A measure of form difference and its application to the study of turbulent motions

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ABSTRACT

The aim of the present paper is twofold, viz., on one hand, to introduce a measure of *form* difference between two configurations of a given number of points as a tool for studying the effect of turbulent motions upon the deformation and spreading of point clusters, and, on the other hand, to derive a statistical relationship between the *deformations* of triangles of a given size, formed by triads of points in a field of random motions, and the properties of that field (eddy diffusivity as a function of size).

As a geometrical by-product, a systemetics of triangle forms is found, which represents the set of all possible triangle forms by means of the points of a sphere. In this representation the form difference, as defined in this paper, appears to be equivalent with the angular distance on the sphere between the points representing the forms of the triangles.

1. Introduction

During a study of two-dimensional point clusters subject to random displacements by a field of turbulent motion, it was felt desirable to have a measure of form difference between two configurations of a certain number of points, one of which originates from the other by random displacements. As a function of time, such a measure of form difference will become a measure of the rate of deformation or distortion of the point cluster. This *deformation* is one of the two complementary characteristic aspects which the process of random displacements shows, the other being the *spreading* of the point cluster. The latter aspect (the one more directly connected with practical problems of eddy diffusion) has been studied in two ways, which we may call the "integral" approach and the "differential" approach, respectively, the former dealing with point densities or with the mean square of the distances of all points to their common centre of gravity, and its change in time, the latter dealing with what L. F. Richardson has called "neighbour diffusivity", which characterizes the statistics of mutual displacements of pairs of particles (RICHARDSON, 1926; see also STOMMEL, 1949).

In measuring the spreading of a point cluster, e.g. by means of a series of photographs, it is essential that, in evaluating the observations, one uses exact measures of distance or at least that one has exact knowledge of the ratios of the scales of successive pictures of the point cluster. When studying air-photos of swarms of papers floating on the sea, it occurred to the writer that, instead of trying to find the exact ratios of their scales (which differed slightly because of unavoidable changes of height of the helicopter from which the pictures were taken), one might as well try to measure distortions or *form differences* between successive pictures of certain groups of points as an effect of the turbulent displacements, "form" being something nondimensional and, consequently, independent of scale (Figure 1).

Following this line of thought it was found possible to define a measure of form difference in a simple way, which moreover opens the possibility of relating it quantitatively to the concepts of spreading and eddy diffusivity.

2. A measure of form difference

This measure is applicable to configurations of any number (>2) of points. We shall use triangles as an illustration of its definition.

Let the points of the two configurations to be compared be called A_1, A_2, A_3 and B_1, B_2, B_3, respectively (Fig. 2). In comparing the configurations with each other it is essential that

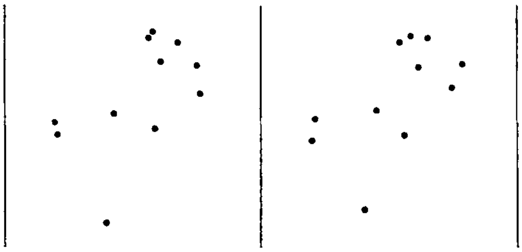

FIG. 1. Deformation of a point cluster by random motions.

A_1 corresponds with B_1, A_2 with B_2 and A_3 with B_3. The two configurations are supposed to lie in the same plane.

Since "form" is not influenced by translation, rotation and multiplication, we may apply a translation, a rotation and a multiplication to one of the configurations in such a way that the sum of the squares of the distances between corresponding pairs of points becomes a minimum. In other words: we try to find a sort of best fit by these operations. For reasons of symmetry we prefer to apply multiplications to *both* configurations. So, instead of multiplying $A_1A_2A_3$ by f or $B_1B_2B_3$ by f^{-1} we multiply $A_1A_2A_3$ by $f^{\frac{1}{2}}$ and $B_1B_2B_3$ by $f^{-\frac{1}{2}}$. The translation and the rotation are applied to $B_1B_2B_3$.- Denoting by $\mathbf{r}_{1,2,3}$ and $\mathbf{s}_{1,2,3}$ the radius-vectors to $A_{1,2,3}$ and $B_{1,2,3}$ from the centres of gravity of $A_1A_2A_3$ and $B_1B_2B_3$, respectively, and by $\mathbf{s}^*_{1,2,3}$ the corresponding radius-vectors in the second triangle *after* the above-mentioned rotation, we find that the condition for the minimum sum of the squared distances requires that

(1) the translation be such that the centres of gravity coincide;

(2) the rotation be such that

$$\sum_j \mathbf{r}_j \times \mathbf{s}_j^* = 0; \tag{1}$$

(3) the multiplications be such that

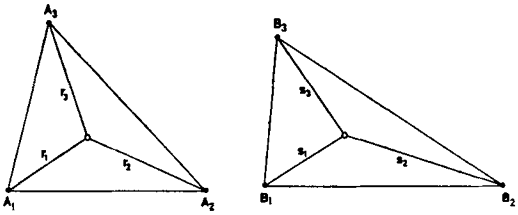

FIG. 2. Triad of points at two successive moments.

$$f\sum_j r_j^2 = f^{-1}\sum_j s_j^2. \tag{2}$$

The proof of the above statements is given in the appendix to this paper. Attention is drawn to the fact that equation (1) yields two solutions: if $\mathbf{s}_j^*$ is a solution, then $-\mathbf{s}_j^*$ is also a solution. However, if the first one gives a minimum of the squares sum, then the second gives a maximum.

