Convective instability in a two-layer fluid heated uniformly from above

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(Manuscript received November 26, 1963)

ABSTRACT

A fluid in which the dynamic and thermodynamic coefficients (μ, α, $\varkappa$, c_p etc.) vary vertically may become unstable when it is heated uniformly from above, assuming that the basic stratification is gravitationally stable and that the thermal expansion coefficient is positive. This "anti-convection" is studied in the special case of a fluid composed by two semi-infinite layers, each of which has constant coefficients. In the theory the Boussinesq approximation is used for each layer separately, and the interface is assumed to be plane. These assumptions seem justified in the present problem.

Instability is favoured when the thermal expansion coefficient and the conductivity are large in one of the layers and small in the other layer. A necessary condition for instability is that the ratio $\mu_1 \alpha_1 c_{p_2}/\mu_2 \alpha_2 c_{p_1}$ is larger than 9 (or smaller than 1/9), where μ is the dynamic viscosity, α the thermal expansion coefficient, and c_p the specific heat per unit mass. When this condition is fulfilled instability will occur for sufficiently large values of the ratio $\varkappa_1/\varkappa_2$ (small values, respectively), where $\varkappa$ is the thermometric conductivity.

The convection cells are confined to a region next to the interface. Small cells have comparable width and height, while larger cells are flattened in proportion to the 1/6 power of the Rayleigh number. Because of counteracting non-linear effects the convective motion cannot grow very large, the vertical velocities can at most be of order $\varkappa/D$, where D is the scale of the cell. It is suggested that cells with Rayleigh numbers of order unity dominate the picture in finite amplitude situations.

The driving energy comes from a region next to the boundary in the fluid layer having the larger thermal expansion and conductivity. Any motion started in the other layer creates thermal perturbations that are conducted into this region inducing a reversed circulation. When the thermal expansion coefficient is large this reversed circulation becomes strong. Through the frictional coupling it can directly amplify the primary motion and an instability develops.

It should be possible to observe the instability in a laboratory experiment. A mercury-water system seems suitable for this purpose. The theoretical analysis shows that instability occurs when heating is applied at temperatures below 18°C.

Geophysical and astrophysical applications seem possible. As example, the instability may appear at the interface between the atmosphere and the sea. An air-water system is stable under laboratory conditions, but in nature the existing eddy friction and conduction change the conditions considerably, and such as to favour the occurrence of the instability.

1. Introduction

Convection in a fluid heated uniformly from below, the so-called Bénard convection, has been investigated quite thoroughly in the past (see, as example, CHANDRASEKHAR (1961)). As is well known the convection can occur only at sufficiently large values of the Rayleigh number $\mathrm{Ra} = (g\alpha\beta d^4)/(\varkappa\nu)$, where g is the acceleration of gravity, β the adverse temperature gradient, d the depth of the fluid layer, and $\varkappa$,

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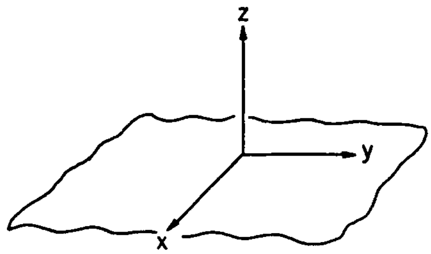

FIG. 1. Coordinates.

$\varkappa$, α, and ν are the coefficients of thermometric conductivity, thermal expansion and kinematic viscosity, respectively. The critical Rayleigh number can be determined by a direct stability analysis, it can also be obtained from an equation expressing the balance between the release of potential energy and the viscous dissipation. The expression for the potential energy release is positive-definite and negative-definite for positive and negative Rayleigh numbers, respectively. In the second case, that corresponds to heating from above, stability is thus proved.

One may now ask whether a similar result holds for a more general fluid, in which the coefficients such as $\varkappa$, α, and ν vary in the vertical direction. At present a study is made of the simplest kind of such fluids, a two-layer fluid in which each layer has uniform coefficients. The fluid is assumed to be unbounded. It will be demonstrated theoretically in Sections II–III that such a fluid, subjected to a uniform heating from above, may become unstable for a certain range of parameter values. It should be noted that the present theory assumes a gravitationally stable basic stratification and a positive coefficient of thermal expansion. (When this coefficient is negative, such as is the case for water in the range 0–4°C, heating from above may, of course, create a convection. This problem is not essentially different from the ordinary Bénard case.)

Although stability has never been proved for fluids with vertically non-uniform coefficients subjected to a heating from above, it has often been argued that instability could be ruled out by general physical arguments. In special, reference has been given to "the lack of a potential energy source". When it now is claimed that instability occurs, a discussion of the physics of the instability process is thus called for, to clear up points such as the above one. This has been attempted in Section IV.

The predicted "anti-convection" has not yet been verified experimentally, but preparations for a laboratory experiment using a mercury-water system are going on. Some remarks on the experimental problem are given in Section V. The possible application to some geophysical and astrophysical situations is also mentioned.

2. Stability analysis: derivation of the characteristic equation in the marginal case

A. Definitions and basic approximations

Let x, y, z be rectangular coordinates with the positive z-direction upward (Fig. 1). The undisturbed interface is at $z = 0$. In the lower layer the fluid is characterized by the constant coefficients ϱ_0, $\varkappa$, α, and ν, in the upper layer by the constant coefficients ϱ_0', $\varkappa'$, α', and ν'. There is applied a uniform heat flux from $z = \infty$; the temperature gradients in the two layers corresponding to this heat flux are denoted by β and β', respectively. At the interface continuity is required in velocity, normal and tangential stresses, and normal heat flux.

