The viscous boundary layer at the free surface of a rotating baroclinic fluid

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ABSTRACT

The properties of the viscous boundary layer at the free surface of a rotating baroclinic fluid are analyzed and compared with those of the well-known Ekman boundary layer at a rigid surface. Although the ageostrophic components of the flow in the free surface boundary layer are weaker than in the Ekman layer, there are problems of practical interest in which their effects are not negligible.

1. Introduction

Geostrophic motion of a baroclinic fluid must satisfy the thermal wind equation, which takes the form

$$2(\mathbf{\Omega}\cdot\nabla)\mathbf{u} = \mathbf{g}x\nabla\varrho/\varrho_0 \quad (1)$$

if $\mathbf{\Omega}$ is the uniform angular velocity of basic rotation of the system, $\mathbf{u}$ the Eulerian flow velocity at a general point P fixed relative to a frame of reference S which rotates with the basic angular velocity, $\mathbf{g}$ the acceleration of gravity (including centrifugal effects), ϱ_0 the mean density of the fluid and ϱ the (small) difference between the actual density, $(\varrho_0+\varrho)$, and the mean density, ϱ_0, at P.

At any part of the bounding surface of the fluid which is free, i.e., subject to no externally applied tangential stresses, continuity of tangential stress requires that the stress just within the fluid should vanish. This boundary condition can only be compatible with the geostrophic relationship given by (1) when the fluid is barotropic (i.e., $\nabla\varrho=0$ everywhere). Otherwise (i.e., when the fluid is baroclinic), ageostrophic motion occurs, and if ν, the coefficient of kinematical viscosity, is not too large, more precisely if an Ekman number

$$E\equiv\delta^2/L^2, \quad \text{where} \quad \delta\equiv(\nu/|\mathbf{\Omega}|)^{\frac{1}{2}}, \quad (2\text{a, b})$$

is much less than unity, L being any length characteristic of the scale of the dynamic pressure field in the fluid, this ageostrophic motion will be restricted to a thin boundary layer extending only a short distance (of order δ) into the fluid from the free bounding surface, with geostrophic motion occurring elsewhere.

It is the purpose of this paper to discuss the structure and properties of the viscous boundary layer at the free surface of a rotating baroclinic fluid. Notwithstanding the possible importance of the study of such boundary layers in geophysical fluid dynamics, their explicit discussion does not seem to have been given previously in the literature.

2. The equations of the problem

Consider an area Σ (say) of the bounding surface of the fluid, and assume that the linear dimensions of Σ are large compared with L (see (2) above). As we are interested in the properties of the boundary layer we restrict attention to the flow in regions remote from the edge of Σ, so that edge effects can be ignored.

It is convenient to make two assumptions about Σ which will greatly simplify the analysis of the boundary layer structure without affecting the main conclusions of the paper in any important way. First of all, we shall suppose that Σ is plane, so that the angle between $\mathbf{\Omega}$ and the normal to Σ is independent of position; secondly, we take this angle to be zero, so that $\mathbf{\Omega}$ is normal to Σ. The first of these assumptions is valid when $|\mathbf{g}|\gg\Omega^2L$, because then the radius of curvature of Σ is much larger than the linear dimensions of the region of interest.

Introduce a Cartesian coordinate system fixed in the rotating reference frame S, having its origin in Σ and in which P has coordinates

(x, y, z), the positive z axis being normal to Σ and directed into the fluid. Hence

$$\mathbf{\Omega} = (0, 0, \Omega), \quad \mathbf{g} = (0, 0, g), \tag{3a, b}$$

the sign of Ω or g being reckoned positive or negative according as the corresponding vector quantity, $\mathbf{\Omega}$ or $\mathbf{g}$, is parallel or antiparallel to the positive z axis.

