# A note on the adjustment towards geostrophic equilibrium in a simple fluid system[1]

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(Manuscript received July 20; revised version September 8, 1964)

ABSTRACT

A simple fluid system is used to study numerically how the atmosphere reaches a geostrophic equilibrium from a non-geostrophic initial state. Most of the previous studies on atmospheric adjustment have emphasized the initial and final states. Furthermore, they have usually considered motions smaller or larger than the *radius of deformation*. This study explores the adjustment process for scales both greater and smaller than the *radius of deformation*. The role of gravity-inertial waves in the adjustment process is clearly shown by the numerical results and by an accompanying theoretical discussion.

Finally, some speculation of the implications of atmospheric adjustment towards the problem of providing initial data (termed initialization) for primitve equation numerical weather prediction models is discussed.

In an effort to better understand the manner in which the atmosphere returns to geostrophic balance from a nongeostrophic initial state, some simple numerical experiments were performed. The model equations are essentially the same as those considered by CAHN (1945) and by OBUKHOV (1949). The Cahn and Obukhov examples, however, primarily explained the initial and final states of the adjustment process, whereas the examples given here were obtained by direct time integration of the equations. Therefore, not only the initial and final states are illustrated, but also the intermediate ones.

The model equations are those for a linearized perturbation in a barotropic atmosphere. The advection terms are assumed to be small so that they are neglected along with all variations of the field variables in the $x$-direction. In addition, the upper surface of this atmosphere, which may be considered as a homogeneous incompressible fluid, is assumed to be free. Movement of the free upper surface produces a small amount of divergence and convergence in the fluid.

In such a simple model the geostrophic balance or stationary state (which will mean steady-state) is characterized by

$$\psi_s = \frac{g}{f} h_s; \quad \chi_s = 0, \tag{1}$$

where $\psi_s$ is the stationary (or steady-state) stream function, $g$ is the acceleration of gravity, $f$ is the Coriolis parameter, $h_s$ is the stationary perturbation height of the fluid and $\chi_s$ is the stationary velocity potential. PHILLIPS (1960) has shown that the stationary velocity potential, or its counterpart, the stationary divergence, may be non-zero if advection terms are included in the equations of motion for the model.

Following OBUKHOV (1949), the divergence equation for such an atmosphere can be simplified to

$$\frac{\partial \chi}{\partial t} = f\psi - gh. \tag{2}$$

The necessary and sufficient condition for placing the height and stream function fields of

[1] This work was carried out while the author was at The Pennsylvania State University, Department of Meteorology, and supported by the Air Force Cambridge Laboratories, Office of Aerospace Research, under contract No. AF 19(604)–7261.

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the model atmosphere into a stationary state is the elimination of the time derivative term in equation (2). In a non-linear system this elimination would correspond to neglecting the total derivative of the divergence in the divergence equation; the result would be a diagnostic relationship between stream function and height fields. To obtain a diagnostic relationship for the quasigeostrophic velocity potential, a diagnostic relationship such as the first equation of (1) must be assumed *ab initio*.

If the atmosphere is not in geostrophic balance gravity-inertial waves will be generated. These are usually unwanted in the initial conditions of numerical weather prediction models because they are not observable. Moreover, the finite difference equations would require extremely refined grid sizes in comparison with those in present use to faithfully reproduce gravity-inertial motions. Because of the difficulty of observing gravity-inertial waves in widely spaced synoptic data, they have been given the label "*meteorological noise*".

The equation governing the process of atmospheric adjustment from a non-geostrophic initial state to an adjusted or geostrophic state turns out to have a form similar to the wave equation except for an additional linear term.

$$\frac{\partial^2 \chi}{\partial t^2} = gH\frac{\partial^2 \chi}{\partial y^2} - f^2\chi. \tag{3}$$

where $H$ is the scale height of our atmosphere (approximately 8 km).

This linear term makes the equation scale-dependent. For small-scale departures from geostrophic balance, the adjustment process is achieved by rapidly moving, nearly pure, gravity waves. For large-scale disturbances the adjustment process is accomplished by nearly pure inertial waves. The gravity waves spread out the mass and energy very rapidly in comparison to the effect of inertial waves. In the limit the group velocity of the gravity waves is equal to their phase velocity (approximately 280 m/sec for external gravity waves). The group velocity of the inertial waves is zero.