Two basic approximations are introduced:

(i) the Boussinesq approximation is applied within each layer, and

(ii) the interface is required to be horizontal.

The first approximation requires that the convection cells have a diameter that is small compared to the basic scale-height of the fluid (SPIEGEL & VERONIS (1960)). This is fulfilled under normal laboratory conditions but may not be valid in geophysical and astrophysical situations.

The second approximation requires some discussion. Obviously it cannot be applied when the layers are gravitationally unstable ($\varrho_0' > \varrho_0$), but we will exclude such cases anyway. In the case where the layers are stably stratified ($\varrho_0' < \varrho_0$) there will still be deformations of the interface, to balance the pressure effects of the motion. The corresponding geometric distortion of the velocity field is not critical (it is a second order effect), but one has to reckon with the energy required for the deformation. If this energy is comparable to, say, the kinetic energy of the cell motion the stability problem may certainly be affected by the approximation. We consider here only a special case to get support for the approximation. Assume that the densities of the two layers as well as their difference can be described by the amplitude ϱ_0, that all velocities can be described by an amplitude U, and that all scales can be described by a wave-length D. The pressure variations Δp along the interface will be of order $\varrho_0 \nu U/D$, from equating the orders of the pressure term and friction term in the equations of motion. These pressure variations call for compensating elevations Δh of the interface, that are of order $\Delta p/g\varrho_0$. The energy needed to build up these elevations are of order $\Delta p\,\Delta h \sim (\Delta p)^2/g\varrho_0 \sim \varrho_0 \nu^2 U^2/gD^2$, counted per unit horizontal area. On the other hand, the kinetic energy of the cell motion is of order $\varrho_0 D U^2$, per unit horizontal area. The ratio of these two energies is ν^2/gD^3. This is normally a very small number. As example, for 1 cm cells in water ν^2/gD^3 is of order 10^{-7}. The picture is not found to change essentially if one makes a more precise estimate, using the results that come out of the later theory. The interface deformations could become appreciable for extremely small cells, but these are anyway stable, as shown in Section III. Large deformations could also occur if the two layers have almost the same density, and this case has to be treated with more care. In the practical applications that are considered the approximation seems to be well justified.

The existence of surface tension is not critical in the present problem, as far as can be judged. In fact, its main effect is only to keep the interface horizontal. This is in contrast with the gravitationally unstable case ($\varrho_0' > \varrho_0$), where surface tension decidedly influences the stability criterion.

B. Basic equations

Under the Boussinesq approximation the equations describing small deviations from the basic state in velocity u, v, w, pressure p, and temperature T are

$$\frac{\partial u}{\partial t} = -\frac{1}{\varrho_0}\frac{\partial p}{\partial x} + \nu\nabla^2 u \tag{1}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\varrho_0}\frac{\partial p}{\partial y} + \nu\nabla^2 v \tag{2}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\varrho_0}\frac{\partial p}{\partial z} + g\alpha T + \nu\nabla^2 w \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4}$$

$$\frac{\partial T}{\partial t} + \beta w = \varkappa\nabla^2 T \tag{5}$$

In our problem we define $\beta = +dT_0/dz$, opposite to the definition in the ordinary Rayleigh problem (to avoid working with negative Rayleigh numbers). Since the heating is applied from above $\beta > 0$.

The above equations apply to the lower layer as they stand; for the upper layer the same equations apply with u, v, w, p, and T replaced by u', v', w', p', and T', respectively. The parameters ϱ_0, α, $\varkappa$, and ν should further be replaced by the values ϱ_0', α', $\varkappa'$ and ν', respectively.

The boundary conditions are, under the assumption of a horizontal interface,

$$u' = u, \quad v' = v, \quad w' = w = 0 \tag{6 a–d}$$

$$\mu'\frac{\partial u'}{\partial z} = \mu\frac{\partial u}{\partial z}, \quad \mu'\frac{\partial v'}{\partial z} = \mu\frac{\partial v}{\partial z} \tag{7 a–b}$$

$$T' = T \tag{8}$$

$$K'\frac{\partial T'}{\partial z} = K\frac{\partial T}{\partial z} \tag{9}$$

all of which are to be applied at $z = 0$. The conditions (6a–d) express the continuity in velocity and the condition that the interface remains horizontal. The conditions (7a–b) express the continuity in tangential stress; μ and μ' are the coefficients of dynamic viscosity. The condition (8) expresses the continuity in temperature, and the condition (9), finally, the continuity in normal heat flux; K and K' are the coefficients of thermal conductivity. Note the relations

$$\mu = \varrho_0\nu, \quad \mu' = \varrho_0'\nu' \tag{10 a–b}$$

$$K = \varrho_0 c_p \varkappa, \quad K' = \varrho_0' c_p' \varkappa' \tag{11 a–b}$$

$$K'\beta' = K\beta \tag{12}$$

where c_p is the specific heat per unit mass and at constant pressure.

C. Equation for the vertical velocity

We put

$$\left.\begin{aligned} u &= U(z) \\ v &= V(z) \\ w &= W(z) \\ p &= P(z) \\ T &= \Theta(z) \end{aligned}\right\} e^{i(k_x x + k_y y)+rt} \qquad (13\text{ a–e})$$

and similarly for u', v', w', p', and T' (with the same k_x, k_y, and r). We will only consider the case of marginal stability, thus we put $r = 0$. One can prove that the principle of exchange of stability is indeed applicable in the present problem, but the proof is not reproduced here. (If one only wants to give an example of instability the question is not critical, anyway.)