Divide the fluid into two regions, region I occupying $z \geqslant z^*$, and region II occupying $0 \leqslant z < z^*$, the only restrictions on z^* being that

$$\delta \equiv (\nu/|\mathbf{\Omega}|)^{\frac{1}{2}} < z^*, \quad L \gg z^* \tag{4a, b}$$

(remembering that $E < 1$, see equation (2) above). Anticipating that the thickness of the region in which viscous forces are significant is of the order of $\delta(= (\nu/|\mathbf{\Omega}|)^{\frac{1}{2}}$ cm) (see equations (26) below), ageostrophic motion arises only in region II, the flow in region I being (by hypothesis) geostrophic. In accordance with convention region I should be termed the "interior region", or, more briefly, the "interior", and region II the "boundary layer".

Suppose for simplicity that the fluid is incompressible. Under this supposition, valid for a highly rotating fluid when $L\Omega/c < 1$, where c is the speed of sound (see HIDE, 1962), the continuity equation reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{everywhere,} \tag{5}$$

where u, v and w denote, respectively, the x, y and z components of $\mathbf{u}$.

Now consider the momentum equations. Throughout region I ($z \geqslant z^*$, the interior region), since by hypothesis the flow there is geostrophic, the horizontal pressure gradient is everywhere balanced by Coriolis forces, so that

$$\left.\begin{aligned} 2\Omega v &= \frac{1}{\varrho_0}\frac{\partial p}{\partial x} \\ \text{and} \qquad 2\Omega u &= -\frac{1}{\varrho_0}\frac{\partial p}{\partial y} \quad (z \geqslant z^*), \end{aligned}\right\} \tag{6a, b}$$

where p denotes the dynamic pressure. To the same degree of approximation, the vertical momentum equation leads to the relationship between p and the density variation ϱ:

$$\frac{\partial p}{\partial z} = g\varrho \quad (z \geqslant z^*). \tag{7}$$

It is readily shown by means of a scale analysis of the hydrodynamical equations of motion that (6) and (7) are valid when:

$$\Delta\varrho/\varrho_0 < 1, \quad (|\Omega|\tau)^{-1} < 1, \quad R \equiv Q/L|\Omega| < 1,$$

and

$$E \equiv (\nu/|\Omega|L^2) < 1 \tag{8a,b,c,d}$$

(see (2)), where $\Delta\varrho$ is the order of magnitude of density variations within the fluid, τ is a time scale characteristic of temporal variations in the relative flow pattern, Q is a typical relative flow speed, R is a Rossby number and the other quantities are as defined previously.

Eliminate p between equations (6) and (7), combine the resulting equation with equation (5), and thus find the following relationships between the velocity and density fields:

$$\left.\begin{aligned} &\frac{\partial u}{\partial z} = -\frac{g}{2\Omega\varrho_0}\frac{\partial \varrho}{\partial y}, \quad \frac{\partial v}{\partial z} = \frac{g}{2\Omega\varrho_0}\frac{\partial \varrho}{\partial x}, \\ &\frac{\partial w}{\partial z} = 0 \quad (z \geqslant z^*). \end{aligned}\right\} \tag{9a, b, c}$$

In treating region II ($0 \leqslant z < z^*$, the boundary layer), viscous stresses cannot be ignored in the equations of motion. However, in virtue of (4b) (see also (8d)), and anticipating the final expressions for the boundary layer flow (see equations (26) and (31)), in computing these stresses we can neglect terms like $\partial^2 u/\partial x^2$, $\partial^2 u/\partial y^2$, $\partial^2 v/\partial x^2$ and $\partial^2 v/\partial y^2$ as small compared with $\partial^2 u/\partial z^2$ and $\partial^2 v/\partial z^2$, and $\nu\varrho_0\,\partial^2 w/\partial z^2$ as small compared with $g\varrho$, so that the horizontal momentum equations are:

$$\left.\begin{aligned} -2\Omega v &= -\frac{1}{\varrho_0}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}, \\ 2\Omega u &= -\frac{1}{\varrho_0}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2} \quad (0 \leqslant z < z^*) \end{aligned}\right\} \tag{10a, b}$$

(cf. equations (6)) and the vertical momentum equation is:

$$\frac{\partial p}{\partial z} = g\varrho \quad (0 \leqslant z < z^*) \tag{11}$$

(cf. equation (7)). *A posteriori* reasoning shows that the neglect of the inertial term, $(\mathbf{u}\cdot\nabla)\mathbf{u}$, as small compared with the viscous term, $\nu\nabla^2\mathbf{u}$, is valid when the Rossby number R is much less than unity, and is thus consistent with the assumption of geostrophic flow in region I (see (8c)).