The length scale that separates the adjustment process for gravity waves from that of inertial waves has been labeled the *radius of deformation* by ROSSBY (1938) and the *critical length* by OBUKHOV (1949). Here, for convenience, a *critical wavelength* (approximately 15,000 km) is defined as

$$\lambda_c \equiv \frac{2\pi\sqrt{gH}}{f}. \tag{4}$$

The time-invariant potential vorticity, which related the initial state to the final balanced state, has some important consequences. Following Obukhov, the time change of potential vorticity takes the form

$$\frac{\partial \Omega}{\partial t} = \frac{\partial}{\partial t}\left[\frac{\partial^2 \psi}{\partial y^2} - \frac{fh}{H}\right] = 0, \tag{5}$$

where $\Omega$ represents the potential vorticity. If the quasigeostrophic assumption (1) is made, then the potential vorticity for the stationary state becomes

$$\Omega_s = \frac{\partial^2 \psi_s}{\partial y^2} - \frac{f^2}{gH}\psi_s. \tag{6}$$

Furthermore, since the potential vorticity is invariant, then the initial and final states can be related by the simple equality

$$\frac{\partial^2 \psi_s}{\partial y^2} - \frac{f^2}{gH}\psi_s = \frac{\partial^2 \psi_i}{\partial y^2} - \frac{fh_i}{H}. \tag{7}$$

Equation (7) has special importance as will now be shown. Let us assume the following variables are harmonic

$$\left.\begin{aligned} \psi_i &= \psi_i' e^{i(2\pi y/\lambda)}, \\ h_i &= h_i' e^{i(2\pi y/\lambda)}, \\ \psi_s &= \psi_s' e^{i(2\pi y/\lambda)}, \end{aligned}\right\} \tag{8}$$

and further assume a critical wavelength (4). Now substitute (8) and (4) into (7) to obtain

$$\psi_s' = \frac{(4\pi^2/\lambda^2)\psi_i' - fh_i'/H}{4\pi^2/\lambda^2 - 4\pi^2/\lambda_c^2}. \tag{9}$$

For wavelengths shorter than $\lambda_c$, (9) shows that

$$\psi_s' \simeq \psi_i' \quad \text{or} \quad \frac{g}{f}h_s' \simeq \psi_i'. \tag{10}$$

Therefore, the final or adjusted stream function is given essentially by the initial stream function and, since the adjusted field is geostrophic, the final heights are proportional to the initial stream function. For the opposite case, where

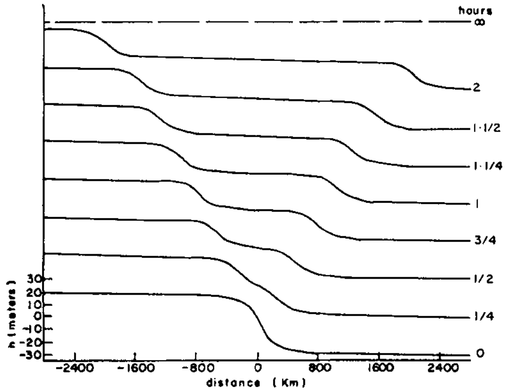


FIG. 1. Example of a small-scale adjustment process. The change of the height field with time is given for the case of zero initial wind field.

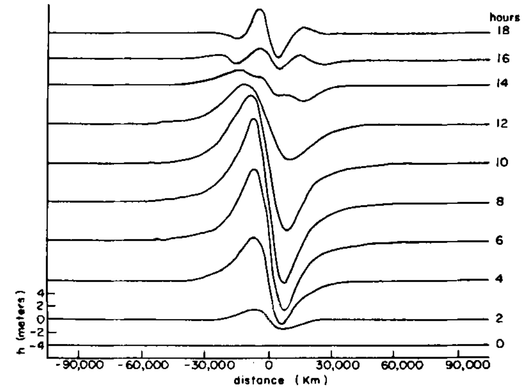


Fig. 2

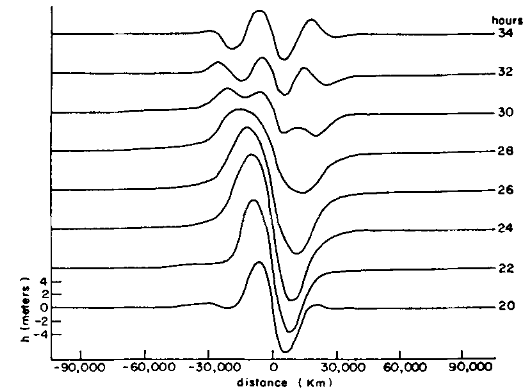


Fig. 3

FIGS 2 and 3. Example of a large-scale adjustment process. The change of the height field with time is given for the case of a jet in the initial wind field.

the wavelength of the disturbance is greater than $\lambda_c$, the final or adjusted stream function is given by

$$\psi_s' \simeq \frac{\lambda_c^2 f h_i'}{H} = \frac{g}{f} h_i'. \tag{11}$$

The equation (11) shows that, when the scale of the disturbance is smaller than some *critical wavelength*, the initial stream-function field gives a good approximation to the adjusted or stationary height field. On the other hand, if the scale of the disturbance is larger than the *critical wavelength*, the initial height field gives a good approximation to the adjusted stream function. The implication for initialization of primitive equation numerical weather preduction models is that the observed height field is a good source of information for the large-scale disturbances, and the wind stream function is a good source for small-scale disturbances from the viewpoint of an adjusted state.