The equations for U, V, W, P, and Θ become

$$\nu\left(\frac{d^2}{dz^2} - k^2\right) U = \frac{ik_x}{\varrho_0} P \qquad (14)$$

$$\nu\left(\frac{d^2}{dz^2} - k^2\right) V = \frac{ik_y}{\varrho_0} P \qquad (15)$$

$$\nu\left(\frac{d^2}{dz^2} - k^2\right) W = \frac{1}{\varrho_0}\frac{dP}{dz} - g\alpha\Theta \qquad (16)$$

$$(k^2 = k_x^2 + k_y^2)$$

$$ik_x U + ik_y V + \frac{dW}{dz} = 0 \qquad (17)$$

$$\varkappa\left(\frac{d^2}{dz^2} - k^2\right)\Theta - \beta W = 0. \qquad (18)$$

As in the ordinary Rayleigh problem one derives the equation for W

$$\left\{\left(\frac{d^2}{dz^2} - k^2\right)^3 - \frac{g\alpha\beta}{\varkappa\nu} k^2\right\} W = 0 \qquad (19)$$

and for the upper layer similarly

$$\left\{\left(\frac{d^2}{dz^2} - k^2\right)^3 - \frac{g\alpha'\beta'}{\varkappa'\nu'}\right\} W' = 0. \qquad (20)$$

One has use later of the equations

$$\left(\frac{d^2}{dz^2} - k^2\right)^2 W = \frac{g\alpha k^2}{\nu}\Theta,$$

$$\left(\frac{d^2}{dz^2} - k^2\right)^2 W' = \frac{g\alpha' k^2}{\nu'}\Theta' \qquad (21\text{ a–b})$$

(from equations (14–17)).

The boundary conditions are, in terms of W and W'

$$W' = W = 0 \qquad (22\text{ a–b})$$

$$\frac{dW'}{dz} = \frac{dW}{dz} \qquad (23)$$

(from $U' = U$, $V' = V$, and the continuity equation)

$$\mu'\frac{d^2 W'}{dz^2} = \mu\frac{d^2 W}{dz^2} \qquad (24)$$

$$\frac{\nu'}{\alpha'}\left(\frac{d^2}{dz^2} - k^2\right)^2 W' = \frac{\nu}{\alpha}\left(\frac{d^2}{dz^2} - k^2\right)^2 W \qquad (25)$$

(from $\Theta' = \Theta$, and (21 a–b))

$$\frac{K'\nu'}{\alpha'}\left(\frac{d^2}{dz^2} - k^2\right)^2\frac{dW'}{dz} = \frac{K\nu}{\alpha}\left(\frac{d^2}{dz^2} - k^2\right)^2\frac{dW}{dz}. \qquad (26)$$

Further it is required that all perturbations should decay to zero both at $z = -\infty$ and $z = +\infty$.

We introduce a new vertical coordinate

$$\zeta = kz \qquad (27)$$

and the Rayleigh numbers

$$\mathrm{Ra} = \frac{g\alpha\beta}{\varkappa\nu k^4}, \quad \mathrm{Ra}' = \frac{g\alpha'\beta'}{\varkappa'\nu' k^4}. \qquad (28\text{ a–b})$$

In the definition of these numbers the inverse wave-number k^{-1} takes the place of the layer depth, and β stands for the direct temperature gradient dT_0/dz.

We further introduce the parameters

$$\varepsilon_1 = \frac{\mu'}{\mu}, \quad \varepsilon_2 = \frac{\nu'}{\nu}\frac{\alpha}{\alpha'}, \quad \varepsilon_3 = \frac{K'}{K}\frac{\nu'}{\nu}\frac{\alpha}{\alpha'}. \qquad (29\text{ a–c})$$

The completely stated mathematical problem is then described by the set of equations

$$\left\{\left(\frac{d^2}{d\zeta^2} - 1\right)^3 - \mathrm{Ra}\right\} W = 0 \qquad (30)$$

$$\left\{\left(\frac{d^2}{d\zeta^2} - 1\right)^3 - \mathrm{Ra}'\right\} W' = 0 \qquad (31)$$

$$\left.\begin{aligned}
&W = W' = 0, \qquad \frac{dW'}{d\zeta} = \frac{dW}{d\zeta} && (32\text{ a–c})\\
&\varepsilon_1 \frac{d^2W'}{d\zeta^2} = \frac{d^2W}{d\zeta^2} && (33)\\
&\varepsilon_2\left(\frac{d^2}{d\zeta^2}-1\right)^2 W' = \left(\frac{d^2}{d\zeta^2}-1\right)^2 W && (34)\\
&\varepsilon_3\left(\frac{d^2}{d\zeta^2}-1\right)^2 \frac{dW'}{d\zeta} = \left(\frac{d^2}{d\zeta^2}-1\right)^2 \frac{dW}{dz}. && (35)
\end{aligned}\right\}\text{ at } z=0$$

It is required that $W \to 0$ when $\zeta \to -\infty$, and that $W' \to 0$ when $\zeta \to +\infty$.

