On parametric values and types of representation in wind-driven ocean circulation studies

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(Manuscript received July 28, 1964)

ABSTRACT

The vertically averaged equations of motion for the barotropic model of wind-driven ocean circulation on the β-plane are discussed. An attempt is made to evaluate the two non-dimensional parameters, viz., the Rossby number, R, and the (bottom) frictional parameter, ε, by comparing observed scales with those derived from simple boundary layer arguments. It is found that ε and R are (upper) bounded for the model if one imposes the scale of the Gulf Stream as the smallest significant scale of the ocean. Maximum values for the bottom-friction coefficient and for the ratio of wind-stress to depth of the model are determined. The general problem of analyzing the equations by approximate (numerical) techniques is discussed and a proposal is made to carry out the analysis by working with limited representations and correspondingly exaggerated values of the parameters. Finally a comparison is made of the error involved in treating Stommel's steady, linear, frictional model by two approximate methods, a limited Fourier representation and a finite-difference scheme. The latter is found to describe the solution of the linear model with considerably less distortion than the former. Plans to treat the fully non-linear, time-dependent system are described.

1. Introduction and summary

In attempting to understand natural phenomena we are usually forced to develop models which involve physical and mathematical simplifications. In the generation of these models it is often necessary to introduce parametric representation of those physical processes which are considered secondary in the study of the particular problem. Mathematical simplifications can also lead to the introduction of parameters which represent processes which are simplified out of the problem. Sometimes it is possible to introduce a numerical value for the parameter with a reasonable certainty that the process is well-represented. More often, however, the numerical value cannot be chosen with any degree of assurance that it is representative of the parameterized process.

If one wishes to study wind-driven ocean circulation in a barotropic system several dimensional parameters appear in the mathematical formulation. For example, a "depth" of the barotropic ocean appears in the model when the equations are integrated (or averaged) over the depth. What relation, if any, does this depth bear to the observed depth? From the vertical integration of the equations the vertical stress term is evaluated at the surface and this is generally equated to the wind-stress. But the "wind-stress" which is used is ordinarily connected to the mean wind system (or a mathematical approximation thereof) and one wonders just how this stress relates to the actual time-varying, fluctuating, wind-stress which acts on the real ocean. Just how is the frictional braking of the driven system to be introduced? These and many other questions arise for several reasons: (a) the physical model is crude; (b) some well-defined parameters (e.g., the wind-stress at the surface) are themselves due to physical processes which are not completely understood; (c) some of the parameters (e.g., bottom frictional coefficient or lateral eddy frictional coefficient) are *ad hoc* parameters which are not well-defined; and (d) even the simplified model cannot be solved in its entirety, hence one cannot compare the theoretical results to observations in order to judge the consistency of the model.

In the present paper a method of attack is outlined for the barotropic wind-driven ocean circulation model. The starting point is the mathematical model for the two-dimensional flow of an ocean of constant depth and on the β-plane. Even though it is not clear precisely how such a model is descriptive of the "real" ocean, it does represent a minimal system which must be understood before one proceeds with a more complete or realistic model. The interactions of the coriolis effect, non-linear effects, time dependence and dissipation are all incorporated in the simplified model. Qualitatively the behavior may differ from that of the real ocean and from that of a more complete model. But it is necessary to have the solutions before one can judge.

We can obtain some information about the possible values of the non-dimensional parameters (viz., the frictional parameter, ε, and the non-linear or Rossby number parameter, R) from a straightforward application of simple boundary layer ideas. From the steady, linear, frictionally controlled model it is found by comparing the theoretical solution with observed scales that, if the Gulf Stream is a phenomenon which is dominated by frictional processes, then it is necessary that the frictional parameter, ε, have a value of about 10^{-2}. If, on the other hand, the Gulf Stream is dominated by non-linear processes, then it is shown from scale considerations that the Rossby number, R, must have a value of 10^{-4}. If both frictional and non-linear processes are important for scales comparable to the lateral scale of the Gulf Stream, then ε and R must have values roughly equal to those given above. If either process is not important for Gulf Stream scales then its parametric value must be somewhat less than the values cited above.