If we now denote by M the minimum squares sum thus obtained, we define a nondimensional *measure of form difference* ψ as follows:

$$\psi = \frac{M}{f\sum r_j^2 + f^{-1}\sum s_j^2} = \frac{\sum (f^{\frac{1}{2}}\mathbf{r}_j - f^{-\frac{1}{2}}\mathbf{s}_j^*)^2}{2f\sum r_j^2} = 1 - \frac{\sum \mathbf{r}_j \mathbf{s}_j^*}{\sqrt{(\sum r_j^2)(\sum s_j^2)}} = 1 - \varphi. \tag{3}$$

In these and all following formulas the summation symbol $\sum$ (written without the summation index) means summation over all j. We note that the value of ψ is not affected by multiplication of the configurations.

It may easily be shown that

$$0 \leqslant \psi \leqslant 1.$$

In the case of perfect similarity we have $\psi = 0$. Instead of ψ we may also use the complementary quantity $\varphi = 1 - \psi$, which may be called the *degree of form resemblance*, being 1 in the case of perfect similarity. Generally, we have:

$$0 \leqslant \varphi \leqslant 1.$$

For computing ψ or φ in any particular case, if the configurations are given, with arbitrary orientations, by the components of their $\mathbf{r}_j$ and $\mathbf{s}_j$ (without rotation) in a Cartesian coordinate system, it is not necessary to solve (1) explicitly, in order to find $\mathbf{s}_j^*$, since it may be shown that

$$(\sum \mathbf{r}_j \mathbf{s}_j^*)^2 = (\sum \mathbf{r}_j \mathbf{s}_j)^2 + (\sum \mathbf{r}_j \times \mathbf{s}_j)^2. \tag{4}$$

Indeed, as will be shown presently, the right hand side of (4) is invariant under rotations of one or both of the configurations about the common centre of gravity; so that, then, equation (4) follows from (1). Applying (4) we can rewrite (3) as follows:

$$\varphi^2 = \frac{(\sum \mathbf{r}_j \mathbf{s}_j)^2 + (\sum \mathbf{r}_j \times \mathbf{s}_j)^2}{(\sum r_j^2)(\sum s_j^2)}. \tag{5}$$

Elementary vector calculus shows that the numerator of formula (5) may be rewritten so as to give the following formula:

$$\varphi^2 = \frac{\sum r_j^2 s_j^2 + 2\sum_{i<j} (\mathbf{r}_i \mathbf{r}_j)(\mathbf{s}_i \mathbf{s}_j) + 2\sum_{i<j} (\mathbf{r}_i \times \mathbf{r}_j)(\mathbf{s}_i \times \mathbf{s}_j)}{(\sum r_j^2)(\sum s_j^2)}. \tag{6}$$

From an inspection of the numerator of this formula, which is equal to the right hand side of (4) (the numerator of (5)), it is now clear that indeed the latter is invariant under rotations of one or both of the two configurations, as was stated above.

Formula (6) is valid quite generally, for configurations of an arbitrary number of points.

For the special case of triangles, which will concern us more particularly, the formula for φ^2 can be simplified by making use of the relations (8) and (9) written below, which follow directly from the basic equations

$$\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 = 0 = \mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_3. \tag{7}$$

The relations meant are:

$$\left.\begin{aligned} \mathbf{r}_1 \times \mathbf{r}_2 = \mathbf{r}_2 \times \mathbf{r}_3 = \mathbf{r}_3 \times \mathbf{r}_1, \\ \mathbf{s}_1 \times \mathbf{s}_2 = \mathbf{s}_2 \times \mathbf{s}_3 = \mathbf{s}_3 \times \mathbf{s}_1, \end{aligned}\right\} \tag{8}$$

$$\frac{\mathbf{r}_1 \mathbf{r}_2}{\sum r_j^2} = \frac{r_3^2}{\sum r_j^2} - \frac{1}{2} \quad \text{and} \quad \frac{\mathbf{s}_1 \mathbf{s}_2}{\sum s_j^2} = \frac{s_3^2}{\sum s_j^2} - \frac{1}{2}, \tag{9}$$

with similar relations for $\mathbf{r}_2\mathbf{r}_3, \mathbf{s}_2\mathbf{s}_3, \mathbf{r}_1\mathbf{r}_3$ and $\mathbf{s}_1\mathbf{s}_3$. Introducing now the following symbols:

$$\chi = \sqrt{12}.\ \frac{\mathbf{r}_1 \times \mathbf{r}_2}{\sum r_j^2} \quad \text{and} \quad \chi' = \sqrt{12}.\ \frac{\mathbf{s}_1 \times \mathbf{s}_2}{\sum s_j^2}, \tag{10}$$

$$\omega_i = \frac{3r_i^2}{\sum r_j^2} - 1 \quad \text{and} \quad \omega_i' = \frac{3s_i^2}{\sum s_j^2} - 1, \tag{11}$$

we may write formula (6) for φ^2 as follows:

$$\varphi^2 = \tfrac{1}{2}(1 + \chi\chi' + \tfrac{2}{3}\sum \omega_j \omega_j'). \tag{12}$$

The sense of the parameter χ is that it is 1 or -1 for an equilateral triangle, the sign depending on the sense of rotation of the circuit $1-2-3-1$, and that, in general,

$$-1 \le \chi \le +1, \tag{13}$$

so that for an arbitrary triangle the value of χ shows the degree of its resemblance to an equilateral triangle; $\chi = 0$ if the three points lie on a straight line. Indeed, applying formula (13) to the comparison of an arbitrary triangle (χ') with an equilateral one ($\chi = 1$, $\omega_1 = \omega_2 = \omega_3 = 0$), we find:

$$\varphi^2 = \tfrac{1}{2}(1 + \chi') \tag{14}$$

for this special case.