The conditions under which the kinematical viscosity, ν, can be regarded as constant are not unrelated to those determining the applicability of the so-called Boussinesq approximation (see (8a)), which we have already invoked in treating the fluid as one of uniform density, ϱ_0, but of variable weight per unit volume, $g(\varrho_0 + \varrho)$ (see SPIEGEL & VERONIS, 1960).

Turning now to the boundary conditions under which (5), (10) and (11) must be solved, consider first the conditions on the surface Σ. First we have the kinematical condition that since (by hypothesis) this surface is plane, the normal component of $\mathbf{u}$ must vanish on Σ. In addition, because Σ is free from externally applied tangential stresses, to ensure continuity of stress $\nu\varrho_0 \partial u/\partial z$ and $\nu\varrho_0 \partial v/\partial z$ must also vanish on Σ. Hence, we must require that

$$w = 0, \quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0, \quad \text{on} \quad z = 0. \qquad (12\text{ a, b})$$

Further conditions arise from the requirement that the boundary layer solutions, valid in $0 \leqslant z < z^*$, should join the interior solutions smoothly at $z = z^*$. Hence, if

$$U^* \equiv -\frac{1}{2\Omega\varrho_0}\left(\frac{\partial p}{\partial y}\right)_{z=z^*}, \quad V^* \equiv \frac{1}{2\Omega\varrho_0}\left(\frac{\partial p}{\partial x}\right)_{z=z^*} \qquad (13\text{ a, b})$$

(see equations (6)), and

$$X^* \equiv -\frac{g}{2\Omega\varrho_0}\left(\frac{\partial \varrho}{\partial y}\right)_{z=z^*}, \quad Y^* \equiv \frac{g}{2\Omega\varrho_0}\left(\frac{\partial \varrho}{\partial x}\right)_{z=z^*} \qquad (14\text{ a, b})$$

(see equations (9)), solutions to (5), (10) and (11) must also satisfy:

$$(u, v, w) = (U^*, V^*, W^*) \quad \text{and} \quad \left(\frac{\partial u}{\partial z}, \frac{\partial v}{\partial z}, \frac{\partial w}{\partial z}\right)$$
$$= (X^*, Y^*, 0) \quad \text{on } z = z^*, \qquad (15\text{ a, b, c, d, e, f})$$

where W^* is the value of w throughout the interior region, $z \geqslant z^*$ (since $\partial w/\partial z = 0$ there, see (9c)).

The dynamic pressure p and density variation ϱ must, of course, be continuous across $z = z^*$. If we assume that the density and pressure fields of the interior region penetrate into the boundary layer without suffering serious modification as a result of the viscous forces present there, we can simplify equations (10) immediately. Because the boundary layer is geometrically thin we can write

$$\varrho(x, y, z) = \varrho(x, y, z^*) + (z - z^*)\left(\frac{\partial \varrho}{\partial z}\right)_{z=z^*} \quad (0 \leqslant z < z^*), \qquad (16\text{ a})$$

$$p(x, y, z) = p(x, y, z^*) + (z - z^*)\left(\frac{\partial p}{\partial z}\right)_{z=z^*}$$
$$= p(x, y, z^*) + g(z - z^*)\varrho(x, y, z^*) \quad (0 \leqslant z < z^*) \qquad (16\text{ b})$$

(see equation (11)). Now eliminate p between (16b) and (10), making use of (13), (14) and (16a), and thus find:

$$-2\Omega[v - V^* - (z - z^*)\,Y^*] = \nu\frac{\partial^2 u}{\partial z^2} \quad (0 \leqslant z < z^*), \qquad (17\text{ a})$$

$$2\Omega[u - U^* - (z - z^*)\,X^*] = \nu\frac{\partial^2 v}{\partial z^2} \quad (0 \leqslant z < z^*). \qquad (17\text{ b})$$

To find the structure of the boundary layer, equations (5) and (17) must be solved for u, v and w under the boundary conditions given by equations (12) and (15).