One point that is clear is that the frictional mechanism and/or the non-linear process must be sufficient to determine the Gulf Stream scale if the model is to be at all applicable to wind-driven ocean circulation because no other scale-determining process exists in the present, simplified model. Therefore, the analysis that one uses must be capable of describing scales as small as the one to be studied (viz., about one per cent of the characteristic scale of the ocean basin).

Thus far the barotropic model has not been successfully treated by analytical methods. Hence, it has been found necessary to resort to approximate (numerical) methods to analyze the problem. One method consists of expanding the stream function into a complete, orthogonal set of representative functions and truncating the representation to some suitable subset which is then analyzed. The other method is to divide the system into a network of grid-points and to determine the value of the stream function at the points.

The practical problem associated with such an analysis is that an inordinately large number of modes or grid points must be kept in order to describe scales as small as 0.01 of the total scale of the system. Hence it is proposed in the present paper that one exaggerate the values of the frictional and non-linear parameters in order to treat cruder systems, i.e., systems whose smallest scales of motion are somewhat larger than 0.01 of the total scale so that the numerical analysis can be done with a coarser grid or a smaller number of normal modes. The relative roles of friction and non-linearity can then be appraised by allowing first one and then the other of the two processes to be dominant. As results are obtained with more limited representations, one can go on to finer representations and thereby eventually obtain results for a system whose scale is 0.01 of the scale of the basin.

The two methods, i.e., a limited number of Fourier modes vs. a grid network, are studied for the linear frictional model and the results are compared to the exact solution. It is found that the grid-point method is considerably superior to the Fourier mode method when the functions are smooth. Furthermore, in the non-linear model the Fourier representation quickly leads to impossibly long equations which cannot be solved on existing computers. Hence, the expansion into normal modes may be adequate for qualitative information but we will very likely have to rely on gridpoint calculations for results which can be directly compared to observation.

2. Equations and parameters

Consider a barotropic, incompressible, ocean of constant depth, D, on a β-plane. If the equations are averaged in the vertical, we derive the following equations for the mathematical model

$$\frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u - fv = -\frac{1}{p}\frac{\partial p}{dx} - Ku + \frac{\tau^x}{D}, \qquad (1)$$

$$\frac{\partial v}{\partial t} + \mathbf{v}\cdot\nabla v + fu = -\frac{1}{p}\frac{\partial p}{\partial y} - Kv + \frac{\tau^y}{D}, \tag{2}$$

$$u_x + v_y = 0, \tag{3}$$

where $v = (u, v)$ are velocity components positive toward the east (x) and north (y) directions respectively; $f(= f_0 + \beta_y)$ is the coriolis parameter varying linearly with latitude and p is pressure. The frictional terms result from the assumption that molecular friction is unimportant and that the only significant large scale friction is due to the vertical stress (per unit mass), $\delta\boldsymbol{\tau}/\delta z$. Then when the equations are averaged in the vertical, we have

$$\frac{1}{D}\int_0^D \frac{\partial \boldsymbol{\tau}}{\partial z}\,dz = \frac{1}{D}[\boldsymbol{\tau}]_0^D.$$

At the surface $(z = D)$ $\boldsymbol{\tau}/D = (\tau^{*}/D)\mathbf{i} + (\tau^y/D)\mathbf{j}$. At the bottom we make the simple assumption following STOMMEL (1948) that the frictional retardation is simply proportional to the velocity, i.e., $(\boldsymbol{\tau}/D)_{z=0} = -K\mathbf{v}$.

Alternatively one can choose a lateral eddy viscosity as MUNK (1950) has done to derive for the frictional terms

$$A\nabla^2\mathbf{v} \tag{4}$$

instead of the terms, $-K\mathbf{v}$, in equations (1) and (2).

Now making use of the two-dimensionality of the above system, we can introduce a stream function, ψ, defined by

$$v = \frac{\partial\psi}{\partial x}, \quad u = -\frac{\partial\psi}{\partial y}$$

and eliminate the pressure, p, by cross-differentiating equations (1) and (2). This yields

$$\nabla^2\psi_t + J(\psi, \nabla^2\psi) + \beta\psi_x = -K\nabla^2\psi + \frac{1}{D}\left(\frac{\partial\tau^y}{\partial x} - \frac{\partial\tau^x}{\partial y}\right), \tag{5}$$

where J is the Jacobian operator and the subscripts on ψ denote differentiation.