The quantities ω_1, ω_2, ω_3 and χ are not independent of each other, but have the following interrelations:

$$\omega_1 + \omega_2 + \omega_3 = 0, \tag{15 a}$$

$$\tfrac{2}{3}(\omega_1^2 + \omega_2^2 + \omega_3^2) = 1 - \chi^2 \tag{15 b}$$

$$= \omega_1^2 + \tfrac{1}{3}(\omega_2 - \omega_3)^2, \tag{15 c}$$

which follow directly from their definitions. The second relation is also found by applying (12) to two identical triangles, which have $\varphi^2 = 1$.

Using relations (15) and introducing two auxiliary quantities λ and μ by means of the following equations:

$$\left.\begin{aligned} \sin\mu &= \chi, \\ \cos\lambda &= \frac{\omega_1}{\sqrt{1-\chi^2}}, \\ \sin\lambda &= \frac{\sqrt{\tfrac{1}{3}}\cdot(\omega_2 - \omega_3)}{\sqrt{1-\chi^2}}, \end{aligned}\right\} \tag{16}$$

we may finally write (12) as follows:

$$2\varphi^2 - 1 = \chi\chi' + \omega_1\omega_1' + \tfrac{1}{3}(\omega_2 - \omega_3)(\omega_2' - \omega_3') \tag{17 a}$$

$$= \sin\mu\sin\mu' + \cos\mu\cos\mu'\cos(\lambda - \lambda'). \tag{17 b}$$

By (16) we can represent all forms of triangles by means of the two auxiliary angles μ and λ, which have the character of a "latitude" ($-90° \le \mu \le 90°$) and a "longitude" (λ) on a sphere. Formula (17b), which is well known from spherical trigonometry, shows that $2\varphi^2 - 1$

may in this representation be interpreted as the cosine of the angular distance between the points (λ, μ) and (λ', μ') on the sphere; φ is the cosine of half that angular distance. The poles of the sphere $(\mu = \pm 1)$ represent equilaterality (the sign depending, as we have seen above, on the sense of rotation of $1-2-3-1$). For some further remarks on this representation see the Appendix.

In the above definition of a measure of form difference, or of form resemblance, it is essential that there is a predetermined one-to-one correspondence between the points of one configuration and those of the other. Thus, these measures change if the numbering of the points of one of the two triangles is changed.

In practice, to compute actual values of φ^2 by means of formula (12) or (17a) (or for more than 3 points: formula (6)) it is sufficient to have, for each of the configurations separately, coordinates of each of its points in an arbitrary Cartesian coordinate system. For the case of triangles one of the following two formulas, corresponding to formulas (12) and (17), respectively, may conveniently be used:

$$2\varphi^2 - 1 = \frac{12(\mathbf{r}_1 \times \mathbf{r}_2)(\mathbf{s}_1 \times \mathbf{s}_2) + \frac{2}{3}\sum_i (3r_i^2 - \sum r_j^2)(3s_i^2 - \sum s_j^2)}{(\sum r_j^2)(\sum s_j^2)} \tag{18 a}$$

$$= \frac{12(\mathbf{r}_1 \times \mathbf{r}_2)(\mathbf{s}_1 \times \mathbf{s}_2) + (3r_1^2 - \sum r_j^2)(3s_1^2 - \sum s_j^2) + 3(r_2^2 - r_3^2)(s_2^2 - s_3^2)}{(\sum r_j^2)(\sum s_j^2)}. \tag{18 b}$$

A few numerical examples are shown in the Appendix.

3. Application to turbulent motions

Our measure of form difference may be applied either to the turbulent distortion of a whole two-dimensional point cluster or to the distortions of the simplest configurational elements of the cluster, i.e. to triads of points forming triangles of various sizes. In the first case one deals with the overall effect of the random displacements, affecting the cluster as a whole. In the second case one can study the effectiveness of turbulent diffusion as dependent on eddy size; in other words, one can analyze the spectrum of eddy diffusion.

The quantitative relation of form change to eddy diffusivity can be found via the concept of spreading, referred to in the introduction.

Suppose we have a cluster of N points. Let the positions of the points with respect to their common centre of gravity at the time $t = t_0$ be represented by the radius-vectors $\mathbf{r}_j$ $(j = 1 \ldots N)$, at any later time by $\mathbf{s}_j$ (which are, thus, functions of time). We denote by σ_0^2 and σ^2 the mean values of all r_j^2 and of all s_j^2, respectively:

$$\sigma_0^2 = N^{-1}\sum r_j^2, \tag{19}$$

$$\sigma^2 = N^{-1}\sum s_j^2 = f^2\sigma_0^2, \tag{20}$$

where f as defined by (19) and (20) may be called the *spreading factor*. Both σ and f are functions of time.

An eddy diffusivity K for the cluster may now be defined (see e.g. H. STOMMEL, 1949) by:

$$2K = d\sigma^2/dt = \sigma_0^2 df^2/dt.$$

More correctly, one may consider an ensemble of experiments (in the same field of turbulence), all starting with the same value of σ_0, and define:

$$2K = d\overline{\sigma^2}/dt = \sigma_0^2 d\overline{f^2}/dt, \tag{21}$$

where a bar on top denotes the ensemble-average. We shall write $\overline{f^2} = f_e^2$.

Equation (21) may be applied not only to a whole cluster but also to triads of points. In the latter case the value of K defined by (21) is to be considered as a *spectral* value corresponding to the value of σ, which is a scale value characteristic of the size of the triangles considered (especially if they are not too far from being equilateral).

Here, now, the link between spreading and distortion comes in.

