n\pi y,\tag{56}$$

$$\sigma_{11}^2=0,\quad \delta_{11}=\left(f_2+\frac{g_1 y}{2}\right)\sin n\pi y+\frac{f^2}{12}\times(2\cos n\pi y-3+\cos 2n\pi y).\tag{57}$$

As in (D) the derivation of (57) yields the requirement that n be even. Also

$$\frac{\partial\delta_{02}}{\partial y^2}+n^2\pi^2\delta_{02}=-\sigma_{02}^2\delta_{00}+\frac{\partial\delta_{01}}{\partial y}=-\sigma_{02}^2 f\sin n\pi y+f_1 n\pi\cos n\pi y+\frac{f}{2}\sin n\pi y+\frac{f}{2}n\pi y\cos n\pi y\quad n=2,4,\dots\tag{58}$$

so that

$$\delta_{02}=f_4\sin n\pi y-\frac{f}{4n\pi}y\cos n\pi y+\sigma_{02}^2\frac{f}{2n\pi}y\cos n\pi y+\frac{fn\pi}{2}\times\left(\frac{1}{4n^2\pi^2}y\cos n\pi y+\frac{y^2}{4n\pi}\sin n\pi y\right)+\frac{f_1}{2}\times y\sin n\pi y\quad n=2,4,\dots.\tag{59}$$

The condition at $y=1$ yields

$$f=4\sigma_{02}^2 f\tag{60}$$

so that

$$\sigma_{02}^2=\frac{1}{4},\quad \delta_{02}=f_4\sin n\pi y+\frac{f_1}{2}y\sin n\pi y+\frac{f}{8}y^2\sin n\pi y.\tag{61}$$

Finally,

$$\frac{\partial^2\delta_{12}}{\partial y^2}+n^2\pi^2\delta_{12}=-\frac{\partial^2\delta_{00}}{\partial x^2}-\frac{\partial\delta_{01}}{\partial y}\frac{\partial\delta_{00}}{\partial y}+\frac{\partial\delta_{11}}{\partial y}-\sigma_{02}^2\delta_{10}-\sigma_{12}^2\delta_{00}=-(f''+\sigma_{12}^2 f)\sin n\pi y-fn\pi\cos n\pi y\left(f_1 n\pi\cos n\pi y+\frac{f}{2}\sin n\pi y+\frac{f}{2}n\pi y\cos n\pi y\right)+f_2 n\pi\cos n\pi y+\frac{g_1}{4}\sin n\pi y+\frac{g_1}{2}n\pi y\cos n\pi y-\frac{f^2}{6}n\pi\sin n\pi y-\frac{f^2}{6}n\pi\sin 2n\pi y.\tag{62}$$

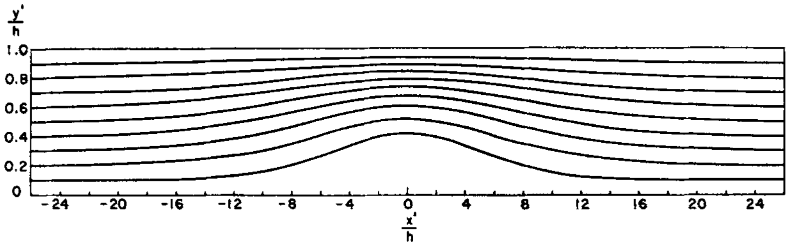

FIG. 1. Solitary wave in a stratified fluid. $\alpha = 0.33$, $\beta = 0.10$.

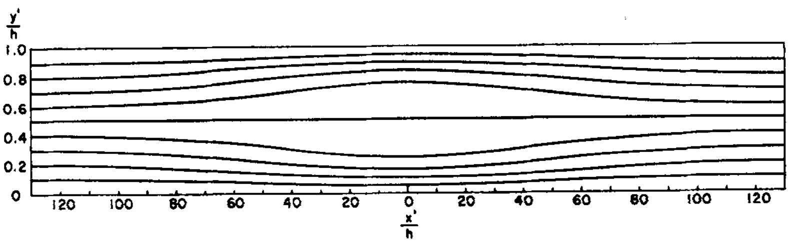

FIG. 2. Solitary wave in a stratified fluid. $\alpha = 0.17$, $\beta = 0.10$.

