SHORTER CONTRIBUTION

A note on the beta-plane approximation

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1. Introduction

The beta-plane approximation was first introduced into meteorological literature by ROSSBY *et al.* (1939), but only in recent years has a systematic attempt been made to justify it. The most notable discussion in this regard was by PHILLIPS (1963). In this note we emphasize some points that may not be clear from Phillips' presentation and extend his analysis somewhat.

In this discussion we are interested in geostrophic motion of the first type PHILLIPS (1963); that is, we exclude the ultra-long waves from consideration. Also, since we are interested in the beta-plane approximation, we will not derive the potential-vorticity equation but will only derive the vorticity equation with the beta-plane approximation included. The potential-vorticity equation can be readily obtained if recourse is made to the hydrostatic relation and the thermodynamic equation as discussed by Phillips and others.

2. Validity of the beta-plane approximation

In this section we discuss the physical conditions for which the beta-plane approximation is valid. The first conditions required are that the flow be quasigeostrophic and that the beta term in the vorticity equation is the same order of magnitude as the vorticity advection terms. It is also required that the time scale is of the order of L/C where L is a scale length and C is a scale velocity. Finally, since we use the logarithm of the pressure as the vertical coordinate, we require the hydrostatic approximation to be valid.

For the purpose of this paper it is considered that the first pair of conditions is of most interest. These conditions can be put into nondimensional form. If f_0 is the Coriolis parameter at latitude θ_0 and β_0 is the beta parameter at the same latitude, the statements that the flow is quasigeostrophic and that the beta term is the same order as the vorticity advection terms are, respectively:

$$\left.\begin{aligned} R_0 &= \frac{C}{f_0 L} < 1, \\ \beta &= \frac{\beta_0 L^2}{C} \sim 1, \end{aligned}\right\} \tag{1}$$

where R_0 and β are the nondimensional Rossby and beta numbers.

For the beta-plane approximation to be valid, the terms in the vorticity equation arising from the curvature of the earth other than the beta term must be negligible. A necessary condition for this result is that $L/a < 1$, where a is the radius of the earth. This condition can be obtained from the two conditions in (1) plus the definitions of f_0 and β_0 in terms of the latitude θ_0. We find

$$L/a = \beta R_0 \tan\theta_0 < 1, \tag{2}$$

which is valid for mid-latitudes where $\tan\theta_0 \sim 1$. Thus equation (2) follows as a natural consequence of (1) and does not have to be made as an additional assumption for mathematical convenience. This equation was derived by LIPPS (1963).

It is to be noted that in our formulation (1) and (2) are valid only in mid-latitudes. However, PHILLIPS (1963) also discusses an expansion which is valid near the pole.

3. The equations of motion

In this section we develop the equations of motion. In order that the present results may be compared directly with those of Phillips we use a Mercator coordinate system centered at latitude θ_0. In the present analysis θ_0 is an arbitrary middle latitude. For Phillips' case $\theta_0 = 45$ degrees. In this system the horizontal coordinates x and y are given by

$$\left.\begin{aligned} x &= a\lambda, \\ y &= a\ln\frac{\cos\theta_0(1+\sin\theta)}{\cos\theta(1+\sin\theta_0)}, \\ a\,d\theta &= \cos\theta\,dy, \\ a\cos\theta\,d\lambda &= \cos\theta\,dx, \end{aligned}\right\} \quad (3)$$

where λ is the longitude and θ the latitude. For the vertical coordinate we use the logarithm of the pressure

$$Z = -\ln p/p_0, \quad (4)$$

where p is the pressure and p_0 is the standard sea-level pressure. The corresponding velocities are given by u, v and $\dot{Z}$ where $\dot{Z} = dz/dt$. The geopotential is denoted by ϕ.

