An exact integral of complete spectral equations for unsteady one-dimensional flow[1]

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ABSTRACT

Finite-amplitude, one-dimensional, isothermal flow of a perfect gas is considered. When the flow is organized initially as a "linear" sound wave, it is equivalent to the solution of an "advection equation". The flow that evolves in this way from an initial state is analyzed by means of a Fourier series in space, with time-dependent coefficients. These coefficients can be determined exactly when the initial velocity is a pure sinusoid. The result is such that the Fourier series in space is a Kapteyn series in time. The series is valid up to the time of shock formation. Throughout this interval the lowest harmonics inferred from severely truncated spectral equations are in good agreement with the exact solution. However, the spectral representation becomes very inefficient as the discontinuous state is approached.

1. Introduction

Numerical integration of the hydrodynamic equations in the spectral domain has been carried out in numerous investigations. Exact solutions of severely truncated spectral equations often can be obtained, but the errors of spectral truncation are difficult to assess. The purpose of the present note is to make a contribution to this problem and to give a result of intrinsic mathematical interest, namely an exact solution of the complete system of nonlinear spectral equations in a special case. This solution is closed, analytic and time-dependent.

2. One-dimensional isothermal flow

The momentum and mass equations for one-dimensional isothermal flow of a perfect gas are, respectively

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) u = -c\frac{\partial \sigma}{\partial x} \qquad (2.1\,a)$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) \sigma = -c\frac{\partial u}{\partial x} \qquad (2.1\,b)$$

Here $\sigma \equiv c \ln(\varrho/\varrho_0)$, where $c \equiv (RT)^{\frac{1}{2}}$ is the sound speed (= constant) and ϱ_0 is a standard density (= constant). If the second equation is multiplied by ± 1 and added to the first, the result is

$$\left[\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right](u \pm \sigma) = 0. \qquad (2.2)$$

The general solution of (2.2) is

$$u + \sigma = f(x - (u + c)t) \qquad (2.3a)$$

$$u - \sigma = g(x - (u - c)t), \qquad (2.3b)$$

where f and g are functions that may be regarded as fixed by initial conditions:

$$f(x) = u(x,0) + \sigma(x,0) \qquad (2.4)$$
$$g(x) = u(x,0) - \sigma(x,0)$$

We shall assume that $u(x,0)$ and $\sigma(x,0)$ can be assigned arbitrarily.

As a special case, consider a disturbance organized initially as a "linear" sound wave moving along $+x$ with speed $+c$. Then $\sigma(x,0) = u(x,0)$, so (2.4) makes $f(x) = 2u(x,0)$ and $g(x) = 0$. Hence $\sigma(x,t) = u(x,t)$ from (2.3b), and (2.3a) gives

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$$u = F(x - (u + c)t), \tag{2.5}$$

where $F(x) \equiv u(x, 0)$. The particular integral (2.5) was obtained first by Poisson in 1808. The distortion implied by (2.5) is the subject of a famous paper by Stokes in 1848 "On a difficulty in the theory of sound".

3. Advection equation

In the flow under consideration the sound speed c is uniform and constant. If the nonlinear terms on the left in (2.1) were absent, (2.5) would be $u = F(x - ct)$; thus, the disturbance $u(x,t) = \sigma(x,t)$ would be stationary in a reference frame moving with speed c. The distortion produced by nonlinear effects therefore is best examined relative to this frame. If $x - ct$ is replaced by x, equations (2.1a) and (2.1b) become identical to

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) u = 0, \tag{3.1}$$

since $\sigma = u$. The general solution of (3.1) is

$$u = F(x - ut), \tag{3.2}$$

as is evident also from (2.5). We shall carry out all further discussion of our problem on the basis of (3.1) and (3.2).

It is clear from (3.1) that $du/dt = 0$ along characteristic curves $dx/dt = u$ in the x, t-plane. Therefore, the characteristics are straight lines, and the motion corresponds to that of "simple waves" (Courant and Friedrichs, 1948). Moreover, (3.1) has *only one family of characteristics*, so the motion is analogous to that of the "kinematic waves" discussed by Lighthill and Witham (1955). It should be noted that in the factor $u + c$ on the right side of (2.5), the term u expresses kinematical effects upon the wave, whereas c expresses dynamical effects. The latter can be eliminated, formally, as in (3.2), because c is uniform in the special problem being considered.