3. The boundary layer at a free surface

At this stage it is convenient to introduce the following quantities:—

$$U \equiv -\frac{1}{2\Omega\varrho_0}\left(\frac{\partial p}{\partial y}\right)_{z=0}, \quad V \equiv \frac{1}{2\Omega\varrho_0}\left(\frac{\partial p}{\partial x}\right)_{z=0}, \qquad (18\text{ a, b})$$

$$X \equiv -\frac{g}{2\Omega\varrho_0}\left(\frac{\partial \varrho}{\partial y}\right)_{z=0}, \quad Y \equiv \frac{g}{2\Omega\varrho_0}\left(\frac{\partial \varrho}{\partial x}\right)_{z=0} \qquad (19\text{ a, b})$$

(see equations (6), (7), (8), (9) and (11)),

$$\hat{U} \equiv U + zX, \quad \hat{V} \equiv V + zY,$$

$$W \equiv w(z = z^*) = w(z > z^*) \qquad (20\text{ a, b, c})$$

(see (9 c)), and

$$\sigma \equiv z/\delta, \qquad (21)$$

where $\delta \equiv (\nu/|\mathbf{\Omega}|)^{\frac{1}{2}}$ (see (2 b)).

Now we are in a position to replace z^*, U^*, V^*, W^*, X^* and Y^* (see equations (4), (13) and (14)) as they appear in equations (15), (16) and (17) by more precisely defined quantities. In (17), replace $u^* + (z - z^*) X^*$ and $v^* + (z - z^*) Y^*$ by $\hat{U}$ and $\hat{V}$ respectively (see equations (20)) and eliminate v between the resulting equations. Thus find the fourth order differential equation in u:

$$\frac{\partial^4 u}{\partial \sigma^4} + 4(u - \hat{U}) = 0, \qquad (22)$$

the dimensionless stretched coordinate σ being used in place of z in the differentiated term. By equations (17 a) and (5), v and w are related to u as follows:

$$v \equiv \hat{V} - \frac{\operatorname{sgn}(\Omega)}{2} \frac{\partial^2 u}{\partial \sigma^2}, \quad \delta^{-1} \frac{\partial w}{\partial \sigma} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \qquad (23\text{ a, b})$$

where $\operatorname{sgn}(\Omega) = \Omega/|\mathbf{\Omega}|$.

These equations for the boundary layer flow apply over the range $0 \leqslant z < z^*$. The value of σ corresponding to $z = z^*$ is z^*/δ, which by (4 a) is very much greater than unity. Hence, without fear of serious error, equations (22) and (23) can be taken to apply for all positive values of σ, i.e., $0 \leqslant \sigma < \infty$, even though we have treated z^* as equal to zero in evaluating the interior flow at the edge of the boundary layer (see (4 b)).

The boundary conditions under which equations (22) and (23) must be solved are as follows:

$$w = 0, \quad \frac{\partial u}{\partial \sigma} = \frac{\partial v}{\partial \sigma} = 0, \quad \text{on} \quad \sigma = 0, \qquad (24\text{ a, b})$$

$$u = \hat{U}, \quad v = V, \quad w = W, \quad \text{on} \quad \sigma = \infty, \qquad (25\text{ a})$$

$$\delta^{-1} \frac{\partial u}{\partial \sigma} = X, \quad \delta^{-1} \frac{\partial v}{\partial \sigma} = Y, \quad \frac{\partial w}{\partial \sigma} = 0, \quad \text{on} \quad \sigma = \infty \qquad (25\text{ b})$$