It is of interest to write down the expression for the rate of release of potential energy in the present system. This is, per unit horizontal area,

$$e_p = g\varrho_0\alpha\int_{-\infty}^{0}\overline{Tw}\,dz + g\varrho_0'\alpha'\int_0^{\infty}\overline{T'w'}dz \tag{36}$$

where the bar means a horizontal average. There is no phase factor between w and T, hence we find $\overline{Tw} = \Theta W\,\overline{\cos^2 k_x x\cos^2 k_y y} = \frac{1}{4}\Theta W$. Expressing W and W' in terms of Θ and Θ' using equation (18) the expression for e_p becomes

$$e_p = \frac{1}{4}\frac{g\varrho_0\alpha\varkappa}{\beta}\int_{-\infty}^{0}\Theta\left(\frac{d^2\Theta}{dz^2}-k^2\Theta\right)dz + \frac{1}{4}\frac{g\varrho_0'\alpha'\varkappa'}{\beta'}\int_0^{\infty}\Theta'\left(\frac{d^2\Theta'}{dz^2}-k^2\Theta'\right)dz.$$

Integrating partially, remembering that the perturbations should decay at large distances from the interface, one finds

$$\begin{aligned}
e_p = &-\frac{1}{4}\frac{g\varrho_0\alpha\varkappa}{\beta}\int_{-\infty}^{0}\left\{\left(\frac{d\Theta}{dz}\right)^2 + k^2\Theta^2\right\}dz\\
&-\frac{1}{4}\frac{g\varrho_0'\alpha'\varkappa'}{\beta'}\int_0^{\infty}\left\{\left(\frac{d\Theta'}{dz}\right)^2 + k^2\Theta'^2\right\}dz\\
&+\frac{1}{4}\frac{g\varrho_0\alpha\varkappa}{\beta}\Theta_0\left(\frac{d\Theta}{dz}\right)_0 - \frac{1}{4}\frac{g\varrho_0'\alpha'\varkappa'}{\beta'}\Theta_0'\left(\frac{d\Theta'}{dz}\right)_0
\end{aligned}\tag{37}$$

where the subscript 0 denotes the value at the interface. This expression contains two integrals that are negative-definite and two boundary terms. Making use of the boundary conditions $\Theta_0' = \Theta_0$, $\beta'^{-1}\,d\Theta'/dz = \beta^{-1}d\Theta/dz$, the two boundary terms can be combined into the form

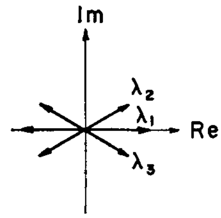

FIG. 2. Location of the λ-roots.

$$e_p^{\text{bound.}} = \frac{1}{4}\frac{g\varrho_0\alpha\varkappa}{\beta}\left(1-\frac{\varrho_0'\alpha'\varkappa'}{\varrho_0\alpha\varkappa}\right)\Theta_0\left(\frac{d\Theta}{dz}\right)_0 \tag{38}$$

In a fluid with uniform coefficients throughout this term vanishes, and the situation must be stable, but in the more general case considered here the term can contribute positively, and eventually dominate. This may occur when $(\varrho_0'\alpha'\varkappa')/(\varrho_0\alpha\varkappa) < 1$, and Θ_0 and $(d\Theta/dz)_0$ are positively correlated, or when $(\varrho_0'\alpha'\varkappa')/(\varrho_0\alpha\varkappa) > 1$, and Θ_0 and $(d\Theta/dz)_0$ are negatively correlated. The form suggests that the ratios α'/α and $\varkappa'/\varkappa$ are the key parameters in the stability problem, and this is verified later.

D. The characteristic equation

We put

$$W = Ce^{\lambda\zeta},\quad W' = C'e^{\lambda'\zeta} \tag{39 a–b}$$

and find $(\lambda^2-1)^3 = \mathrm{Ra}$, $(\lambda'^2-1)^3 = \mathrm{Ra}'$ (40 a–b)

The roots λ_i are six, of which three, say, λ_1, λ_2, and λ_3, have positive real parts, the other three roots are $-\lambda_1$, $-\lambda_2$, and $-\lambda_3$ (Fig. 2).

For W we use the roots λ_1, λ_2, and λ_3, for W' the roots $-\lambda_1'$, $-\lambda_2'$, and $-\lambda_3'$; this will secure that the perturbations decay at large distances from the interface. Since two roots are complex we expect a series of cells away from the interface, but the rapid decay should make all except the first one unimportant. For small Rayleigh numbers the width to depth ratio of the cells should be close to one, for large Rayleigh numbers it becomes $\mathrm{Ra}^{1/6}$, thus the cells are flattened.

Putting

$$\begin{aligned}
W &= C_1e^{\lambda_1\zeta} + C_2e^{\lambda_2\zeta} + C_3e^{\lambda_3\zeta}\\
W' &= C_1'e^{-\lambda_1'\zeta} + C_2'e^{-\lambda_2'\zeta} + C_3'e^{-\lambda_3'\zeta}
\end{aligned}\tag{41 a–b}$$

the boundary conditions (32–35) yield the following characteristic equation, valid for marginal stability,

$$f(\mathrm{Ra}, \mathrm{Ra}', \varepsilon_1, \varepsilon_2, \varepsilon_3) \equiv$$

$$\equiv \begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_1' & \lambda_2' & \lambda_3' \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & -\varepsilon_1 \lambda_1'^2 & -\varepsilon_1 \lambda_2'^2 & -\varepsilon_1 \lambda_3'^2 \\ (\lambda_1^2-1)^2 & (\lambda_2-1)^2 & (\lambda_3^2-1)^2 & -\varepsilon_2 (\lambda_1'^2-1)^2 & -\varepsilon_2 (\lambda_2'^2-1)^2 & -\varepsilon_2 (\lambda_3'^2-1)^2 \\ (\lambda_1^2-1)^2\lambda_1 & (\lambda_2^2-1)^2\lambda_2 & (\lambda_3^2-1)^2\lambda_3 & \varepsilon_3 (\lambda_1'^2-1)^2\lambda_1' & \varepsilon_3 (\lambda_2'^2-1)^2\lambda_2' & \varepsilon_3 (\lambda_3'^2-1)^2\lambda_3' \end{vmatrix} = 0. \quad (42)$$