A discussion of (3.1) would not be complete without reference to the equation

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) u = \nu\frac{\partial^2 u}{\partial x^2}, \tag{3.3}$$

the simplest hydrodynamical-type equation that contains the mechanism of counteraction between diffusion and nonlinear convection. Bateman (1915) pointed out that (3.3) has the exact solution

$$u = c - U\,\mathrm{th}[U(x - ct)/2\nu],$$

where c is an arbitrary, uniform translation speed and U an arbitrary, uniform velocity scale. This gives a smooth "shock" of width ν/U propagating along x with speed c. Cole (1951) noted that (3.3) is invariant under the transformation $x - ct \to x; t \to t; u - c \to u$; hence, without loss of generality we can write $c = 0$ in the preceding solution:

$$u = -U\,\mathrm{th}(Ux/2\nu). \tag{3.4}$$

This is a steady solution of (3.3), in which dissipation in the shock is exactly balanced by convection of kinetic energy from infinity (Cole, 1951).

The following remark is pertinent in connection with (3.4). Let $u = -U\,\mathrm{th}\,kx$ be an initial condition for (3.1), so that the solution of (3.1) is

$$u = -U\,\mathrm{th}\,k(x - ut),$$

in accordance with (3.2). This is an unsteady solution, in which the shock width (initially of order k^{-1}) steadily decreases until a discontinuity is formed. On the other hand, if we regard $u = -U\,\mathrm{th}\,kx$ as an initial condition for (3.3), rather than for (3.1), then there is a limiting steady state—given uniquely by (3.4)—which is independent of k. The preceding statement, which is plausible intuitively, was proved formally by Cole (1951) (in the case $k \to \infty$).

The most remarkable aspect of (3.3) is the existence of a *general*, exact solution. This solution was discovered independently by Hopf (1950) and Cole (1951), and will be given at the end of the next section. Lighthill (1956) has discussed the matter further, and in particular has shown exactly the circumstances under which (3.3) is a valid approximation for sound waves of finite amplitude.

Another work which bears a close relation to the present one is the well-known investigation of mathematical models of turbulent flow by Burgers (1939, 1948), who studied equations similar to (3.3). *Statistical* properties of this equation have been examined by Burgers (1952), Reid (1957), and Ogura (1957).

It is evident that (3.1) is a special case of

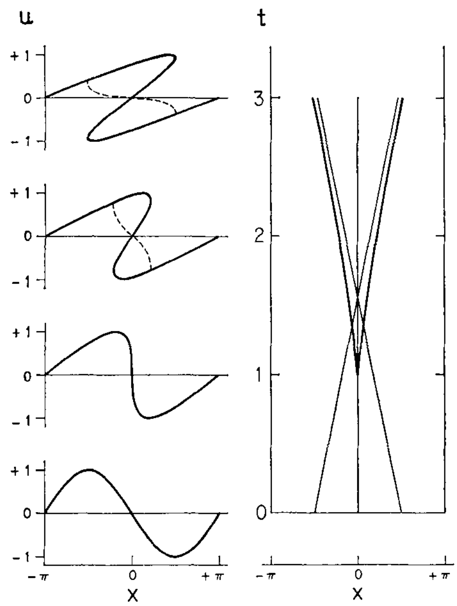

FIG. 1. Left panel: profiles of u as a function of x for selected t. Broken line in triple-valued region shows single-valued composite function u^* represented by Fourier series (5.6). Right panel: x, t-diagram showing characteristics $u = 0$ and $u = \pm 1$. Heavy curve with cusp at $t = 1$ is envelope of characteristics.

(3.3). The significance of equation (3.1) in the present investigation is simply that the non-linear term $u\partial u/\partial x$ contains the most rudimentary form of the mechanism whereby the spectrum of motion is modified through inertial exchange between spectral components of different scales. From the point of view of numerical integration, this equation has been studied by PLATZMAN (1954), OBUKHOV (1957), BENTON (1958), and by PHILLIPS (1960), who suggested the name "advection equation".

4. Sinusoidal solution

The special case $u(x,0) = -U\sin kx$ will be considered throught the sequel. The general solution (3.2) therefore gives

$$u = -U\sin k(x-ut).$$

We adopt the constants U and k^{-1} as units of velocity and length; thus

$$u = -\sin(x-ut), \tag{4.1}$$

on the understanding that u, x, t now correspond to what previously were u/U, kx, kUt.

The right panel of Fig. 1 is an x, t-diagram which shows a few typical characteristics of (4.1). Also shown (by heavy lines) is the envelope of the "cusp region", inside which the solution is triple valued (three characteristics through each point). The left panel of Fig. 1 shows the x, u-configuration of the solution at selected times.