We shall define an *ensemble-value* φ_e of φ, not by just averaging φ or φ^2, but by averaging the numerator and the denominator of formula (5) or (6) or (18a,b) separately. Assuming as before that all experiments of the ensemble start with the same value of $\sum r_j^2$ and using formula (18a), we find:

$$2\varphi_e^2 - 1 = \frac{12\overline{(\mathbf{r}_1 \times \mathbf{r}_2)(\mathbf{s}_1 \times \mathbf{s}_2)} + \frac{2}{3}\sum_i \overline{(3r_i^2 - \sum r_j^2)(3s_i^2 - \sum s_j^2)}}{(\sum r_j^2)(\overline{\sum s_j^2})}$$

$$= \frac{\sqrt{12} \cdot \chi\overline{(\mathbf{s}_1 \times \mathbf{s}_2)} + \frac{2}{3}\sum_i \omega_i \overline{(3s_i^2 - \sum s_j^2)}}{\overline{\sum s_j^2}}. \tag{22}$$

In order to evaluate the meaning of this formula we shall introduce *relative displacement vectors* $\mathbf{w}_j$ defined by:

$$\mathbf{s}_j = \mathbf{r}_j + \mathbf{w}_j. \tag{23}$$

These $\mathbf{w}_j$ now have a number of statistical properties following from the facts that, according to (7),

$$\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3 = 0 \tag{24}$$

and that the true turbulent displacements are random.

Denoting these true turbulent displacements by $\mathbf{W}_j$ we may write

$$\mathbf{w}_j = \mathbf{W}_j - \tfrac{1}{3}(\mathbf{W}_1 + \mathbf{W}_2 + \mathbf{W}_3). \tag{25}$$

Averaging equations (25) and using the fact that, because of the randomness of the $\mathbf{W}_j$,

$$\overline{\mathbf{W}}_j = 0,$$

we find that also

$$\overline{\mathbf{w}}_j = 0 \quad \text{for each} \quad j. \tag{26}$$

In the same manner it may be shown that, since there is no correlation between $\mathbf{W}_j$ and $\mathbf{r}_1$ or $\mathbf{r}_2$ or $\mathbf{r}_3$, we have also:

$$\overline{\mathbf{r}_i \mathbf{w}_j} = 0, \tag{27}$$

$$\overline{\mathbf{r}_i \times \mathbf{w}_j} = 0. \tag{28}$$

On the other hand, although (because of the assumed homogeneity of the turbulence) $\overline{W_1^2} = \overline{W_2^2} = \overline{W_3^2}$, we may *not*, in general, expect $\overline{w_1^2}$, $\overline{w_2^2}$ and $\overline{w_3^2}$ to be equal. This is because these quantities depend on the values of $\overline{\mathbf{W}_1\mathbf{W}_2}$, $\overline{\mathbf{W}_2\mathbf{W}_3}$ and $\overline{\mathbf{W}_3\mathbf{W}_1}$ which differ among each other if the original triangle is not equilateral. In the extreme case, for instance, where this triangle degenerates because two of its points, say 2 and 3, coincide, we have always: $\mathbf{w}_2 = \mathbf{w}_3 = -\tfrac{1}{2}\mathbf{w}_1$ (according to (24)), so that

$$\overline{w_2^2} = \overline{w_3^2} = \tfrac{1}{4}\overline{w_1^2} \tag{29}$$

in this special case.

For the vectorial product of any two of the $\mathbf{w}_j$ we have the following statistical relation:

$$\overline{\mathbf{w}_i \times \mathbf{w}_j} = 0, \tag{30}$$

since for reasons of symmetry (because of the homogeneity of the turbulence) we must expect that $\overline{\mathbf{w}_i \times \mathbf{w}_j} = \overline{\mathbf{w}_j \times \mathbf{w}_i}$.

For the ensemble-averages of the s_j^2, finally, we have

$$\overline{s_j^2} = \overline{r_j^2} + 2\overline{\mathbf{r}_j\mathbf{w}_j} + \overline{w_j^2} = \overline{r_j^2} + \overline{w_j^2}, \tag{31}$$

$$f_e^2 \sum r_j^2 = \sum \overline{s_j^2} = \sum r_j^2 + \sum \overline{w_j^2}. \tag{32}$$

Turning back now to formula (22), we shall first consider the two most simple cases of (a) equilateral initial triangles, and (b) initial "triangles" having two points coinciding.

(a) Equilateral initial triangles have $\chi = 1$ and $\omega_1 = \omega_2 = \omega_3 = 0$, so that formula (22) yields simply

$$2\varphi_e^2 - 1 = \frac{\sqrt{12} \cdot \overline{\mathbf{s}_1 \times \mathbf{s}_2}}{\sum \overline{s_j^2}}.$$

Now it follows from (28) and (30) that

$$\sqrt{12} \cdot \overline{\mathbf{s}_1 \times \mathbf{s}_2} = \sqrt{12} \cdot \overline{(\mathbf{r}_1 + \mathbf{w}_1) \times (\mathbf{r}_2 + \mathbf{w}_2)}$$
$$= \sqrt{12} \cdot \mathbf{r}_1 \times \mathbf{r}_2 = \sum \mathbf{r}_j^2,$$

so that we find:

$$2\varphi_e^2 - 1 = \frac{\sum r_j^2}{\sum \overline{s_j^2}} = \frac{1}{f_e^2} \tag{33}$$

for the case of equilateral initial triangles.

(b) For the other special case mentioned above ($\chi = 0$, $\omega_1 = 1$, $\omega_2 = \omega_3 = -\tfrac{1}{2}$) it is not even necessary to apply formula (22), since from the fact that the two points which coincide will coincide forever it is clear that there can be no form change: $\varphi_e^2 = 1$ at all times.