(see equations (12), (13), (14), (15), (18), (19) and (20)). Solutions which satisfy these conditions are readily shown to be:

$$u = \hat{U} + (\tfrac{1}{2}\delta) e^{-\sigma} \{(X + \operatorname{sgn}(\Omega) Y) \cos \sigma - (X - \operatorname{sgn}(\Omega) Y) \sin \sigma\}, \qquad (26\text{ a})$$

$$v = \hat{V} + (\tfrac{1}{2}\delta) e^{-\sigma} \{(Y - \operatorname{sgn}(\Omega) X) \cos \sigma - (Y + \operatorname{sgn}(\Omega) X) \sin \sigma\}, \qquad (26\text{ b})$$

$$w = W(1 - e^{-\sigma} \cos \sigma) \qquad (26\text{ c})$$

(see equations (18) to (21)), where

$$W = -(\tfrac{1}{2}\delta^2 \operatorname{sgn}(\Omega)) \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) \qquad (27)$$

(see (20 c) and (2 b)).

The corresponding expressions for the two horizontal components of the viscous stress are:

$$\varrho_0 \nu \frac{\partial u}{\partial z} = \varrho_0 (\nu |\mathbf{\Omega}|)^{\frac{1}{2}} \frac{\partial u}{\partial \sigma} = \varrho_0 \nu \{X(1 - e^{-\sigma} \cos \sigma) - \operatorname{sgn}(\Omega) Y e^{-\sigma} \sin \sigma\}, \qquad (28\text{ a})$$

$$\varrho_0 \nu \frac{\partial v}{\partial z} = \varrho_0 (\nu |\mathbf{\Omega}|)^{\frac{1}{2}} \frac{\partial v}{\partial \sigma} = \varrho_0 \nu \{Y(1 - e^{-\sigma} \cos \sigma) + \operatorname{sgn}(\Omega) X e^{-\sigma} \sin \sigma\}, \qquad (28\text{ b})$$

and if the (horizontal) ageostrophic flux vector has x and y components

$$\left.\begin{aligned} F_x &\equiv \int_0^{z^*} (u - \hat{U})\, dz = \left(\frac{\nu}{|\mathbf{\Omega}|}\right)^{\frac{1}{2}} \int_0^{\infty} (u - \hat{U})\, d\sigma, \\ F_y &\equiv \int_0^{z^*} (v - \hat{V})\, dz = \left(\frac{\nu}{|\mathbf{\Omega}|}\right)^{\frac{1}{2}} \int_0^{\infty} (v - \hat{V})\, d\sigma \end{aligned}\right\} \qquad (29)$$

(so that $W = -(\partial F_x/\partial x + \partial F_y/\partial y)$, see (20 c)), by (26) and (23 b)

$$F_x = (\tfrac{1}{2}\delta^2 \operatorname{sgn}(\Omega)) Y = \nu Y/2\Omega, \qquad (30\text{ a})$$

$$F_y = -(\tfrac{1}{2}\delta^2 \operatorname{sgn}(\Omega)) X = -\nu X/2\Omega. \qquad (30\text{ b})$$

4. Discussion

Equations (26) to (30) constitute a complete description of the boundary layer at a free surface. The physical interpretation of these equations will be facilitated by comparing them with the corresponding equations for the viscous boundary layer arising near $z=0$ when a rigid surface, fixed in S, is in contact with the fluid at $z=0$ (see PRANDTL, 1952). These equations can be found by solving equations (22) and (23) under the same boundary conditions on $\sigma=\infty$ as in the free surface case (see equations (25)), the same kinematical boundary condition on $\sigma=0$ (see (24a)), but with the no-slip condition, $u=v=0$ on $\sigma=0$ in place of the vanishing stress condition (see (24b)); thus:

$$u=\hat{U}-e^{-\sigma}(U\cos\sigma+\operatorname{sgn}(\Omega)V\sin\sigma),\quad \text{(rigid surface)}, \tag{31 a}$$

$$v=\hat{V}-e^{-\sigma}(V\cos\sigma-\operatorname{sgn}(\Omega)U\sin\sigma)\quad \text{(rigid surface)}, \tag{31 b}$$

$$w=W(1-e^{-\sigma}(\cos\sigma+\sin\sigma))\quad \text{(rigid surface)} \tag{31 c}$$