The equations are non-dimensionalized by taking

$$\frac{\partial}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x'}, \quad \frac{\partial}{\partial y} = \frac{1}{L}\frac{\partial}{\partial y'}, \; \frac{\partial}{\partial t'} = \beta L\frac{\partial}{\partial t'},$$

$$\boldsymbol{\tau} = W\boldsymbol{\tau}', \; \psi = \frac{W}{D\beta}\psi'.$$

The primed quantities are dimensionless. Equation (5) then becomes

$$\nabla^2\psi_t + RJ(\psi, \nabla^2\psi) + \psi_x = -\varepsilon\nabla^2\psi = \frac{\partial\tau^y}{\partial x} - \frac{\partial\tau^x}{\partial y}, \tag{6}$$

where the primes have been dropped. Here

$$R = \frac{W}{\beta^2 DL^3}, \quad \varepsilon = \frac{K}{\beta L}. \tag{7}$$

In the non-dimensionalization the length scale, L, may correspond either to the dimension of the basin or to the scale of the wind (or to both if they are comparable in magnitude). The non-dimensional driving term, $\delta\tau^y/\delta x - \delta\tau^x/\delta y$, is then of order one. *If* the non-dimensionalization normalizes the variables, one can deduce information about the types of balances which obtain. On the other hand it is possible that the above non-dimensionalization does not normalize the variables and one must then go into a more detailed, mechanistic, account.

We note that R depends on four dimensional variables, W, β, D and L. Two of these, viz., β and L, are well defined external parameters. W and D, on the other hand, are involved in the averaging processes and it is not *a priori* clear what values one must choose for them. Similarly, ε depends on the frictional parameter, K, which represents a complicated physical process and which also results from the type of averaging which has been employed. Therefore a specific value for ε is not immediately available.

3. Choices of parameters for specific models

Throughout this section we consider an ocean basin which is square in shape and has dimension L. The results may be modified quantitatively for basins of other shapes but qualitatively the remarks below should hold for the mathematical model described by equation (6) and subject to the condition

$$\psi = 0 \quad \text{on the boundaries} \quad x = 0.1; \; y = 0.1 \tag{8}$$

a. *Sverdrup balance*

Assume that the circulation is steady and that both R and ε are small. Then in mid-oceanic regions the SVERDRUP (1947) balance

$$v = \frac{\partial \psi}{\partial x} = \frac{\partial \tau^y}{\partial x} - \frac{\partial \tau^x}{\partial y} \tag{9}$$

obtains. This can be integrated with respect to x to yield

$$\psi = \int_1^x \left(\frac{\partial \tau^y}{\partial x} - \frac{\partial \tau^x}{\partial y} \right) dx \tag{10}$$

for the value of the stream function at any point in the interior provided that (9) holds up to the eastern boundary. Expression (10) yields explicit information about ψ; in particular it sets the amplitude of ψ at order unity. The Sverdrup balance has been assumed to be valid in some interior region of the ocean by most investigators who have used boundary layer methods.

(b) *Stommel's linear, frictional model*

Assume $\partial\psi/\partial t = 0$, R is negligibly small and

$$\frac{\partial \tau^y}{\partial x} = 0, \quad \frac{\partial \tau^x}{\partial y} = \sin y.$$

Equation (6) can then be written

$$\varepsilon \left(\frac{d^2\psi}{dx^2} - \psi \right) + \frac{d\psi}{dx} = -1, \tag{11}$$

where ψ has been taken proportional to $\sin y$.

If ε is small, the solution in the interior is

$$\frac{d\psi}{dx} = -1 \tag{12}$$

and near the boundary, $x = 0$, we have

$$\frac{d}{dx} = \frac{1}{\varepsilon} \frac{d}{d\xi}$$

with $d/d\xi = O(1)$. To order ε equation (11) becomes

$$\frac{d^2\psi}{d\xi^2} + \frac{d\psi}{d\xi} = 0.$$

Thus the complete boundary layer solution for ψ is

$$\psi = 1 - x - e^{-x/\varepsilon} \tag{13}$$

which satisfies the equation and the boundary conditions to order ε.