In general, starting with any particular form of the triangle (excluding only the last mentioned case, where the triad in fact has degenerated into a couple), we must expect φ_e^2 to decrease in time towards $\tfrac{1}{2}$ as a limit (not of $\overline{\varphi^2}$, of course, but of φ_e^2), so that $2\varphi_e^2 - 1$ will decrease towards 0. Indeed, we remember from section 2 that $2\varphi^2 - 1$ can be interpreted as $\cos\delta$, where δ is, on the sphere representing all possible forms of triangles, the angular distance between the two points representing the form of the initial triangle and the form of the triangle at a later time. Now, in a large ensemble of experiments we may, after a sufficiently long time of spreading, expect the triangles to show all possible forms, in other words the value of $\cos\delta$ to be scattered over all possible values between 1 and -1, negative values having the same probability as positive values.

The rate of decrease of φ_e^2 or $2\varphi_e^2-1$ with increasing f_e may, however, be different for different initial forms. In order to work out formula (22) further, we shall therefore assume the initial triangles (apart from having the same value of $\sum r_j^2$) to have the same form, characterized by χ and ω_1 (the values of ω_2 and ω_3 are then also fixed according to equations (15), leaving only two possibilities, one of which is, however, merely the mirror image of the other, obtained by interchanging 2 and 3).

Introducing for $\mathbf{s}_j$ in (22) the substitution (23) and applying the relations (28), (30), (31) and (15), we find

$$2\varphi_e^2-1=\frac{(\chi^2+\frac{2}{3}\sum\omega_j^2)\sum r_j^2+\frac{2}{3}\sum_i\omega_i(3\overline{w_i^2}-\sum\overline{w_j^2})}{\sum r_j^2+\sum\overline{w_j^2}}$$

$$=\frac{\sum r_j^2+\frac{2}{3}(\sum_i\omega_i m_i)(\sum\overline{w_j^2})}{\sum r_j^2+\sum\overline{w_j^2}}, \tag{34}$$

where parameters m_i have been introduced with the following meaning (compare the definition of ω_i):

$$m_i=\frac{3\overline{w_i^2}}{\sum\overline{w_j^2}}-1,\quad m_1+m_2+m_3=0. \tag{35}$$

Considering, as simplest applications, once more the two special cases mentioned before, we easily find: for the degenerated case (b), where, according to (29), $m_1=1=\omega_1$, $m_2=m_3=-\frac{1}{2}=\omega_2=\omega_3$, formula (34) yields $2\varphi_e^2-1=1$, as it should be; for the case (a) of equilateral initial triangles formula (34), with $\omega_{1,2,3}=0$, yields the result (33), of course.

In order to apply (34) quantitatively for arbitrary triangle forms ($\omega_{1,2,3}$) we shall have to know more about the parameters m_1, m_2 and m_3 as depending on ω_1, ω_2, ω_3 and on time ($\sum r_j^2$ is supposed to be fixed). From the definitions of w_j and m_i (formulas (25) and (35)) it may easily be derived that

$$\sum\overline{w_j^2}=\tfrac{2}{3}(\sum\overline{W_j^2}-\sum_{i<k}\overline{\mathbf{W}_i\mathbf{W}_k})$$

and that

$$m_1=\frac{3\overline{\mathbf{W}_2\mathbf{W}_3}-\sum_{i<k}\overline{\mathbf{W}_i\mathbf{W}_k}}{\sum\overline{W_j^2}-\sum_{i<k}\overline{\mathbf{W}_i\mathbf{W}_k}}.$$

Considering that

$$\overline{W_1^2}=\overline{W_2^2}=\overline{W_3^2}=W^2$$

and introducing correlations R_j defined by

$$R_j=\frac{\overline{\mathbf{W}_i\mathbf{W}_k}}{W^2}\quad(i\neq j\neq k\neq i)$$

we find: $$\sum\overline{w_j^2}=2W^2(1-\tfrac{1}{3}\sum R_j), \tag{36}$$

$$m_i=\frac{R_i-\frac{1}{3}\sum R_j}{1-\frac{1}{3}\sum R_j}. \tag{37}$$

The correlations $R_{1,2,3}$ are determined by the distances $a_{1,2,3}$ between the pairs of points concerned, which in turn depend on the $\omega_{1,2,3}$ in the following way:

$$a_i^2=2\sum r_j^2-3r_i^2=(1-\omega_i)\sum r_j^2. \tag{38}$$

Except when two of the three points coincide to make one of the a_i vanish, it follows that $R_i<1$ and, from (37), $m_i<1$. On the other hand it follows from the original definition (35) of m_i that $m_i>-1$, since $\overline{w_i^2}>0$ (one of the $\overline{w_i^2}$, say $\overline{w_1^2}$, could only vanish if w_1 would always be zero, which would mean, according to (25), that always $2\mathbf{W}_1=\mathbf{W}_2+\mathbf{W}_3$; this, however, would imply that $R_2+R_3=2$, which is impossible unless all three points would coincide). Thus we have

$$-1<m_i\leq 1.$$

Because of the homogeneity of the field of turbulence, the functional dependence of R_i on a_i is the same for $i=1$, 2 and 3, so that we may write:

$$R_i=R(a_i^2),$$

where the function $R(a^2)$ still depends on time.