The equation is seen to be real. Developing the determinant one arrives, after some computing work, to the equation

$$L\varepsilon_2\varepsilon_3 + M\varepsilon_1\varepsilon_3 + N\varepsilon_1\varepsilon_2 + P\varepsilon_1 + Q\varepsilon_2 + R\varepsilon_3 = 0 \quad (43)$$

where L, M, N, P, Q and R are functions of λ_i and λ_i', i.e. of Ra and Ra$'$. The expressions are

$$\left.\begin{aligned} L(\mathrm{Ra}, \mathrm{Ra}') &= 3\,\mathrm{Ra}^{1/3}\,\mathrm{Ra}'^{4/3}\cdot \\ &\cdot \{(\lambda_2-\lambda_1)(\lambda_3'+I_3\lambda_2'+I_2\lambda_1') \\ &+ (\lambda_3-\lambda_1)(I_2\lambda_3'+\lambda_2'+I_3\lambda_1')\} \\ M(\mathrm{Ra}, \mathrm{Ra}') &= i3\sqrt{3}\,\mathrm{Ra}^{2/3}\,\mathrm{Ra}' \\ &\cdot \{-(\lambda_2-\lambda_1)(I_2\lambda_3-\lambda_1) \\ &+ (\lambda_3-\lambda_1)(I_3\lambda_2-\lambda_1)\} \\ N(\mathrm{Ra}, \mathrm{Ra}') &= -3\,\mathrm{Ra}^{2/3}\,\mathrm{Ra}' \\ &\cdot (\lambda_1+I_2\lambda_3+I_3\lambda_2)(\lambda_1'+\lambda_2'+\lambda_3') \end{aligned}\right\} \quad (44\,\text{a–c})$$

where $\quad I_2 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad I_3 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$

further

$$\begin{aligned} P(\mathrm{Ra}, \mathrm{Ra}') &= L(\mathrm{Ra}', \mathrm{Ra}) \\ Q(\mathrm{Ra}, \mathrm{Ra}') &= M(\mathrm{Ra}', \mathrm{Ra}) \\ R(\mathrm{Ra}, \mathrm{Ra}') &= N(\mathrm{Ra}', \mathrm{Ra}) \end{aligned} \quad (45\,\text{a–c})$$

L is always positive, M and N always negative (for Ra, Ra$' > 0$). The magnitudes increase monotonically with Ra and Ra$'$.

In the limiting case of small Ra and Ra$'$ the expressions (44a–c) reduce to

$$\begin{aligned} L &= \tfrac{27}{64}\,\mathrm{Ra}\,\mathrm{Ra}'^2 \\ M &= -\tfrac{27}{2}\,\mathrm{Ra}\,\mathrm{Ra}' \\ N &= -\tfrac{27}{2}\,\mathrm{Ra}\,\mathrm{Ra}'. \end{aligned} \quad (46\,\text{a–c})$$

In the other limiting case of large Ra and Ra$'$ one finds

$$\begin{aligned} L &= 3\,\mathrm{Ra}^{1/2}\,\mathrm{Ra}'^{3/2} \\ M &= -12\,\mathrm{Ra}^{5/6}\,\mathrm{Ra}'^{7/6} \\ N &= -9\,\mathrm{Ra}\,\mathrm{Ra}'. \end{aligned} \quad (47\,\text{a–c})$$

As in the general case, P, Q, and R are found from (45a–c).

For given Rayleigh numbers, i.e. for given values of the coefficients $L, --R$, equation (43) represents a second order surface in the $\varepsilon_1 - \varepsilon_2 - \varepsilon_3$-space (Fig. 3).

The characteristic equation $f(\mathrm{Ra}, \mathrm{Ra}', \varepsilon_1, \varepsilon_2, \varepsilon_3) = 0$ is invariant under the transformation $\mathrm{Ra} \rightleftarrows \mathrm{Ra}'$, $\varepsilon_i \to \varepsilon_i^{-1}$, corresponding to an interchange of all primed and unprimed quantities. The stability properties are thus unchanged when the two layers are interchanged. However, a complete physical interchange of all properties cannot be made. An originally stable stratification ($\varrho_0' < \varrho_0$) becomes unstable after the interchange, and the theory does not apply in the second case. Nevertheless, the transformation $\mathrm{Ra} \rightleftarrows \mathrm{Ra}'$, $\varepsilon_i \to \varepsilon_i^{-1}$ gives us new physically possible cases. This is because the ratio ϱ_0'/ϱ_0 does not enter as a single parameter but only in com-

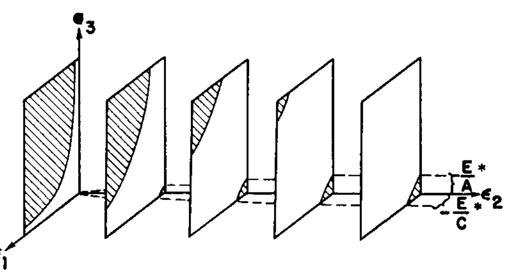

FIG. 3. Geometric representation of equation (43). The unstable region is shaded.