The significance of the boundary layer solution is that the exponential term in (13) is negligible outside the boundary layer but is O(1) in the boundary layer. If one identifies the latter region with the Gulf Stream which has a characteristic scale[1] of the order of 50 km, then corresponds to the ratio of the Gulf Stream scale to the scale of the basin, or

$$\varepsilon \approx \frac{50 \text{ km}}{5000 \text{ km}} = 0.01. \tag{14}$$

Substituting $\beta = 2 \times 10^{-13}\text{cm}^{-1}\text{sec}^{-1}$, $L = 5 \times 10^8$ cm, $\varepsilon = 0.01$ into (7), we derive

$$K = \varepsilon\beta L = 10^{-6}\text{sec}^{-1}. \tag{15}$$

Thus we can say that if a linear, steady, barotropic, frictionally controlled model is meaningful and if the ocean circulation is of the boundary layer type, then by relating the frictional parameter to observed variables, we deduce $\varepsilon \approx 0.01$, $K = 10^{-6}\text{sec}^{-1}$.

It is necessary to operate in this inverse fashion because we do not know what the value of K should be (or even if it is a meaningful parameter to use). The ultimate test, of course, is in how well the model relates to the observed circulation and whether it can be used for predictive purposes. Stommel's model does yield a flow with westward intensification but it also has concomitant undesirable features such as the north-south symmetry of the flow field. Therefore, a more complete model is called for.

(c) *Munk's lateral friction model*

The only difference between MUNK's (1950) model and Stommel's is that the former replaces the terms $-K\nabla^2\psi$ by $A\nabla^4\psi$, where A is an eddy coefficient. It is necessary to add additional boundary conditions and Munk has used $\delta\psi/\delta n = 0$ at the boundaries, i.e., that the tangential velocity or transport vanishes.

[1] By "scale" we mean the distance across which significant changes take place. The width of the Gulf Stream would be π times the scale.

In this case, the term $\varepsilon_L \nabla^4 \psi$ replaces $-\varepsilon \nabla^2 \psi$ in equation (6) where $\varepsilon_L = A/\beta L^3$. The boundary layer solution has the form $\exp(-x/2\varepsilon_L^{\frac{1}{3}}) f(x,y)$ where $f(x,y)$ is of order one. Thus the thickness of the boundary layer is $2\varepsilon_L^{\frac{1}{3}}$. If we use the same argument as in § (b) we derive

$$\varepsilon_L^{\frac{1}{3}} = 2\left(\frac{A}{\beta L^3}\right)^{\frac{1}{3}} = 0.01, \tag{16}$$

so that with $\beta = 2 \times 10^{-13}\,\mathrm{cm^{-1}sec^{-1}}$ and $L = 5 \times 10^8\,\mathrm{cm}$, we deduce

$$A = 3 \times 10^6\,\mathrm{cm^2 sec^{-1}}. \tag{17}$$

Munk used a somewhat larger value for A. His analysis yields a more detailed structure for the western boundary layer (or Gulf Stream) with countercurrents on the off-shore side of the stream. However, the north-south symmetry is present here as it was in Stommel's model. One must introduce additional processes to improve the results.

(d) *Considerations of non-linearity*

Analyses by MUNK, GROVES & CARRIER (1950), FOFONOFF (1954), CHARNEY (1955), MORGAN (1956), CARRIER & ROBINSON (1962), MOORE (1963), BRYAN (1963) and VERONIS (1963) have taken into account the inertial terms in the equations of motion. In each of these accounts it is shown that the north-south symmetry of the linear, frictional models can be eliminated by non-linear processes. We confine our attention here to a discussion of the parameters in equation (6) with specific attention to the values which R must have in order that non-linearities be significant.

Suppose that $\psi = O(1)$ in equation (6). Then the amplitudes of the various terms are determined by the coefficients, R and ε, and by the magnitudes of the derivatives. In particular, the considerations of the previous two sections showed that in the linear frictional problems one could deduce information about the size of ε by relating the size of derivatives to ε in such a way that the frictional terms balanced the β term (ψ_x) in the boundary layer.