Since R_i (for a fixed lapse of time) decreases with increasing distance a_i, it follows from (37) and (38) that $m_i < m_k$ if $\omega_i < \omega_k$ (supposed that $\sum r_j^2$ is given). In general, we may state that the m-values in formula (34) tend to increase or decrease with increasing or decreasing ω-values. To be more precise: if we assume that $R(a^2)$ decreases with increasing a^2 and that dR/da^2 increases with increasing a^2 (see Figure 3), it may be proved that

$$0 < \sum m_i \omega_i < \sum \omega_i^2 < \tfrac{3}{2}, \tag{39}$$

except (a) in the case of equilaterality, where $\sum m_i \omega_i = \sum \omega_i^2 = 0$, and (b) in the case of degeneration, when two points coincide, making $\sum m_i \omega_i = \sum \omega_i^2 = \frac{3}{2}$ (see (29)). The proof is given in the Appendix.

It follows now from (34) and (39) that, if we except the case of degeneration, we have:

$$1 > \frac{\sum r_j^2 + \frac{2}{3}(\sum \omega_j^2)(\sum \overline{w_j^2})}{\sum r_j^2 + \sum \overline{w_j^2}} >$$

$$> 2\varphi_e^2 - 1 > \frac{\sum r_j^2}{\sum r_j^2 + \sum \overline{w_j^2}} = f_e^{-2} \tag{40}$$

or: $$f_e^{-2} + \tfrac{2}{3}\sum \omega_j^2 (1 - f_e^{-2}) > 2\varphi_e^2 - 1 > f_e^{-2},$$

the $>$-signs changing into signs of equality in the case of equilaterality ($\sum \omega_j^2 = 0$).

It is clear that with increasing time t the $R(a_i^2)$ diminish and tend towards zero for very large t. According to (37), the same then holds for the m_i. It then follows from (34) and (32) that $2\varphi_e^2 - 1$ decreases also and tends towards zero for large t, as we had already concluded before on more qualitative grounds.

The relation between this decrease of $2\varphi_e^2 - 1$ and the increase of f_e depends on the $\omega_{1,2,3}$ and on the function $R(a^2)$ and will not, in general, be a simple one, except in the case of equilateral initial triangles, where (33) applies. If, however, the deviation from equilaterality is not too large, $\sum \omega_j^2$ being sufficiently small, formula (33) might still be used as a good approximation.

By applying formula (33) it is possible to find the spreading factor f_e, for a certain initial size measure σ_0, from two successive pictures of a two-dimensional point cluster even if the scales of the pictures are different and their ratio is not known. In order to compute φ_e^2 to this end, one may use a sufficiently large number of triads of points of the cluster, forming triangles of about the same size which are not too far from being equilateral. It is only essential that the points are numbered as individuals and that each point keeps its number from one picture to a following. The ensemble value φ_e^2 is, according to its definition, obtained from formula (18a) or (18b) by averaging the numerator and the denominator separately and taking the quotient of these averages. If in each picture the positions of the points are measured with respect to an arbitrary Cartesian coordinate system, the evaluation of φ_e^2 is then a matter of straightforward computation, for which modern computing techniques may be used.

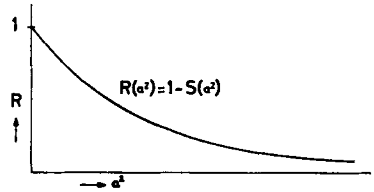

FIG. 3. Correlation of displacements as a function of squared distance.

Having found f_e^2 from φ_e^2 for a number of successive points of time, the eddy diffusivity K may be calculated from the change of f_e^2 by using equation (21):

$$2K = \sigma_0^2 df_e^2/dt = \sigma_0^2 d\Phi_e^{-1}/dt, \tag{21}$$

where we have written

$$\Phi_e = 2\varphi_e^2 - 1.$$

For a sufficiently small lapse of time $\Delta t = t - t_0$, we find:

$$f_e^2 \approx 1 + 2K\sigma_0^{-2}\Delta t \tag{41}$$

and $$2\varphi_e^2 - 1 = \Phi_e \approx 1 - 2K\sigma_0^{-2}\Delta t. \tag{42}$$

If we assume $$K = k\sigma^m, \tag{43}$$

we find from (41) and (42):

$$f_e^2 \approx 1 + 2k\sigma_0^{m-2}\Delta t,$$

$$\Phi_e \approx 1 - 2k\sigma_0^{m-2}\Delta t.$$

With $m = \frac{4}{3}$ these formulas become:

$$f_e^2 \approx 1 + 2k\sigma_0^{-\frac{2}{3}}\Delta t,$$

$$\Phi_e \approx 1 - 2k\sigma_0^{-\frac{2}{3}}\Delta t.$$

More generally we may assume a power law for the increase of $\overline{\sigma^2} = \sigma_e^2 = f_e^2 \sigma_0^2$, as follows:

$$\sigma_e = \varrho t^n. \tag{44}$$

Substituting this for σ in (43) and using (21), we obtain:

$$2k\varrho^m t^{mn} = d\sigma_e^2/dt = 2n\varrho^2 t^{2n-1},$$

from which it follows that

$$n = \frac{1}{2-m}$$

and $$\sigma_e = ((2-m)kt)^{1/(2-m)}.$$

By putting (44) we fixed the zero point of time, so that t_0 is now determined by

$$\sigma_0 = ((2-m)kt_0)^{1/(2-m)},$$

from which follows:

$$t_0 = \frac{\sigma_0^{2-m}}{k(2-m)} = \frac{\sigma_0^2}{(2-m)K_0}.$$

Hence $$f_e^2 = \frac{\sigma_e^2}{\sigma_0^2} = \left(\frac{t}{t_0}\right)^{2/(2-m)} = \left(1 + \frac{\Delta t}{t_0}\right)^{2/(2-m)}$$

and $$\Phi_e = \left(1 + \frac{\Delta t}{t_0}\right)^{-2/(2-m)}.$$

With $m = \frac{4}{3}$ these formulas become:

$$f_e^2 = \left(1 + \frac{\Delta t}{t_0}\right)^3,$$

$$\Phi_e = \left(1 + \frac{\Delta t}{t_0}\right)^{-3},$$

where $$t_0 = \tfrac{3}{2}k^{-1}\sigma_0^{\frac{2}{3}} = \tfrac{3}{2}\sigma_0^2 K_0^{-1}.$$

Details and results of a practical application of the theory outlined above, an application already referred to in the introduction, will be published later.