The solution procedure for the adjustment problem has been outlined in detail by OBUKHOV (1949). A straightforward application of a centered-finite difference approximation to the wave equation (3) is used as described by RICHTMYER (1957) on a grid of 700 points. The differential equations for $h$ and $\psi$ were similarly expressed in terms of finite differences. The initial conditions for equation (3) are equation (2) and $\chi_i = f(y)$.

Fig. 1 shows an example of a jet in a flow. It is similar to the example chosen by CAHN (1945). The jet is implied in the height field by the steep gradient which is given by

$$h_i = -c\left[\frac{1}{2} + \frac{1}{\pi} \quad \text{arc tan} \quad (a(y-y_0))\right], \tag{12}$$

where $C = 50$ m, $a = 10^{-5}$ m$^{-1}$, $y_0$ = center point. The corresponding initial wind field was zero. A phase velocity equal to the speed of an external gravity wave was also used to determine the proper time step. We see that the steep gradient in the height field disappears with time. The adjustment is accomplished by the transfer of mass from one side of the fluid to the other side. The adjustment for this small-scale case is also very rapid. The opposite situation which correspond to Cahn's experiment was also run, where the jet was in the wind field and there was no initial steep gradient in the height field. Again the gravity wave *pulses* moved out from the region of geostrophic imbalance to bring about geostrophic balance.

The above case of atmospheric adjustment is an example of a small-scale situation. A large-scale case is shown in the Figs. 2–3. The frequency of inertial waves is much smaller than that of gravity waves. The adjustment process is much slower because nearly pure inertial waves have zero group-velocity. The initial situation shown in these two figures is that of a jet in the initial wind field, where

$$\psi_i = -\frac{gc}{f}\left[\frac{1}{2} + \frac{1}{\pi}\arctan \quad (a(y-y_0))\right], \quad h_i = 0 \quad (13)$$

and $\chi_i = 0$. The constants for this case are given as $C = 500$ m, $a = 10^{-7}$ m$^{-1}$, $f = 10^{-4}$ sec$^{-1}$.

The jet in Figs. 2–3 turns with an inertial period of approximately 17 hours. The height field never seems to quite catch up to the moving momentum field. We note that, even for this large-scale disturbance, the waves are not purely inertial but seem to contain small components of nondispersive gravity waves, thus the energy can gradually spread. The rotation of the momentum field would probably continue until the adjusted height field approached the initial height field. We see at 34 hours that the height field still shows large contrasts so that it will take some time for the heights to approach the zero limit. We should likewise note that there seems to be a near perfect interchange of potential and kinetic energy between the height field and the wind field in this large-scale example.

Because our atmosphere adjusts only with external gravity and inertial waves, the *critical wavelength* is quite large. For a more realistic atmosphere in which density variations (especially in the vertical) are considered (Bolin, 1953), the gravity waves can be of the slower-moving, larger-amplitude, internal type, and the *critical wavelength*, which is directly proportional to the gravity wave speed, will be correspondingly smaller. This is important because Figs. 2–3 may give the false impression that the initial height field only approximates the adjusted height field for extremely large disturbances. Moreover, internal gravity waves probably dominate the external variety as the usual adjustment mechanism in the real atmosphere.

More sophisticated models of atmosphere adjustment have been proposed and discussed by Monin & Obukhov (1959), Monin (1958) and Matsumoto (1961) but without extensive numerical results of this type.

Customarily the balance equation of numerical weather prediction is solved from the observed height field, largely because measurements of height are more plentiful than measurements of the vorticity of the observed wind. This simple adjustment study implies that the usual method of solving the balance equation is probably satisfactory for large-scale but not as satisfactory for small-scale processes, because, for synoptic analyses, we are usually interested *a priori* in the adjusted state. Of course, the smaller-scale motions can only be analyzed in regions such as North America, with a dense network of data.

## Acknowledgements

The author wishes to express his thanks to Professors R. T. Duquet and H. A. Panofsky for their advice and criticisms of this study. In particular the author would like to thank Professor E. F. Danielson for the time spent with him in many stimulating discussions on this subject.

The author appreciates the financial support of this work by the Air Force Cambridge Laboratories, Office of Aerospace Research, under contract No. AF 19(604)-7261.

The computer facilities of the computation center of the Pennsylvania State University were used for this study.

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