Consider the non-linear terms. If the boundary layer is inertially controlled, we can deduce the same information about R as we did about ε previously. There are three x-derivatives in the non-linear terms and one in the β-term. Thus the non-linear terms are important provided that $R(\delta^2/\delta x) = O(1)$. This implies simply that x derivatives must have a magnitude of order $R^{-\frac{1}{2}}$. In other words, the variation in the x direction takes place over a scale, l, which equals $R^{\frac{1}{2}}L$. Thus

$$\frac{l^2}{L^2} = R \tag{18}$$

and the Rossby number, R, can be represented as the ratio of the square of the significant (l) scale to the square of the dimension, L, of the basin. Substituting $l = 50$ km, $L = 5000$ km, we have

$$R = 10^{-4} \tag{19}$$

as the condition necessary for the non-linear terms to be of order unity in the Gulf Stream region. From equation (7) we see that that means

$$W/D = 5 \times 10^{-4}\,\mathrm{cm\,sec^{-2}}. \tag{20}$$

For a wind-stress of 1 dyne cm^{-2} we find that $D = 20$ m or conversely for a depth of 5 km we have $W = 250\,\mathrm{cm^2sec^{-1}}$! The introduction of π in appropriate places can cut down on the exaggerated values of D and W by a factor of 10 but one is still left with $W/D = 5 \times 10^{-5}\,\mathrm{cm\,sec^{-2}}$. This is large compared to $W/D = 2 \times 10^{-6}$ which results from taking $W = 1\,\mathrm{cm^2sec^{-2}}$ and $D = 5$ km.

The conclusion is that if a barotropic model is related meaningfully to large scale ocean circulation, and if the inertial terms are to be taken as comparable to the β-term in the vorticity equation, then one must choose a value of the ratio W/D which is at least an order of magnitude larger than would result if one used the observed mean wind-stress and the total depth to evaluate W/D.

4. A proposal for analyzing the barotropic model

We note from the foregoing discussion that an investigator who wishes to analyze the barotropic wind-driven ocean circulation model is faced with several problems. Since some of the dimensional parameters are due to physical processes which are not well-understood or well-defined and others come in because of the simplifying procedure of vertical averaging, it is not clear what values one can assign to these parameters *a priori*. However, the specific values which one chooses will determine the model behavior. Note, for example, the distinct qualitative difference between Stommel's linear, frictional model (i.e., ε is small and $R = 0$) and Fofonoff's steady, frictionless, non-linear model ($\varepsilon = 0$ and R is small). The former produces a western boundary current, the latter a northern boundary current. A model which has both processes acting and which can also have time-dependent behavior will very likely yield quite different results from either.

The question that remains is whether one should choose the values of ε and R such that the system is essentially frictional and is modified by non-linear effects or vice-versa. Certainly the two ranges of the parameters, viz., $\varepsilon/R^{\frac{1}{2}} > 1$ or $\varepsilon/R^{\frac{1}{2}} < 1$, will give quite different results. That has been made clear in two preliminary investigations by the author (VERONIS (1963), VERONIS (1964)). It is also clear that an attempt to study the two ranges will tax even the biggest and fastest computers available at present if one chooses the parametric values to border on the values ($\varepsilon \leqslant 10^{-2}$, $R \leqslant 10^{-4}$) which would determine scales as small as that of the Gulf Stream.

The proposed method of attack is the following: Choose an appropriate representation, either a certain number of normal modes or a certain grid size in a network. Such a limited representation can describe scales which are predetermined by the representation. Hence, the ranges of the parameters, ε and R, which can be treated is limited since values which would determine smaller scales than those available cannot be considered. Now vary the values of ε and R so that either the frictional or the non-linear process (or both) is dominant at the smallest scale which can be described. Once the relative roles of the two processes are understood in this fashion, go to a finer representation and smaller values of ε and R using the information which has been obtained from the coarser system. Hopefully, the results will tend asymptotically toward the same qualitative behavior for given ranges of ε and R. Thus one could finally treat the desired values of ε and R, with a maximum amount of information from the coarser systems.