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Appendix 1

The following deduction deals with triads of points but it applies equally well to configurations of an arbitrary number of points.

Let the origin of a Cartesian coordinate system be placed in the centre of gravity of triangle $A_1A_2A_3$. The radius-vectors to A_1, A_2 and A_3 from their centre of gravity (the origin) are denoted by $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, their components being (x_1, y_1), (x_2, y_2) and (x_3, y_3), respectively. Let the centre of gravity of $B_1B_2B_3$ be (a, b) and let the radius-vectors from that point to B_1, B_2 and B_3 be $\mathbf{s}_1$, $\mathbf{s}_2$ and $\mathbf{s}_3$. The coordinates of B_j ($j = 1,2,3$) may then be written $a + s_j \cos\beta_j$ and $b + s_j \sin\beta_j$, where β_j is the azimuth of the radius-vector $\mathbf{s}_j$. When applying a rotation to $B_1B_2B_3$ the values of the azimuth are replaced by $\beta_j + \gamma$, the angle of rotation being γ. For reasons of symmetry we apply multiplications to both $A_1A_2A_3$ and $B_1B_2B_3$: the first triangle is multiplied by $f^{\frac{1}{2}}$, the second one by $f^{-\frac{1}{2}}$. The sum of the squares of the distances A_1B_1, A_2B_2 and A_3B_3 becomes now

$$S = \sum\{(a + f^{-\frac{1}{2}} s_j \cos(\beta_j + \gamma) - f^{\frac{1}{2}} x_j)^2 + (b + f^{-\frac{1}{2}} s_j \sin(\beta_j + \gamma) - f^{\frac{1}{2}} y_j)^2\}.$$

The minimum condition

$$\delta S = 0$$

gives the following equations for a, b, f and γ:

$$\frac{\partial S}{\partial a} = 0; \quad \frac{\partial S}{\partial b} = 0; \tag{i}$$

$$\frac{\partial S}{\partial f} = 0; \tag{ii}$$

$$\frac{\partial S}{\partial \gamma} = 0. \tag{iii}$$

Equations (i) yield: $a = b = 0$ (coincidence of the centres of gravity). When these values are substituted, equation (iii) yields:

$$\sum\{x_j s_j \sin(\beta_j+\gamma) - y_j s_j \cos(\beta_j+\gamma)\} = 0, \quad \text{(iv)}$$

which may be written as

$$\sum \mathbf{r}_j \times \mathbf{s}_j^* = 0, \quad \text{(v)}$$

where $\mathbf{s}_j^*$ means the radius-vector to B_j *after* the rotation. This condition is invariant with respect to multiplication.

Finally, equation (ii) yields:

$$f^2 = \frac{\sum s_j^2}{\sum r_j^2}. \quad \text{(vi)}$$

This condition is invariant with respect to rotation of one or both of the two configurations.

Appendix 2

The formula (5) which has been found for φ^2 may also be written very elegantly by means of complex vector notation. Denoting Cartesian components of $\mathbf{r}_j$ and $\mathbf{s}_j$ by x_j, y_j and ξ_j, η_j, respectively, and writing

$$x_j + iy_j = z_j, \quad \xi_j + i\eta_j = \zeta_j,$$

$$x_j - iy_j = z_j^\dagger, \quad \xi_j - i\eta_j = \zeta_j^\dagger,$$

we find that formula (5) is transformed into

$$\varphi^2 = \frac{(\sum z_j \zeta_j^\dagger)(\sum z_j^\dagger \zeta_j)}{(\sum z_j z_j^\dagger)(\sum \zeta_j \zeta_j^\dagger)}. \quad \text{(5 a)}$$

Appendix 3

A few words may be added to what has been said about the representation of triangle forms by points on a sphere.

The definition of the "latitude" μ, as contained in (16), does not make any difference between the three points, it has no preference for any one of them, but depends only on their sense of rotation. In the definition of the "longitude" λ, however, one of the points plays a special role. In (16) this is point 1 and the longitude thus defined might well have been called $\lambda^{(1)}$, in order to draw attention to this special choice. We might equally well have chosen point 2 or 3 and have defined a $\lambda^{(2)}$ or $\lambda^{(3)}$ as follows:

$$\cos \lambda^{(2)} = \frac{\omega_2^2}{\sqrt{1-\chi^2}}, \quad \sin \lambda^{(2)} = \frac{\sqrt{\tfrac{1}{3}} \cdot (\omega_3 - \omega_1)}{\sqrt{1-\chi^2}};$$

$$\cos \lambda^{(3)} = \frac{\omega_3^2}{\sqrt{1-\chi^2}}, \quad \sin \lambda^{(3)} = \frac{\sqrt{\tfrac{1}{3}} \cdot (\omega_1 - \omega_2)}{\sqrt{1-\chi^2}}.$$

Choosing $\lambda^{(2)}$ or $\lambda^{(3)}$ instead of $\lambda^{(1)}$ means only a coordinate transformation on the sphere, a change of zero meridian so to say. It is easily seen that the relation between $\lambda^{(2)}$, $\lambda^{(3)}$ and $\lambda^{(1)}$ is:

$$\lambda^{(1)} = \lambda^{(2)} + 120° = \lambda^{(3)} - 120°.$$

In fact, going around the sphere along a parallel of latitude χ = constant, one finds the same form three times, at longitudes that differ mutually by 120°, only the numbering of the points being different in the three cases. Since in the formulas for ψ and φ the numbering is essential (the numbers of the points expressing their unlosable identities), these three forms do show form difference with respect to each other, as expressed by these formulas or by the angular distance between the representative points of the sphere.