(cf. equations (26)), where

$$W=(\tfrac{1}{2}\delta\operatorname{sgn}(\Omega))\left(\frac{\partial V}{\partial x}-\frac{\partial U}{\partial y}\right)\quad \text{(rigid surface)} \tag{32}$$

(cf. equation (27)), and

$$\varrho_0\nu\frac{\partial u}{\partial z}=\varrho_0(\nu|\boldsymbol{\Omega}|)^{\frac{1}{2}}\frac{\partial u}{\partial\sigma}=\varrho_0\nu\{X+\delta^{-1}e^{-\sigma}\times[(U-\operatorname{sgn}(\Omega)V)\cos\sigma+(U+\operatorname{sgn}(\Omega)V)\sin\sigma]\}\quad \text{(rigid surface)}, \tag{33 a}$$

$$\varrho_0\nu\frac{\partial v}{\partial z}=\varrho_0(\nu|\boldsymbol{\Omega}|)^{\frac{1}{2}}\frac{\partial v}{\partial\sigma}=\varrho_0\nu\times\{Y+\delta^{-1}e^{-\sigma}[(U+\operatorname{sgn}(\Omega)V)\cos\sigma-(U-\operatorname{sgn}(\Omega)V)\sin\sigma]\}\quad \text{(rigid surface)} \tag{33 b}$$

(cf. equations (28)), and

$$F_x=-(\delta/2)[U+\operatorname{sgn}(\Omega)V],$$

$$F_y=-(\delta/2)[V-\operatorname{sgn}(\Omega)U],\quad \text{(rigid surface)} \tag{34}$$

(cf. equations (30)).

Equations (26) and (31) confirm one conclusion of the general arguments presented in Section 1 above, namely, that although in the rigid surface case a viscous boundary layer arises irrespective of whether the fluid is barotropic or baroclinic (i.e., $\nabla\varrho=0$ or $\nabla\varrho\neq0$), only when the fluid is baroclinic ($\nabla\varrho\neq0$) does a viscous boundary layer arise at a free surface.

The ageostrophic flow in a free-surface boundary layer is much weaker, by a factor of $E^{\frac{1}{2}}$, than the corresponding flow in a rigid-surface case (see equations (2), (26) and (31)). The length associated with the decaying exponential and the harmonic components of the variation with respect to z of the ageostrophic flow is equal to $\delta(=(\nu/|\boldsymbol{\Omega}|)^{\frac{1}{2}}$, see (2b)) irrespective of whether Σ is rigid or free. The form of this variation is readily visualized by considering the case when the geostrophic flow is in the x direction (a quite general case since the orientation of the x and y axes is arbitrary), so that Y, V and $\hat{V}$ can be set equal to zero in equations (26) and (31).

It is noteworthy that the dependence on σ of the horizontal stress, components $\varrho_0\nu\,\partial u/\partial z$ and $\varrho_0\nu\,\partial v/\partial z$, in the free surface case is identical with that of the horizontal velocity, components u and v, in the rigid surface case. This is not surprising, having regard for the *raisons d'être* of these viscous boundary layers. The viscous layer is required at a free surface to reduce the horizontal stress from its value ($\varrho_0\nu X$, $\varrho_0\nu Y$) corresponding to the thermal wind just inside the geostrophic interior, to zero at $z=0$. (Since $X=Y=0$ when $\nabla\varrho=0$ (see (19)), there is no need for a boundary layer at the free surface of a barotropic fluid.) The layer at a *rigid* surface is required to reduce the horizontal velocity from its geostrophic value (U, V) just inside the geostrophic interior, to zero at $z=0$.