The solution is

$$\delta_{12} = -\left(\frac{g_1}{4} - f'' - \sigma_{12}^2 f - \frac{n\pi f^2}{6}\right)\frac{y\cos n\pi y}{2n\pi} + \frac{n\pi}{2} g_1\left(\frac{1}{4n^2\pi^2}\, y\cos n\pi y + \frac{y^2}{4n\pi}\sin n\pi y\right) + \frac{f_2}{2} y \sin n\pi y + \frac{5f^2}{36}\sin 2n\pi y - \frac{ff_1}{2} + \frac{1}{6} ff_1 \cos 2n\pi y - \frac{f^2 y}{4} - \frac{f^2 n^2\pi^2}{4} \times \left[-\frac{y}{3n^2\pi^2}\cos 2n\pi y + \frac{4}{9n^3\pi^3}\sin 2n\pi y\right] + f_5 \sin n\pi y + \frac{1}{3} ff_1 \cos n\pi y. \tag{63}$$

The condition at $y = 1$ yields

$$f'' + \sigma_{12}^2 f - \frac{n\pi f^2}{6} = 0. \tag{64}$$

This has a solution

$$f = \frac{9\sigma_{12}^2}{n\pi}\operatorname{sech}^2\left(\frac{x}{2}\, i\,\sigma_{12}\right) \tag{65}$$

vanishing at $|x| = \infty$, if σ_{12}^2 is negative. Thus

$$\frac{9\sigma_{12}^2}{n\pi} = -1 \tag{66}$$

and, therefore, to the present order,

$$\frac{\delta'}{h} = -\alpha \operatorname{sech}^2 \frac{x'}{6h}(n\pi\alpha\beta^2)^{\frac{1}{2}} \sin n\pi \frac{y'}{h} \qquad n = 2, 4, \ldots, \tag{67}$$

$$\sigma^2 = n^2\pi^2 + \frac{1}{4}\beta^2 - \frac{n\pi}{9}\alpha\beta^2. \tag{68}$$

Notice that if we seek other solutions by putting $\gamma = \alpha^3\beta, \alpha\beta^3$, etc., all subsequent cases will lead to Eq. (64) with the f''-term missing. It follows immediately that $f = 0$, and that there are no other solutions of the kind we are looking for.

The solutions of (43) and (67) represent disturbances with maximum amplitude at $x = 0$, dying off monotonically and symmetrically on both sides. The speeds of propagation are determined by (44) and (68) respectively. They are internal solitary disturbances[1] similar to those discussed by KEULEGAN (1953), LONG (1956, 1964 a) and PETERS & STOKER (1959).

[1] Dr. Brooke Benjamin has made a very general, elegant investigation of internal solitary waves. His study is not yet published.

Two examples, corresponding to $n = 1$ and $n = 2$, are shown in Figs. 1 and 2. Since β is always positive, we see from (43) and (67) that $\alpha > 0$. The wave in the lowest portion of the channel is one of elevation if n is odd, and one of depression if n is even. We recognize that if closed streamlines appear in the fluid, we have a local situation in which density increases with height. An estimate of the amplitude of the maximum stable wave, therefore, is

$$\alpha_{\max} = \frac{1}{n\pi}. \tag{69}$$

The waves are an example of some of the earlier remarks of this paper. Neither exists if the term involving β is neglected[1] in Eq. (15), i.e., if the Boussinesq approximation is made. Notice too that in some of the cases considered above, in which there is no solitary wave, it may be possible, nevertheless, to find solutions for an infinite train of finite-amplitude, internal waves.

[1] This was noticed independently by Dr. Benjamin in a private communication to the author.

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