If, however, we forget about the numbering of the points, we find all possible forms of triangles (the mirror image of any form being still considered as a different form) on a part of the sphere covering a longitude interval of 120°. All possible isosceles triangle forms, for instance, are found on the meridians $\lambda^{(1)} = 0°$ and $\lambda^{(1)} = \pm 60°$.

Figure 4 shows a number of triangle forms corresponding to certain combinations of latitude and longitude.

Appendix 4

The proof of the statement that

$$0 < \sum m_j \omega_j < \sum \omega_j^2,$$

supposed that not all three ω_j vanish and that all of them are smaller than 1, runs as follows.

We introduce

$$S_j = 1 - R_j, \quad p_j = 1 - \omega_j.$$

Then we have

$$m_i = -3S_i/\sum S_j + 1, \quad \sum p_j = 3$$

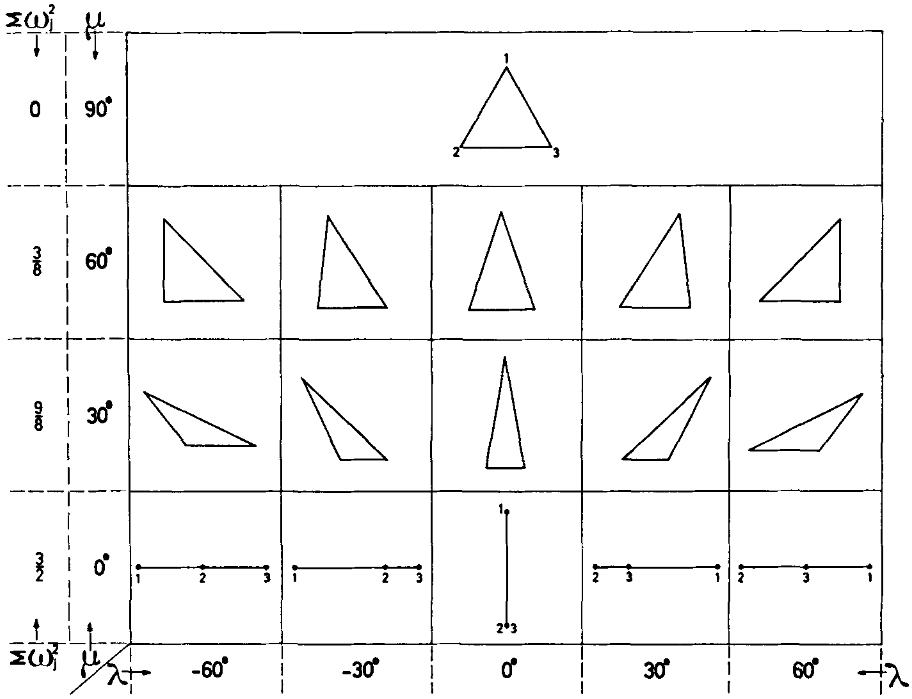

Fig. 4. Examples of triangle forms, expressed by means of λ and μ.

and $$a_i^2 = p_i \sum r_j^2,$$

where $\sum r_j^2$ is supposed to be fixed. The latter fact means, according to the supposition we have made concerning R_j, that S_j increases monotonously with p_j but dS_j/dp_j decreases with increasing p_j (see Fig. 3).

Now let first all three ω_j be different. Then we may choose the numbering so that

$$0 < p_1 < p_2 < p_3. \tag{vii}$$

If we then write $S_j = c_j p_j$,

we have $$0 < S_1 < S_2 < S_3 \tag{viii}$$

and (see Fig. 3)

$$c_1 > c_2 > c_3 > 0. \tag{ix}$$

Now we can write:

$$\sum m_j \omega_j = -\sum m_j p_j = \frac{3\sum p_j S_j}{\sum S_j} - 3. \tag{x}$$

From (vii) and (viii) it may be easily deduced that $\sum p_j S_j > \sum S_j > 0$, from which it follows that, according to (x),

$$\sum m_j \omega_j > 0,$$

provided that not all three ω_j vanish.

On the other hand we may, instead of (x), also write

$$\sum m_j \omega_j = \frac{3\sum c_j p_j^2}{\sum c_j p_j} - 3, \tag{xi}$$

whereas $$\sum \omega_j^2 = \sum p_j^2 - 3.$$

It may now be deduced from (vii) and (ix) that

$$\frac{\sum c_j p_j^2}{\sum c_j p_j} < \frac{\sum p_j^2}{\sum p_j} = \frac{\sum p_j^2}{3},$$

from which it follows that, according to (xi),

$$\sum m_j \omega_j < \sum \omega_j^2,$$

provided that not all three ω_j vanish (i.e. that the triangle is not equilateral) and that all of them are smaller than 1 (i.e. that no two of the three points coincide).

If, instead of (vii),

$$0 < p_1 = p_2 < p_3$$

or $$0 < p_1 < p_2 = p_3,$$

the proof runs quite similarly.