Since each value of u is propagated along x with speed u, it follows that the wave crest ($u>0$) is propagated toward x-positive and the trough ($u<0$) toward x-negative. Moreover, if $|u_1| > |u_2|$, then u_1 is propagated more rapidly than u_2, so the slope $S \equiv \partial u/\partial x$ of the wave profile becomes steeper where S is negative initially, and flatter where S is positive initially. This process may be examined quantitatively by computation of the change of slope along a characteristic:

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} = -S^2$$

by (3.1.). Integration yields

$$S = (t + S_0^{-1})^{-1}, \tag{4.2}$$

where S_0 is the initial value of S. This equation shows that for each value of u where $S_0 < 0$, the slope becomes increasingly negative until at a critical time $t_c \equiv -S_0^{-1}$ it becomes infinite, after which it is positive and declines to zero (ultimately as t^{-1}). The minimum value of t_c occurs for $u = 0$ at $x = 0$, since there $-S_0^{-1} = \sec x$ has its minimum value $= 1$, as is easily seen from (4.1). It follows that the cusp in the x, t-plane is at the point $x = 0$, $t = 1$.

For $t > 1$ the solution (4.1) is triple-valued in an expanding zone between the two points where the slope is infinite—one forward-moving and one backward-moving. To obtain the coordinates of these points, write (4.1) parametrically:

$$u = -\sin\theta \tag{4.3a}$$

$$x = \theta - t\sin\theta. \tag{4.3b}$$

From this we find $S = (t - \sec\theta)^{-1}$ for the slope at any point. Hence, where the slope is infinite,

$$x = \theta - \tan\theta.$$

$$t = \sec\theta, \tag{4.4}$$

which gives the envelope of characteristics (shown by the two heavy curves in the right panel of Fig. 1). On the envelope, $u = (1 - t^{-2})^{\frac{1}{2}}$,

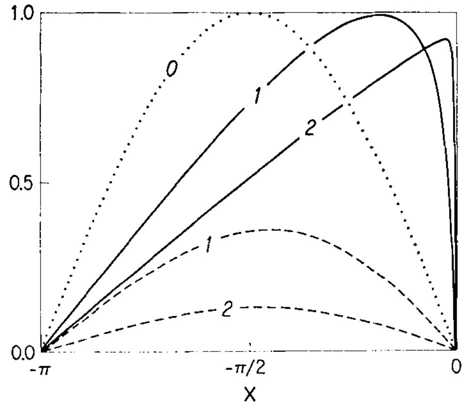

FIG. 2. Solutions (4.8) of (3.3) at $t=1$ and $t=2$, for Reynolds number $R=1$ (broken curves) and $R=100$ (solid curves). Dotted curve shows initial condition.

so the points where S is infinite gradually approach the crest and trough of the wave, and the regions where $S<0$ are squeezed into an x-interval that approaches zero (at a rate $\sim t^{-1}$). Thus, sharp corners are formed at crest and trough as the wave flattens out.

Further mathematical detals of the solution in the cusp region can be ascertained without difficulty in principle. However, these details have no significance in the solution of a physical problem, since at $t=1$, shock conditions must be imposed which render the solution single-valued subsequently. In this connection it is illuminating to refer to the exact solution of (3.3) which was discovered by HOPF (1950) and COLE (1951), because the diffusion term in this equation ensures a smooth transition at the shock. If $\phi(x,t)$ is a solution of the linear diffusion equation

$$\frac{\partial\phi}{\partial t}=\frac{1}{R}\,\frac{\partial^2\phi}{\partial x^2}, \tag{4.5}$$

then an exact solution of the nonlinear equation (3.3) is

$$u\,(x,\,t)=-\frac{2}{R}\cdot\frac{1}{\phi}\,\frac{\partial\phi}{\partial x}. \tag{4.6}$$

Here $R\equiv U/\nu k$ is a Reynolds number for the macro-scales U and k^{-1} in terms of which x and t are rendered dimensionless. The initial condition $u(x,0)=-\sin x$ corresponds to $\phi(x,0)=\exp(-\tfrac{1}{2}R\cos x)$, in accordance with (4.6), so the problem posed by (4.5) is that of the conductive redistribution of heat in a bar with insulated ends, the initial temperature distribution being $\phi(x,0)$. This has the solution

$$\phi\,(x,\,t)=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx \tag{4.7}$$

$$a_n\equiv(-)^n I_n(\tfrac{1}{2}R)\exp(-n^2t/R),$$

where I_n is the exponentially increasing Bessel function of the second kind. Substitution in (4.6) gives

$$u\,(x,\,t)=\frac{2R^{-1}\sum_{n=1}^{\infty}na_n\sin nx}{\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx} \tag{4.8}$$

which (in slightly modified notation) is a result given explicitly by COLE (1951).