The nondimensional variables are defined by:

$$\left.\begin{aligned} &X = (x\cos\theta_0)/L, \quad Y = (y\cos\theta_0)/L, \\ &Z = -\ln p/p_0, \\ &\tau = tC/L, \quad U = u/C, \quad V = v/C, \\ &R_0 W = \dot{Z}L/C, \quad \Phi = \phi/f_0 CL. \end{aligned}\right\} \quad (5)$$

We now write down the momentum and continuity equations to order R_0 and neglect higher terms:

$$\left.\begin{aligned} &R_0\left(\frac{\partial U}{\partial \tau} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y}\right) \\ &\quad = -(1+R_0 bY)\frac{\partial \Phi}{\partial X} + V(1+R_0\beta Y), \\ &R_0\left(\frac{\partial V}{\partial \tau} + U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y}\right) \\ &\quad = -(1+R_0 bY)\frac{\partial \Phi}{\partial Y} - U(1+R_0\beta Y), \\ &R_0\left(\frac{\partial W}{\partial Z} - W\right) \\ &\quad = -(1+R_0 bY)\left(\frac{\partial V}{\partial X} + \frac{\partial V}{\partial Y}\right) + R_0 bV, \end{aligned}\right\} \quad (6)$$

where we have defined:

$$b \equiv R_0^{-1}L/a\tan\theta_0 = \beta\tan^2\theta_0 \approx 1. \quad (7)$$

The factors $(1 + R_0\beta Y)$ represent the nondimensional Coriolis parameter and its variation with latitude. The b terms arise due to the curvature of the spherical coordinates and cancel to leading order when the vorticity equation is derived. Many papers in which the basic equation is the vorticity equation start with the momentum and continuity equations as given in (6) but without the b terms. In these papers the vorticity equation is obtained in its correct form because the b terms cancel. However, the consistent form of the former equations is as given in (6). If spherical coordinates are used instead of Mercator coordinates, the form of the b terms changes in some cases but they again cancel when the vorticity equation is derived.

In order to obtain the vorticity equation we first consider equations (6) to leading order. We find:

$$\left.\begin{aligned} 0 &= -\frac{\partial \Phi}{\partial X} + V, \\ 0 &= -\frac{\partial \Phi}{\partial Y} - U, \\ 0 &= \frac{\partial V}{\partial X} + \frac{\partial V}{\partial Y}. \end{aligned}\right\} \quad (8)$$

Thus the momentum equations are consistent with the continuity equation and we see that Φ is the stream function for the flow.

Now, by taking the cross derivatives of the momentum equations in (6) and using the corresponding continuity equation we obtain for the vorticity equation to leading order:

$$\frac{\partial \zeta}{\partial \tau} + U\frac{\partial \zeta}{\partial X} + V\frac{\partial \zeta}{\partial Y} + \beta V = -W + \frac{\partial W}{\partial Z} \quad (9)$$

where ζ is the vorticity. If we use that Φ is the stream function for the horizontal flow we finally obtain:

$$\frac{\partial}{\partial \tau}\nabla^2\Phi + \frac{\partial(\Phi, \nabla^2\Phi + \beta Y)}{\partial(X, Y)} = -W + \frac{\partial W}{\partial Z}. \quad (10)$$

As mentioned previously, we could derive the potential-vorticity equation as discussed by Phillips and others. However, since the purpose of this paper is to discuss the beta-plane approximation we stop the development here.

This development of the vorticity equation is similar to that given by Phillips. However in his case he expanded about 45 degrees latitude. But we see from (6) and (7) that the b and β terms cancel at this latitude in both the momentum equations to the order of approximation implied in (6). As a result the bY term that appeared in his vorticity equation originated in the continuity equation. In the present analysis Mercator coordinates are expanded about an arbitrary middle latitude θ_0. It is evident in the present analysis that the beta term in the vorticity equation results from the variation of the Coriolis parameter with latitude. The other curvature terms identically cancel to leading order when this equation is derived. Only at $\theta_0 = 45$ degrees where $b = \beta$ can one argue that the beta term in (9) and (10) originates in the continuity equation. At any other latitude one must take the point of view that this term is a result of the variation of the Coriolis parameter with latitude. A similar result is obtained if one uses spherical instead of Mercator coordinates.

4. Summary

In conclusion we note the following main points: (*a*) The primary assumptions necessary for the validity of the beta-plane approximation are that the flow is quasigeostrophic and that the beta term in the vorticity equation is the same order of magnitude as the vorticity advection terms. Then $L/a \sim R_0 < 1$ follows as a consequence. (*b*) The momentum and continuity equations to order R_0 are given by (6). (*c*) In the derivation of the vorticity equation the b terms in (6) cancel to leading order. The sole curvature effect remaining in the vorticity equation results from the variation of the Coriolis parameter with latitude. That this term exists is due to the combined effects of the earth's curvature and rotation. This is the significance first given to β by Rossby 25 years ago.

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