* Errata: $-\frac{E}{A}$ and $-\frac{E}{C}$ should be deleted.

bination with other parameters. It is possible to vary ϱ_0'/ϱ_0 without changing the parameters Ra, Ra′ and the ε_i:s, if the ratios μ'/μ, α'/α, and $\varkappa'/\varkappa$ are adjusted properly. Thus new gravitationally stable cases are included among the transformed cases.

3. The critical equation, with a discussion

The Rayleigh numbers Ra and Ra′ contain the wave-number k. This is a free parameter in our problem, since no walls are assumed. If we let k vary from zero to infinity, Ra and Ra′ will vary from infinity to zero, with a fixed ratio

$$\varepsilon_4 = \frac{\mathrm{Ra}'}{\mathrm{Ra}} = \frac{\alpha'}{\alpha}\frac{\beta'}{\beta}\frac{\varkappa}{\varkappa'}\frac{\nu}{\nu'} = \frac{K}{K'}\frac{\varkappa}{\varkappa'}\frac{\alpha'}{\alpha}\frac{\nu}{\nu'}. \quad (48)$$

ε_4 is thus one more characteristic parameter of the problem. For given values of the parameters ε_1, ε_2, ε_3, and ε_4 we may have stability for all k-values, or we may have instability for at least some k-value. The surface dividing these two regions in the $\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4$-space represent the critical equation. It marks the parameter values for which instability first sets in.

Consider now the second order surface represented in Fig. 3. If Ra and Ra′ are given *positive* increments ΔRa and ΔRa′, the displaced surface will enclose the original surface without intersection, in the positive octant of interest, thus adding a new region of instability. The proof of this, that hangs on the exact form of the functions $L(\mathrm{Ra},\mathrm{Ra}')$, $M(\mathrm{Ra},\mathrm{Ra}')$, and $N(\mathrm{Ra},\mathrm{Ra}')$, is not reproduced here. It follows from this result that, for given values of ε_1, ε_2, ε_3, and ε_4, instability for one wave-number implies instability for all smaller wave-numbers. The largest scales will become unstable first.

The critical equation marking the first onset of instability can accordingly be derived from the characteristic equation (43) by letting Ra and Ra′ tend to infinity with their ratio (ε_4) fixed. Using the expressions derived for L, $- -$ R in the limit of large Rayleigh numbers the critical equation becomes[1]

$$g(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) \equiv \varepsilon_2\varepsilon_3\varepsilon_4^{1/2} - 4\varepsilon_1\varepsilon_3\varepsilon_4^{1/6} - 3\varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_4^{-1/2} - 4\varepsilon_2\varepsilon_4^{-1/6} - 3\varepsilon_3 = 0. \quad (49)$$

When $g>0$ one is in the unstable region, when $g<0$ in the stable region (in rewriting (49) in the following we will retain the sign unchanged to preserve this property).

It may be remarked that for sufficiently small values of Ra and Ra′, i.e. for sufficiently large k-values, the situation is always stable. In this case (43) reduces to

$$-\varepsilon_1\varepsilon_3 - \varepsilon_1\varepsilon_2 - \varepsilon_2 - \varepsilon_3 = 0 \quad (50)$$

and the left hand side is always negative.

Returning to the critical equation (49), we rewrite this in terms of four new parameters that are better separated, physically:

$$F = \frac{\mu'}{\mu},\ E = \frac{\varrho'\alpha'}{\varrho\alpha},\ H = \frac{c_p'\varrho'}{c_p\varrho},\ C = \frac{\varkappa'}{\varkappa}. \quad (51\text{ a–d})$$

These may, in turn, be called the friction ratio, the expansion ratio, the heat ratio, and the conductivity ratio. One has

$$\varepsilon_1 = F,\ \varepsilon_2 = \frac{F}{E},\ \varepsilon_3 = \frac{CHF}{E},\ \varepsilon_4 = \frac{E}{C^2HF} \quad (52\text{ a–d})$$

and the critical equation can be written in the form

$$g^*(C,E,F,H) \equiv E\left\{\left(\frac{FH}{E}\right)^{1/2} - \frac{3H}{E}\right\}\cdot C - 4E\left(\frac{FH}{E}\right)^{5/6}C^{2/3} - 4\left(\frac{FH}{E}\right)^{1/6}\cdot C^{1/3} + \left(\frac{FH}{E}\right)^{1/2} - 3F = 0. \quad (53)$$

If (53) is considered as a critical equation for C one sees that critical values do exist when $FE/H>9$ or $FE/H<1/9$, as this makes the first and the last term positive, respectively. In the first case instability occurs for all C-values larger than a certain critical value, in the second case for all C-values smaller than a certain critical value. These critical values depend, of course, on E, F, and H. The most effective way to produce instability is to make E and C both very large or very small. Regarding the friction ratio F and the heat ratio H these should not be made extreme, as this is seen to

[1] A more direct derivation of equation (49) can be given by making Ra, Ra′$\to\infty$ in (42). One finds that $\lambda_1\to\mathrm{Ra}^{1/6}$, $\lambda_2\to J_2\mathrm{Ra}^{1/6}$, $\lambda_3\to J_3\mathrm{Ra}^{1/6}$, where $J_2 = \frac{1}{2}+i\sqrt{3}/2$, $J_3=\frac{1}{2}-i\sqrt{3}/2$. After dividing down the R-factors and introducing $\varepsilon_4 = \mathrm{Ra}'/\mathrm{Ra}$, the determinant is easily reduced to give equation (49).