Fig. 2 shows the solution (4.8) at $t=1$ and 2, for Reynolds numbers $R=1$ (broken lines) and $R=100$ (solid lines).[1] The dotted line shows the initial condition $u=-\sin x$. For low Reynolds number the diffusion term in (3.3) dominates, and the result is very nearly a pure-harmonic decay of the initial disturbance. This is expressed in (4.8) by rapid convergence of the series, through operation of the exponential factor in a_n. No shock is formed. On the other hand, for high Reynolds number the convection term in (3.3) dominates initially, and produces a shock. This is expressed in (4.8) by very slow convergence of the series—in other words, by spectral cascade of a significant amount of energy. In the shock zone (of width $\sim R^{-1}$) there is approximate balance between convection and diffusion. Outside this zone the solution coincides very nearly with that of the inviscid equation (3.1). In fact, after shock formation ($t>1$) the solution of (3.3) can be approximated closely by (3.4) in the shock zone, and (4.1) elsewhere.

5. Fourier spectrum

The solution (4.1) keeps $u=0$ at all points where x is an integral multiple of π. Therefore, *prior to the time of shock formation* ($t<1$), this solution has a well-defined Fourier sine-series representation

[1] The curves in Fig. 2 were not obtained from the analytic solution (4.8). It is simplier to compute $\phi(x,t)$ by direct numerical integration of (4.5), and then $u(x,t)$ from (4.6) by numerical differentiation of ϕ.

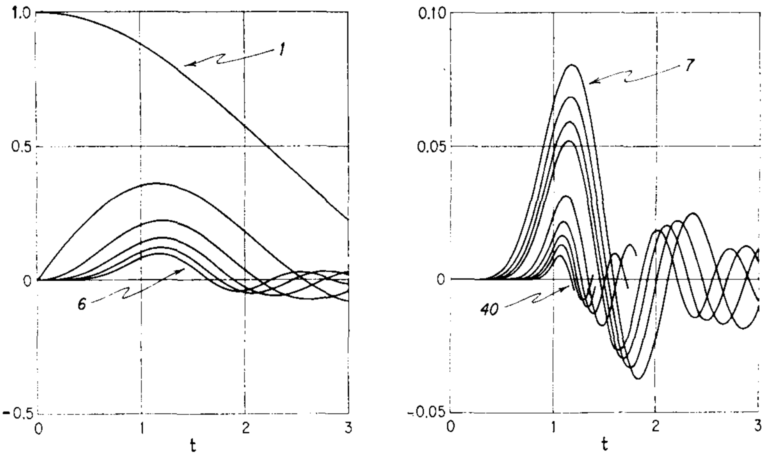

FIG. 3. Fourier coefficients $u_n(t)$ evaluated from (5.5). Left panel: $n = 1(1)6$; right panel: $n = 7(1)10(5)30(10)40$. (Note that ordinate scales of left and right panels differ by a factor of 10).

$$-u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin nx, \tag{5.1}$$

the transform of which is

$$u_n(t) = -\frac{1}{\pi}\int_{-\pi}^{+\pi} u(x,t) \sin nx\, dx. \tag{5.2}$$

(Note that u_n is defined as the spectrum of *minus* u.) It is remarkable, and one of the main points of this note, that in spite of the implicit nature of the solution (4.1), the preceding integral for $u_n(t)$ can be evaluated explicitly.

By partial integration

$$u_n(t) = -\frac{1}{n\pi}\int_{x=-\pi}^{x=\pi} \cos nx\, du, \tag{5.3}$$

and after substitution of the parametric equations (4.3),

$$u_n(t) = \frac{2}{n\pi}\int_0^{\pi} \cos n(\theta - t\sin\theta) \cos\theta\, d\theta. \tag{5.4}$$

Bessel's integral for $J_n(z)$ is

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)\, d\theta.$$

If this is inserted on the right side of the recurrence formula

$$J_n(z) = \frac{z}{2n}[J_{n+1}(z) + J_{n-1}(z)],$$

the result is a modified form of Bessel's integral:

$$J_n(z) = \frac{z}{n\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta) \cos\theta\, d\theta.$$

Comparison with (5.4) leads at once to

$$u_n(t) = 2J_n(nt)/nt. \tag{5.5}$$

Equation (5.5) is a simple, closed formula for the Fourier coefficients of $-u(x,t)$.