BRYAN's (1963) numerical attack has gone part of the way in treating the problem. In all of his cases the frictional boundary layer was thicker than the inertial boundary layer (in the southern half-basin which is the only region where a boundary layer thickness can be strictly defined). Results by the author (1963, 1964) have shown that quite different behavior can be expected when the inertial boundary layer is thicker than the frictional one. Hence, such a grid-point calculation is needed and in fact has been begun but with a more limited number of grid points than that considered by Bryan.

5. A comparison of grid-point and Fourier methods of analysis for a linear problem

To get an idea of how the Fourier mode method compares in accuracy and descriptive capacity with the grid point method it is instructive to look at a linear model whose solution is known. Consider Stommel's model as described in section 3 but where the wind-stress curl is given by -sin$x$sin$y$. The solution is again proportional to sin$y$. Hence, the equation

$$\varepsilon\left(\frac{d^2\psi}{dx^2} - \psi\right) + \frac{d\psi}{dx} = -\sin x \tag{21}$$

must be solved, subject to the boundary conditions

$$\psi = 0 \quad \text{on} \quad x = o, \pi. \tag{22}$$

It is a trivial problem to solve this exactly and the solution will simply be written down.

$$\psi = \frac{1}{1+4\varepsilon^2}\left\{2\varepsilon \sin x + \cos x + \frac{1}{e^{\pi D_1} - e^{\pi D_2}} \right.$$
$$\left. \times \left[(1+e^{\pi D_2})e^{D_1 x} - (1+e^{\pi D_1})e^{D_2 x}\right]\right\}, \tag{23}$$

where

$$D_1 = \frac{-1-\sqrt{1+4\varepsilon^2}}{2\varepsilon}, \quad D_2 = \frac{-1+\sqrt{1+4\varepsilon^2}}{2\varepsilon}. \tag{24}$$

Eqation (21) can also be solved approximately by writing

$$\psi = \sum_1^N A_n \sin nx, \tag{25}$$

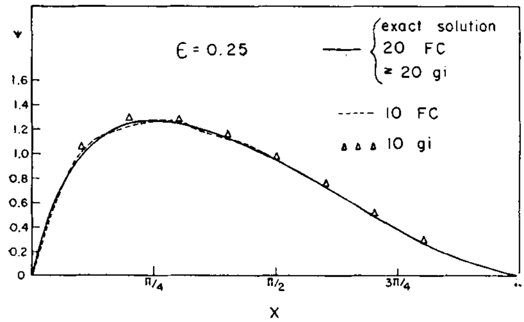

FIG. 1. The stream function, ψ, vs. the east-west coordinate, χ, for $\varepsilon = 0.25$. See Figure 3 for explanation of the figure.

where N is some integer. The stream function expressed in this fashion satisfies the boundary conditions and satisfies the necessary conditions for differentiating a Fourier series.

The solution for $N = 10$ and $N = 20$ is compared to the exact solution in figures 1, 2 and 3 for values of ε equal to 0.25, 0.1 and 0.05 respectively. It is seen that for $\varepsilon = 0.25$ the system with $N = 10$ is very close to the exact solution. That would be expected in this case since the function to be described is smooth and has a smallest characteristic scale which is considerably larger than the smallest scale which can be described by the Fourier series with $N = 10$. When $\varepsilon = 0.1$ a considerable distortion is introduced by the Fourier representation. For $N = 10$ we note that the smallest Fourier scale is comparable to the smallest scale which the exact solution has. Hence the tenth wave number has a large amplitude and this is reflected all the way across the basin so that there are alternatively northward and southward moving bands of water. There is an unavoidable northward moving current at the eastern boundary.

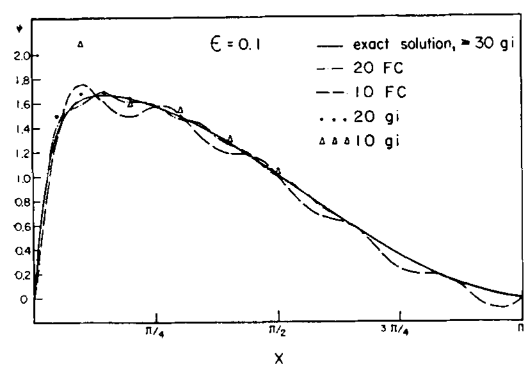

FIG. 2. The stream function, ψ, vs. the east-west coordinate, χ, for $\varepsilon = 0.1$. See Figure 3 for explanation of the figure.