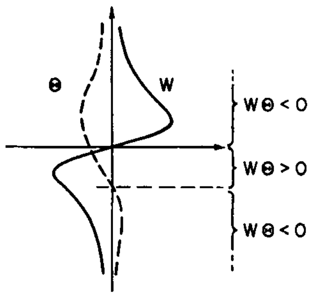

FIG. 4. Profiles of W and Θ. The release of potential energy is proportional to $W\Theta$, and to the thermal expansion coefficient.

stabilize the fluid. Their optimum values depend, of course, on the values chosen for C and E in each case.

4. Physical nature of the instability

To get an understanding of the mechanism that produces the instability it is helpful to look at a simplified case. Assume a vertical velocity of the form

$$\left.\begin{aligned} w &= W(z) \\ w' &= W'(z) \end{aligned}\right\} \cdot e^{i(k_x x + k_y y)} \left.\begin{aligned} &= Aze^{\gamma z} \\ &= Aze^{-\gamma z} \end{aligned}\right\} \cdot e^{i(k_x x + k_y y)}. \tag{54 a–b}$$

This gives the W-profile shown in Fig. 4. At the interface we have continuity in the velocity and also in the stress, provided the dynamic friction coefficients are the same in both layers. The temperature perturbations created by this field of motion are found from equation (18):

$$\left.\begin{aligned} \varkappa\left(\frac{d^2}{dz^2} - k^2\right)\Theta &= \beta Aze^{\gamma z} \\ \varkappa'\left(\frac{d^2}{dz^2} - k^2\right)\Theta' &= \beta' Aze^{-\gamma z} \end{aligned}\right. \tag{55 a–b}$$

with the boundary conditions

$$\left.\begin{aligned} \Theta' &= \Theta \\ \frac{1}{\beta'}\frac{d\Theta'}{dz} &= \frac{1}{\beta}\frac{d\Theta}{dz}. \end{aligned}\right\} \text{ at } z=0 \tag{56 a–b}$$

The solution is of the form

$$\begin{aligned} \Theta &= (a+bz)\,e^{\gamma z} + c\,e^{kz} \\ \Theta' &= (a'+b'z)\,e^{-\gamma z} + c'\,e^{-kz} \end{aligned} \tag{57 a–b}$$

Determining the coefficients by use of (55) and (56) and evaluating the temperature perturbation at the interface one finds

$$\Theta_{z=0} = a + c = \frac{A}{k(k+\gamma)^2} \cdot \frac{\dfrac{1}{\varkappa} - \dfrac{1}{\varkappa'}}{\left(\dfrac{1}{\beta} + \dfrac{1}{\beta'}\right)}. \tag{58}$$

This is positive if $\varkappa' > \varkappa$, negative if $\varkappa < \varkappa'$. Thus the temperature perturbations near the interface have the sign determined by the advection in the layer of smallest conductivity. The resulting temperature field is sketched in Fig. 4, in the case where the lower layer has the larger conductivity. The total field of the streamlines and isotherms is shown in Fig. 5, in the case of twodimensional motion. In the lower fluid one finds a region next to the interface where W and Θ are positively correlated, i.e. warm fluid rises and cold fluid sinks. Potential energy is thus released. If the coefficient of thermal expansion is made large enough in the lower layer one may obtain sufficient energy to make up for the potential energy needed to drive the cells in the upper layer as well as for the dissipation in the two layers.

One can consider the amplification of the motion in a sequence about as follows: Start from the conductive basic state plus a small superimposed cellular motion in the upper layer. This motion will produce a slight warming in regions of downward motion, cooling in regions of upward motion. The effects are conducted across the boundary into the lower layer. The horizontal temperature variations thus created in the lower layer must start a reversed circulation there. If the thermal expansion coefficient is large this motion will become strong. Through the frictional coupling at the interface it amplifies the original cellular motion in the upper layer, and an amplifying cycle is closed.

The idea that a combination of a large thermal expansion and a large conductivity in the same layer will favour an instability is in accordance with the result of the formal stability analysis. It can also be verified that this situation is one making the boundary term (38) in the expression for the potential energy release positive.

It seems that the "anti-convection" will be limited by finite amplitude effects much more strongly that the ordinary Bénard convection. In the Bénard convection there is a positive

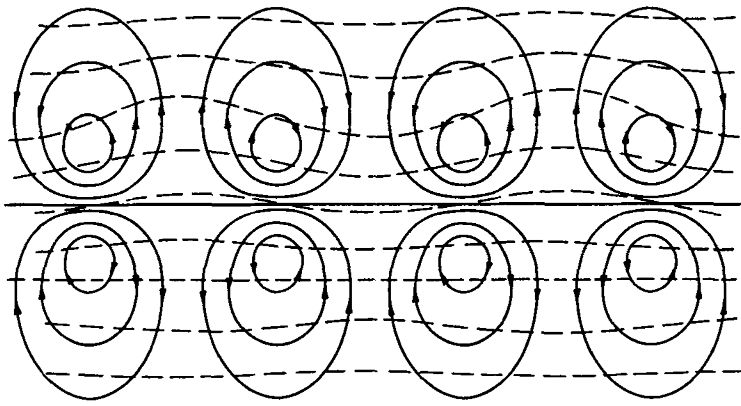

FIG. 5. Streamlines and isotherms in the simplified model.

release of potential energy in the main part of the fluid even at very strong motions. In our case a positive release can be stopped and reversed everywhere if the counteracting advective effects in the lower layer become too strong. The maximum vertical velocities to be expected should come out by balancing the second order advective term $w(\partial T/\partial z)$ and the conductive term $\varkappa(\partial^2 T/\partial z^2)$, this velocity is of order $\varkappa/D$, where D is the wave-length. For 1 cm cells the value is around $1.5 \cdot 10^{-3}$ cm per second in water, $6 \cdot 10^{-2}$ cm per second for mercury.