In view of (5.5) the Fourier expansion (5.1) is explicitly

$$u = -2\sum_{n=1}^{\infty} (nt)^{-1} J_n(nt) \sin nx. \tag{5.6}$$

Regarded as a series of Bessel functions, this is a Kapteyn series, the distinctive feature of which is presence of the order n as a factor in the argument of J_n. It is a curious fact that (5.6) is completely equivalent to Bessel's solution of Kepler's problem concerning the anomalies of a planetary orbit—the problem which led Bessel to define the functions that now bear his name (see WATSON, 1944, § 17.2). The parameters of Kepler's problem and their formal relations to those of (4.3) are the "mean anomaly" $M \equiv x$, the "eccentric anomaly" $E \equiv \theta$, and the orbital eccentricity $\varepsilon \equiv t$, in terms of which (4.3*b*) is $M = E - \varepsilon \sin E$. Kepler's problem is inversion of the latter relation: to express E explicitly in terms of M. Writing this as $E = M + \varepsilon \sin E$ and introducing (5.6) for $\sin E$ ($= -u$ by (4.3*a*)), we find

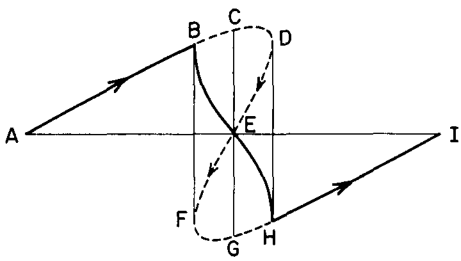

FIG. 4. Schematic profile of u as a function of x for $t > 1$. Broken curve: original, triple-valued u; solid curve: single-valued composite u^* represented by series (5.6).

$$E = M + 2\sum_{n=1}^{\infty} n^{-1} J_n(n\varepsilon) \sin nM,$$

which is Bessel's solution of Kepler's problem.

Fourier coefficients $u_n(t)$ were computed from (5.5) for $0 \leqslant t \leqslant 3$. The left panel of Fig. 3 shows results for $n = 1(1)6$, the right panel (on an expanded scale) for $n = 7(1)10(5)30(10)40$. All $u_n(t)$ are zero initially with exception of $u_1(0) = 1$. As t increases from zero, $u_1(t)$ gradually declines as energy cascades through the spectrum. Fig. 3 exhibits this process in a striking manner. In each harmonic the energy increases monotonically throughout the "physical" range $0 \leqslant t < 1$ and reaches a maximum very soon after $t = 1$. (If t_n denotes the time of incidence of this first maximum, then $t_n \to 1^+$ in the limit $n \to \infty$.) After the first maximum $u_n(t)$ is oscillatory and decays to zero (as $t^{2/3}$ ultimately).

When $t > 1$ the function $u(x,t)$ determined by (4.1) is multiple-valued, as shown in Fig. 1. However, the series (5.6) is convergent and single-valued for all t. The meaning of this series therefore must be reconsidered for $t > 1$. It is possible to sum (5.6) as a Kapteyn series in the manner described by WATSON (1944, § 17.7), but a simpler route is to examine the integral (5.3) from which $u_n(t)$ was obtained. In order to make the transition from (5.2) to (5.3), it is necessary to regard (5.2) as a line integral, as indicated by arrows in Fig. 4. Now, on the interval of the x-axis in which u is triple-valued, dx is positive along BCD and FGH, but negative along DEF. Therefore, the effect of transformation from (5.2) to (5.3) is such that on this interval u has the composite value $u^* \equiv u' - u'' + u'''$, where $u' \equiv u$ on BCD, $u'' \equiv u$ on DEF, and $u''' \equiv u$ on FGH. This composite value is indicated by the solid curve BEH in Fig. 4 (and by the broken curves in the left panel of Fig. 1).

The series (5.6) for $t > 1$ therefore represents the single-valued function indicated by the solid curve ABEHI in Fig. 4. Since the composite function u^* in the interval BEH does not satisfy the basic differential equation (3.1), the preceding analysis must not be construed as having any physical meaning.

At $t = 1$, the limit of the "physical" range of validity of (5.5), the spectral distribution of u_n for large n may be found from the asymptotic formula (WATSON, 1944, § 8.2)

$$J_n(n) \approx \Gamma(\tfrac{1}{3})/(48)^{1/6}\pi n^{\frac{1}{3}}. \tag{5.7}$$

According to (5.5), the corresponding $u_n \sim n^{-4/3}$. The energy spectrum at $t = 1$ therefore has the distribution $n^{-8/3}$. This shows an attenuation stronger than that for statistical equilibrium ($n^{-5/3}$) by a factor n^{-1}.