In figure 3 the smallest scale of the exact solution is smaller than that which can be described by the ten-component system and comparable to that of the twenty-component description. The ten-component case now shows a violent oscillation and is essentially valueless for description. However, it still maintains some of the gross behavior of the exact solution. Even the twenty-component description has a sizable oscillation, comparable in magnitude to that of the ten-component one in figure 2.

Thus we see that, when the smallest scale of the physical system is comparable to the smallest scale of the descriptive system, the latter tends to distribute the small scale behavior (and its accompanying physical effects!) throughout the domain. What effect this has on the resulting dynamical picture can be estimated only when the original system is analyzed by some other means and a comparison is made. Certainly in the present model the effect is no more serious than to distribute

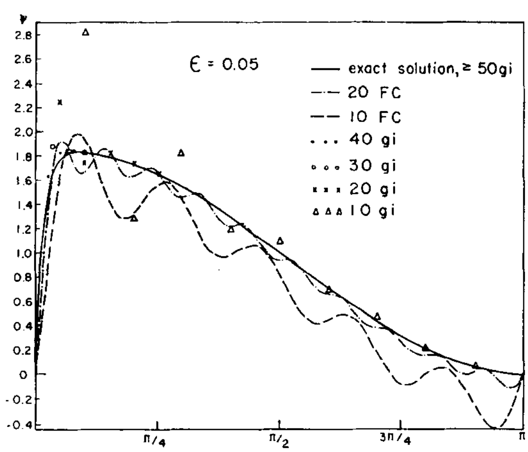

FIG. 3. The stream function, ψ, vs. the east-west coordinate, χ, for $\varepsilon = 0.05$. The solid line represents the exact solution and the solution by the finite difference method for 50 or more grid intervals (gi in the figure) The solutions by a Fourier series representation is shown by the dashed (20 Fourier components) and dash-dot (10 Fourier components) curves. Point values for ψ for 10, 20, 30 and 40 grid intervals are indicated by the symbols. When the approximate solution is essentially the same as the exact solution, the points and curves of the former are not plotted.

the frictional dissipating process through more of the region. When non-linearity is present the effect may be more serious and undoubtedly less traceable.

The finite difference form of equation (21) is written as

$$\left[\frac{1}{(\Delta x)^2}+\frac{1}{2\varepsilon\Delta x}\right]\psi_{i+1}-\left[\frac{2}{(\Delta x)^2}+1\right]\psi_i$$

$$+\left[\frac{1}{(\Delta x)^2}-\frac{1}{2\varepsilon\Delta x}\right]\psi_{i-1}=-\frac{\sin x_i}{\varepsilon}, \qquad (26)$$

where $$\frac{\partial^2\psi}{\partial x^2}=\frac{1}{(\Delta x)^2}(\psi_{i+1}-\partial\psi_i+\psi_{i-1})$$

and $$\frac{\partial\psi}{\partial x}=\frac{1}{2\Delta x}(\psi_{i+1}-\psi_{i-1}).$$

The solution can be obtained in an elegant fashion by the method described in RICHTMEYER (1957). The results are compared with those of the previous two methods in figures 1, 2 and 3.

Because of the smoothness of the exact solution the grid-point method yields extremely accurate results over most of the range; in fact, even a sparse network is inadequate only in the vicinity of boundary layer. As one would expect, the major error is introduced in those regions where the function changes most rapidly and when the number of grid intervals (denoted by *gi*) is small. The interesting aspect of the finite-difference scheme is that the representation is considerably better for the finer networks even when the value of ε puts a correspondingly stronger demand on it. Thus the 20 grid-point network yields more accurate results over most of the range in figure 2 than the 10 grid-point network does in figure 1. The conclusion from a comparison of the two methods is that the finite-difference scheme is definitely superior in its descriptive capacity and especially so if the function is smooth over large regions.

Acknowledgements

I wish to thank Dr. Victor Ooyama for pointing out to me both the method and the Richtmeyer reference for solving the second order finite difference system. The research was supported by the National Science Foundation through contract GP 2564.

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