According to equation (27), an exchange of fluid occurs between the geostrophic interior region and the viscous boundary layer, the local speed of flow out of the interior region, $-W$, resulting from this so-called "boundary layer suction" being proportional to the product of E (see equation (2)) and the derivative with respect to z of the z component of the vorticity of the geostrophic interior flow evaluated at $z = 0$. There is a simple relationship between W and ϱ, obtained by combining the thermal wind relationship (equation (19)) with equation (27), namely

$$-W = \frac{g\nu}{4\Omega^2\varrho_0}\left(\frac{\partial^2\varrho}{\partial x^2} + \frac{\partial^2\varrho}{\partial y^2}\right)_{z=0}. \qquad (35)$$

Boundary layer suction is well known in the rigid surface case (see CHARNEY & ELIASSEN, 1949; ROBINSON, 1960; HOLOPAINEN, 1961; GREENSPAN & HOWARD, 1963; BARCILON, 1964); according to equation (32) $-W$ is then proportional to the product of $E^{\frac{1}{2}}$ and the z component of the vorticity of the geostrophic interior flow evaluated at $z = 0$.

5. Critique and conclusion

The results of Section 3 above are relationships between local quantities (see (26)–(30), also (31)–(35)) and one might question their usefulness in the treatment of complete systems. After all, in the equations of hydrodynamics we already possess quite basic relations between local quantities. Unfortunately, these basic equations, which are essentially non-linear and generally intractable, do not lead directly to the kind of physical insight and intuition which are indispensable when dealing with complete systems, even including highly idealized theoretical models.

In the study of the hydrodynamics of rotating fluids the thermal wind equation (equation (1)), which reduces to the celebrated Proudman-Taylor theorem ($(\mathbf{\Omega}\cdot\nabla)\mathbf{u} = 0$) when the fluid is barotropic, occupies a central position and can often lead directly to useful results (see TAYLOR, 1923; HIDE, 1960, 1962). Insight into the effects of boundaries will come from a thorough understanding of boundary layer flow and it is in this context that the present paper is offered as a serious contribution to the subject.

The present work stems from two unpublished theoretical studies (HIDE, 1963) of the axisymmetric regime of thermal convection in a rotating vertical annulus of liquid (HIDE, 1953). In the first of these—of the thermal structure of the convecting system—it turns out that even when the upper surface of the liquid is free, advection of heat in the geostrophic interior region by secondary motions there resulting from boundary layer suction can dominate thermal conduction. In the other theoretical study, in which the stability of the axisymmetric regime to non-axisymmetric disturbances is investigated, boundary layer suction can also play a significant, and in some circumstances dominant, rôle. The application of inviscid boundary conditions in the treatment of either problem can lead to quite erroneous results and it was this discovery in the first of these problems that led to the present investigation.

In the discussion of the thermal structure problem, it was found that unless $PR < 1$, where

$$P \equiv \nu/\varkappa \qquad (36)$$

is a Prandtl number ($\varkappa$ being the thermometric conductivity of the fluid), vertical heat advection induced by boundary layer suction at a free surface cannot be ignored in discussing the thermal balance of the interior region. (The corresponding result for a rigid surface is $RP < E^{\frac{1}{2}}$.)

In the second problem—that of the stability of the axisymmetrical regime—the accurate treatment of the field of w, the vertical motion, is of critical importance. In the absence of all viscous effects and when the x and y components of $\mathbf{u}$ are quasi-geostrophic, $|w| \sim QR$ (see (8c)). It follows from equation (27) that unless $E < R$ (see equations (8c, d)), equivalent to $QL/\nu \gg 1$, it is not legitimate to apply inviscid boundary conditions at a free surface, even though viscosity can safely be ignored in the equations governing the interior flow. (The corresponding result for a rigid surface is $E^{\frac{1}{2}} < R$, equivalent to $QL/\nu \gg (|\mathbf{\Omega}|L^2/\nu)^{\frac{1}{2}}$.)

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