The rate of transfer of energy across the spectrum at wave number n is

$$\varepsilon_n \equiv -\frac{d}{dt}\sum_{k=1}^{n}[u_k(t)]^2. \tag{5.8}$$

Therefore, for large n we have

$$\frac{d\varepsilon_n}{dn} \approx \varepsilon_n - \varepsilon_{n-1} = -\frac{d}{dt}[u_n(t)]^2.$$

With (5.5) for $u_n(t)$, this gives at $t = 1$:

$$d\varepsilon_n/dn \approx -8n^{-1}J_n(n)\,[J_n'(n) - n^{-1}J_n(n)]. \tag{5.9}$$

According to WATSON (1944, § 8.2), for large n,

$$J_n'(n) \approx 3^{1/6}\Gamma(\tfrac{2}{3})/2^{\frac{1}{3}}\pi n^{\frac{2}{3}}.$$

Hence, in view of (5.7), the first term in the bracket of (5.9) dominates the second, and we have

$$d\varepsilon_n/dn \approx -(8/\pi\sqrt{3})\,n^{-2},$$

with the aid of $\Gamma(\tfrac{1}{3})\Gamma(\tfrac{2}{3}) = 2\pi/\sqrt{3}$. Integration with respect to n yields

$$\varepsilon_n \approx (8/\pi\sqrt{3})\cdot n^{-1}. \tag{5.10}$$

The dependence of ε_n upon n exhibits the degree to which the spectrum is nonstationary. The result given in (5.10) was confirmed by direct numerical evaluation from (5.8), using (5.5) for $u_n(t)$. This gave $n\varepsilon_n = 1.364$ at $n = 45$ and 1.371

at $n = 49$, which shows $n\varepsilon_n$ to be approximately independent of n. The numerical value of the factor in (5.10) is 1.470.

6. Invariants

After multiplication by mu^{m-1}, equation (3.1) may be written

$$\frac{\partial}{\partial t} u^m = -\frac{m}{m+1} \frac{\partial}{\partial x} u^{m+1};$$

but (4.1) keeps $u = 0$ at $x = -\pi$ and at $x = 0$, so integration with respect to x gives

$$\frac{\partial}{\partial t} \int_{-\pi}^{0} u^m dx = 0, \qquad (6.1)$$

for any $m > -1$. Since the integral in (6.1) is invariant, it may be evaluated at $t = 0$, where $u = -\sin x$. Therefore, after integration with respect to t,

$$M_m \equiv \pi^{-\frac{1}{2}} \frac{\Gamma\left(\frac{m+2}{2}\right)}{\Gamma\left(\frac{m+1}{2}\right)} \int_{-\pi}^{0} u^m dx = 1, \qquad (6.2)$$

where Γ denotes the gamma-function.

Equation (6.2) gives a class of (normalized) integral invariants of the original differential equation (3.1). Two in particular are of interest here: namely,

$$M_1 \equiv \tfrac{1}{2} \int_{-\pi}^{0} u\, dx = 1 \qquad (6.3a)$$

$$M_2 \equiv \frac{2}{\pi} \int_{-\pi}^{0} u^2 dx = 1. \qquad (6.3b)$$

The first of these is proportional to the integrated momentum (over a half wave length), the second to the kinetic energy.

Spectral forms of (6.3) are obtained by substituting the series (5.6) into the integrals:

$$M_1 = 2 \sum_{n=1,3,5,\dots}^{\infty} J_n(nt)/n^2 t = 1 \qquad (6.4a)$$

$$M_2 = 4 \sum_{n=1,2,3,\dots}^{\infty} [J_n(nt)/nt]^2 = 1. \qquad (6.4b)$$

The identity (6.4a) is a special case of a Kapteyn series known as Meissel's expansion (WATSON, 1944, § 17.23), while (6.4b) is a special case of what WATSON (1944, § 17.6) refers to as a Kapteyn series of the second kind. (The whole of the foregoing analysis can be regarded as a devious but elementary proof of these identities.)

It was noted previously that when $t > 1$ the series (5.6) represents the composite function u^* illustrated by the solid curve in Fig. 4. In the interval BEH this function does not satisfy the basic differential equation (3.1). Terefore, when $t > 1$ the identities (6.4) are no longer valid and the series have no physical significance. Nevertheless, it is worth recording that M_1 for $t > 1$ can be evaluated in an elementary manner, as follows.