For large cells the limiting velocities become, of course, very small and these cells, although they become first unstable in the linear approximation, may never be observed in practice. Small cells, with Rayleigh numbers much less than one, have been found to be stable. In an actual convection under supercritical conditions it may well be that the picture is dominated by cells having Rayleigh numbers of order one. These will probably also have the strongest growth rate.

5. Laboratory experiments. Application to geophysical and astrophysical situations

One may feel astonished that the "anti-convection" has not been discovered experimentally. The reason must be, firstly, that the phenomenon occurs only in fluid systems of a very special composition, and, secondly, that the convective motions, when they occur, are quite weak. However, in a properly designed experiment there should be possible to demonstrate the effect.

Of the fluid systems that may be considered a mercury-water combination seems most suitable to work with. Mercury has a relatively large mass expansion and good conductivity. The mass expansion of water can be made as small as is required by working close to 4°C, its conductivity is further poor compared to that of mercury. Gas-fluid combinations have been considered, but these are less suitable, at least at normal temperatures and pressures (generally the parameter FE/H is small, the parameter C large).

For the mercury-water system the parameters F, E, H, and C have been computed in the interval 5°C–30°C, their variation with temperature can be judged from Fig. 6. In the same figure is drawn the curve for g^*, the left hand side of the critical equation (53). The zero value of g^*, marking the temperature below which instability will occur when a heat flux is applied, is found to lie around 18°C. Thus, unless one wants to work with very supercritical situations it is not necessary to go very close to the 4°-limit.

In the present experiment the determination of the transition is more difficult than in the ordinary Bénard convection. The heat flux plotted against the applied temperature difference gives little information. With the boundaries far removed the heat flux should be completely independent of the presence of a convection, and in practice one cannot certainly expect any sharp bend in the curve. The cell motion and the temperature variations near the interface can, of course, be detected using the different optical techniques that have been

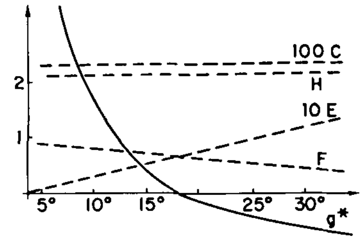

FIG. 6. Variation of the parameters F, E, H, and C, and of the function g^* with the temperature for a water-mercury system. The system is unstable when $g^* > 0$.

developed in connection with convection experiments, and this will also be attempted in the first experiments that are planned. Ordinary tracers cannot be used to visualize the motion, since any small contamination of the water or the mercury seems to create an almost rigid surface film that prevents the momentum exchange between the layers. In fact, one of the main difficulties of the experiment is the extreme cleanness required.

One may consider some applications of the "anti-convection" to geophysical and astrophysical situations. One should look for conditions where the heat flux is applied in the direction of the gravity force and where strong variations in thermal expansion and conductivity are expected. In planetary atmospheres such conditions may certainly be found. There exists also the possibility that critical conditions can be reached in stars, which during certain stages in their development are expected to be heated from an outer zone (private communication by E. Spiegel). The treatment of all such cases is, however, complicated by the presence of radiative and turbulent effects. Although the present theory is not strictly applicable in these cases it may perhaps serve as a rough first approximation if the coefficients of viscosity and conductivity are replaced by appropriately computed effective values. In some cases at least the introduction of a turbulence seems to bring the conditions closer to criticality.

As example, an air-water system is characterized by the following parameter values (at 20°C):

$$F = 0.00180,\ E = 0.0214,\ H = 0.000290,$$

$$C = 145,\ \frac{EF}{H} = 0.133.$$

This system is definitely stable, according to equation (53). If one turns to a turbulent air-water system representing, say, the conditions at the interface of the atmosphere and a lake or the sea, F and C will change drastically. The eddy coefficient for vertical friction is generally in the range 100–1000 g $cm^{-1}s^{-1}$ in the lower atmosphere; for the surface layer of a lake or the sea somewhat lower values, in the range 10–100 g $cm^{-1}s^{-1}$, are common. Assuming that the mechanism of turbulent exchange is the same for momentum and heat, the eddy conductivity should be of the order of the eddy viscosity divided by the density. Computing the friction ratio F and conductivity ratio C one finds that they may well take on values of the order 10 and 10,000 respectively. With these values one finds $g^* > 0$ and an instability may be expected. In the case of large scale convection in the atmosphere-ocean system one may further have to include rotation effects. A preliminary study of this problem has given the surprising result that a strong rotation makes the system more unstable. The instability further occurs in the form of low-frequency oscillations (overstability). This oscillatory behaviour is explained by a reversal of the frictional coupling in the strongly rotating case. A detailed discussion of this case will appear in a second paper.

Acknowledgement

The present study has been supported by the Swedish Natural Science Research Council and by the Office of Naval Research, U.S.A., through the contracts No. 62558-3033 and No. 2196(00).

REFERENCES

CHANDRASEKHAR, S., 1961, Hydrodynamic and Hydromagnetic Stability, Oxford, at the Clarendon Press.

SPIEGEL, E., and VERONIS, G., 1960, On the Boussinesq Approximation for a Compressible Fluid, *Astrophysical Journal*, Vol. **131**, No. 2, pp. 285–97.