With reference to Fig. 4,

$$M_1^* \equiv \tfrac{1}{2} \int_{-\pi}^{0} u^* dx = \tfrac{1}{2} \left(\int_{ABC} + \int_{EFG} \right) u\, dx$$

$$= \tfrac{1}{2}([ux]_A^C + [ux]_E^G) - \tfrac{1}{2} \left(\int_{ABC} + \int_{EFG} \right) x\, du.$$

The integrated parts are zero because $u = 0$ at A and $x = 0$ at C, E and G. In $x du$ we substitute $x = \theta + ut$ and $u = -\sin\theta$, from (4.3); then

$$M_1^* = \tfrac{1}{2} \left(\int_{ABC} + \int_{EFG} \right) (\theta \cos\theta\, d\theta - tu\, du).$$

After performing the indicated quadratures, we can write the result in the following parametric form:[1]

$$M_1^* = \cos\phi + \tfrac{1}{2}\phi \sin\phi \qquad (6.5)$$
$$t = \phi/\sin\phi,$$

where $\phi \equiv u_C t (= -\theta_C = \theta_G)$.

The points D and F eventually will move out of the interval $-\pi < x < \pi$, and this interval will be invaded then by forward- and backward-moving lobes from intervals $-3\pi < x < -\pi$ and $\pi < x < 3\pi$, respectively (since u is periodic in x with period 2π). When this happens, the result in (6.5) is no longer valid. The limit of validity of (6.5) therefore is the time at which D and F reach π and $-\pi$, respectively. This time is found by writing $x = \pm\pi$ in (4.4); the result is $t = 4.60$.

[1] This result also can be obtained by summing the series in (6.4a) as a Kapteyn series (WATSON, 1944, § 17.7). However, the method used above is simplier.

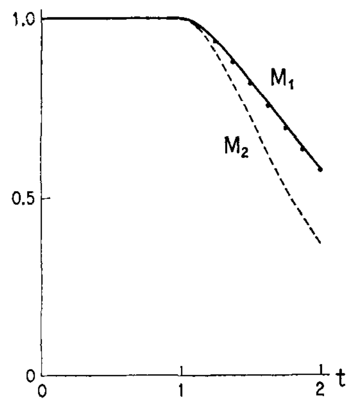

FIG. 5. Linear and quadratic integral invariants. Solid curve: $M_1 = 1$ for $0 \leqslant t \leqslant 1$ and $M_1 = M_1^*$ from (6.5) for $t \geqslant 1$. Dots show values of M_1 computed from 24 terms of the series in (6.4). Broken line shows M_2 computed in the same manner.

It follows that the Kapteyn series M_1 in (6.4*a*) is equal to the M_1^* of (6.5) for $1 \leqslant t \leqslant 4.60$. The solid curve in Fig. 5 shows M_1 in the range $0 \leqslant t \leqslant 2$, computed from $M_1 = 1$ for $0 \leqslant t \leqslant 1$ and from M_1^* for $1 \leqslant t \leqslant 2$. The dots show results of direct numerical computation of M_1 and M_2 from 24 terms of the series in (6.4).

7. Spectral dynamics

The Fourier coefficients $u_n(t)$ satisfy an infinite system of quasilinear, ordinary differential equations, obtained by taking the Fourier transform of the original equation (3.1) after substituting expansion (5.1). The result can be written

$$\frac{2}{n}\frac{du_n}{dt} = \tfrac{1}{2}\sum_{k=1}^{n-1} u_k u_{n-k} - \sum_{k=1}^{\infty} u_n u_{n+k}. \qquad (7.1)$$

The first few equations in this set are

$$\left.\begin{aligned}
\frac{2}{1}\frac{du_1}{dt} &= -(u_1u_2 + u_2u_3 + u_3u_4 + \cdots) \\
\frac{2}{2}\frac{du_2}{dt} &= \tfrac{1}{2}u_1^2 - (u_1u_3 + u_2u_4 + \cdots) \\
\frac{2}{3}\frac{du_3}{dt} &= u_1u_2 - (u_1u_4 + u_2u_5 + \cdots) \\
\frac{2}{4}\frac{du_4}{dt} &= u_1u_3 + \tfrac{1}{2}u_2^2 - (u_1u_5 + u_2u_6 + \cdots).
\end{aligned}\right\} \qquad (7.2)$$

It is remarkable that the exact, time-dependent solution of the infinite system of spectral-dynamic equations (7.1) is given by the closed formula (5.5).[1]

If the integrals defined in (6.3) are computed directly from the expansion (5.1), then

$$M_1 = \sum_{n=1,3,5,\ldots}^{\infty} n^{-1} u_n(t) = 1 \qquad (7.3a)$$

$$M_2 = \sum_{n=1,2,3,\ldots}^{\infty} [u_n(t)]^2 = 1 \qquad (7.3b)$$

By inspection of (7.2) it is easy to verify that the sums in (7.3) are, in fact, invariant. Moreover, since initially $u_1 = 1$ and all other $u_n = 0$, it is also clear that each sum has the value unity. Indeed, in view of (5.5), equations (7.3) are merely a restatement of (6.4).

The assertion that (5.5) gives the solution of (7.1) certainly is valid in the interval $0 \leqslant t \leqslant 1$. For $t > 1$ the coefficients (5.5) represent the "composite" function u^* (Fig. 4), as shown previously. It is not evident *a priori* what function is represented by the solution of (7.1) for $t > 1$; however, it is easy to show that this function cannot be u^*. This follows from the fact that (7.3*a*, *b*) are valid inferences from (7.1) *for all t*. In other words, the function represented by the solution of (7.1) has the property $M_1 = M_2 = 1$ for all t. On the other hand, the function u^* represented by (5.5) is such that $M_1 < 1$ and $M_2 < 1$ for $t > 1$ (Fig. 5). It follows that (5.5) gives the solution of (7.1) only in the interval $0 \leqslant t \leqslant 1$.

Truncated spectral equations have been used in numerous investigations of nonlinear fluid mechanics. It is therefore of interest to compare our exact solution (5.5) of the *complete* spectral equations with solutions of tuncated equations. If only one component is retained, (7.2) reduces to $du_1/dt = 0$ and we get $u_1(t) = 1$. If two components are retained, (7.2) reduces to

$$du_1/dt = -\tfrac{1}{2}u_1u_2$$
$$du_2/dt = \tfrac{1}{2}u_1^2.$$

The exact solution of these equations for initial conditions $u_1 = 1$, $u_2 = 0$ is

[1] Burgers' spectral equations have quadratic terms identical to those of (7.1), and admit certain exact *steady-state* solutions (BURGERS, 1948).

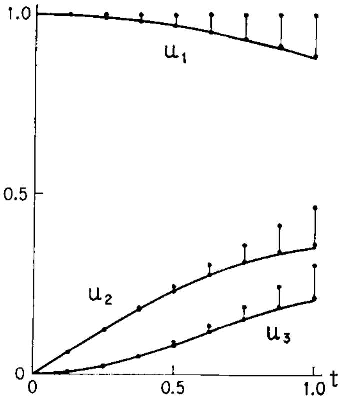

Fig. 6. Solid curves: exact $u_n(t)$ in the "physical" interval $0 \leqslant t \leqslant 1$ from (5.5). Dots: first and second spectral approximations.

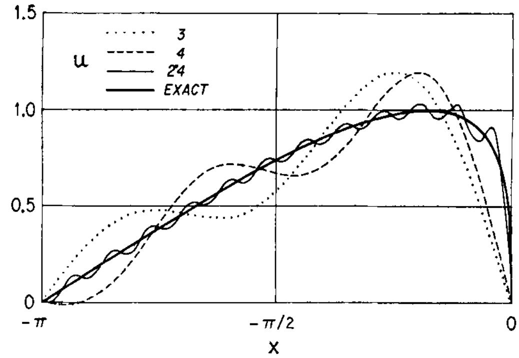

Fig. 7. Profiles of u as a function of x at $t=1$. Exact profile is shown by heavy, solid curve. The other curves show spectral approximations obtained by numerical integration of truncated spectral equations with 3, 4 and 24 terms.

$$u_1 = \operatorname{sech} \tfrac{1}{2}t$$

$$u_2 = \operatorname{th} \tfrac{1}{2}t.$$

(Thus, the initial configuration $u = -\sin x$ is converted to $u = -\sin 2x$ in the limit $t \to \infty$.)

The solid curves in Fig. 6 show the exact coefficients

$$u_1 = 2J_1(t)/t$$

$$u_2 = 2J_2(2t)/2t$$

$$u_3 = 2J_3(3t)/3t$$

in the "physical" interval $0 \leqslant t \leqslant 1$. The dots show first and second spectral approximations. The preceding analysis gives both approximations for u_1 and the first for u_2. The second approximation for u_2 and the two approximations for u_3 were obtained by direct numerical integration of three- and four-component truncations of (7.2).

Agreement between exact u_1, u_2, u_3 and the corresponding second spectral approximations is remarkable. However, at shock time $t=1$ the function $u(x, t)$ represented by only three harmonics is very poor, and there is only slight improvement by addition of the next harmonic, as shown in Fig. 7. Even 24 terms do not suffice to give a smooth representation of the exact solution at $t=1$. These circumstances are not surprising from the standpoint of harmonic analysis, since at $t=1$ the function $u(x, t)$ has infinite slope at $x=0$.

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