On Rossby waves in spiral galaxies

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ABSTRACT

Galactostrophic eddy velocities are used to eliminate the dependence of the potential vorticity equation on the velocities. The resultant time dependent equation involves the mass densities alone.

1. Introduction

A galaxy may in the large be regarded as a rotating, self gravitating fluid and may be discussed from the viewpoint of Eulerian fluid dynamics. The transfer of momentum by random motions of stars is equivalent to a non-isotropic pressure, which together with the self gravitation produces the large scale force fields that are included in our dynamical description. In the model galaxy considered, all the mass lies in the galactic plane. Cylindrical geometry is approximated by Cartesian geometry, and we treat the mean differential rotation as constant except when differentiated with respect to the radial coordinate. These two approximations are, strictly speaking, only applicable to flow between narrowly separated concentric cylinders. Besides the above simplifying approximations, we shall make a balance assumption that the magnitude of substantial derivatives of the eddy fields are much less than the rotational frequency of the galaxy. If this is so, the Coriolis forces will be approximately in balance with the sum of the gravitational and pressure forces and the flow will be quasi-nondivergent. In general the dynamic equations governing the eddy motions of a rotating galaxy have for a given wavelength three possible frequencies. In a coordinate system moving with the mean rotation, two of these are high frequency or self gravitational-acoustic-inertial modes, and the third is low frequency or Rossby mode. TOOMRE (1964) considered the stability of the former and concluded that our own galaxy is either stable against such motions or is on the verge of instability over a short range of wavelengths about half the galactic radius. Our balance assumption in effect filters these waves out of the dynamic equations leaving only the low frequency mode. Such motions may possibly be unstable even when the high frequency motions are stable. If this is the case, one would expect irregular motions obeying our assumption to occur in actual spiral galaxies.

Meteorological motions responsible for large scale weather systems with time scales of several days are in essence such a low frequency mode. The high frequency or gravitational-inertial modes are predominantly not excited. In such a case one may use the development of CHARNEY (1948), who showed how one might obtain equations describing the evolution of large scale motions of the earth's atmosphere, without including the high frequency modes of oscillation. This powerful simplification together with the simultaneous development of high speed computers has been the basis of numerical weather prediction up to the present day. See PHILLIPS (1963) and THOMPSON (1961).

The technique as applied to rotating galaxies is to form an equation for the vorticity normal to the galactic plane. The continuity equation is used to eliminate the velocity divergence which occurs in the vorticity equation, thus yielding a potential vorticity equation. The eddy velocities as a consequence of an assumed balance may be approximately expressed in terms of the pressures and gravitational forces. The gravitational potential may be expressed in terms of the density distribution by inversion of the Poisson equation. To proceed further without introducing the added complication of considering the thermodynamics of the random star motions, we shall neglect the spatial variation of the random motion variances and assume that the random motion covariance is negligible. With these assumptions, the pressurelike forces become directly proportional to the density gradients, and the viscositylike terms are then omitted.

A particular time dependent solution is obtained for the resulting approximate equation. It appears that the anisotropic pressures corresponding to the difference between radial and tangential random motions acts to destroy leading arms and to excite trailing arms when the galaxy is stable to high frequency waves.

2. Notation

We use the following symbols

G $= 6.670 \times 10^{-8}$cgs = universal gravitational constant

ϖ = radial distance from the axis of rotation (center of mass)

λ = polar angle in the galactic plane

z = coordinate normal to galactic plane

ϖ_0 = a constant radius

λ_0 = a constant polar angle

x $= \varpi_0(\lambda - \lambda_0)$

y $= \varpi_0 - \varpi$

t = time

ϱ_3 = the three dimensional mass density field

$\delta(z)$ = a Dirac delta function

ϱ $= \int_{-\infty}^{\infty} \varrho_3 dz$ = the surface density of galactic disk

Ω $= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{d\lambda}{dt}\right)d\lambda$ = the mean differential rotation (which is assumed to be positive)

U $= \Omega\varpi_0$ = basic tangential velocity

u $= \frac{dx}{dt} = \varpi\frac{d\lambda}{dt} - U$ = the eddy fluid velocity about the axis of rotation

$v = \frac{dy}{dt} = -\frac{d\varpi}{dt}$ = the inward radial fluid velocity

ζ = relative vorticity in the axial (z) direction $= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

q = galactostrophic potential vorticity, defined in the text

A $= -\frac{1}{2}\frac{d\Omega}{d\varpi}$ = Oort's constant A

B $= -\frac{1}{2}\left(2\Omega + \frac{d\Omega}{d\varpi}\right)$ = Oort's constant B

$[\sigma_x^2]$ = the random star motion variance for the x-direction

$[\sigma_y^2]$ = the random star motion variance in the y-direction

$[\sigma_x\sigma_y]$ = the random star motion covariance

$(\quad)_0 = \frac{1}{2\pi}\int_0^{2\pi}(\quad)d\lambda$ evaluated at $\varpi = \varpi_0$

f $= (2\Omega)_0 = 2(A-B)_0$

f^* $= (2\Omega)_0 + \left(\varpi\frac{\partial\Omega}{\partial\varpi}\right)_0 = -(2B)_0$ = mean vorticity $\varpi = \varpi_0$

β $= -\left(\frac{\partial}{\partial\varpi}2\Omega\right)_0$

γ $= -\left(\frac{\partial\varrho}{\partial\varpi}\right)_0$

ϱ' $= \varrho - \frac{1}{2\pi}\int_0^{2\pi}\varrho d\lambda$ = eddy part of surface density

ϕ = the gravitational potential

ϕ' $= \phi - \int^{\varpi}\Omega^2\varpi d\varpi - \int_0^{2\pi}\int^{\varpi}\left(\frac{1}{\varrho}\frac{\partial}{\partial\varpi}\frac{\varrho}{\varpi}[\sigma_y^2] + \frac{1}{\varrho}\frac{\partial\varrho}{\varpi\partial\lambda}[\sigma_x\sigma_y]\right)d\varpi\frac{d\lambda}{2\pi}$

ψ = galactostrophic stream function for eddy velocities

ζ_g $= \nabla_2^2\psi$ = galactostrophic vorticity

$\frac{D}{Dt} = \frac{\partial}{\partial t} + (U+u)\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$ = substantial derivative.

3. The basic equations

When one neglects certain physically unimportant curvature terms and assumes a linear variation of the differential rotation, the model equations governing the large scale eddy flows in a flat disk galaxy are the following approximate versions of the horizontal equations of motion, the continuity equation, and the gravitational equation.

$$\frac{Du}{Dt} - (f^* + \beta y)v + \frac{\partial\phi'}{\partial x} + \frac{1}{\varrho}\frac{\partial}{\partial x}\varrho[\sigma_x^2] + \frac{1}{\varrho}\frac{\partial}{\partial y}\varrho[\sigma_x\sigma_y] = 0 \tag{1}$$

$$\frac{Dv}{Dt} + (f+\beta y)u + \frac{\partial \phi'}{\partial y} + \frac{1}{\varrho}\frac{\partial}{\partial x}\varrho[\sigma_x\sigma_y] + \frac{1}{\varrho}\frac{\partial}{\partial y}\varrho[\sigma_y^2] = 0 \tag{2}$$

$$\frac{D\varrho}{Dt} + \varrho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{3}$$

$$\frac{\partial^2 \phi'}{\partial x^2} + \frac{\partial^2 \phi'}{\partial y^2} + \frac{\partial^2 \phi'}{\partial z^2} = 4\pi G\varrho'\delta(z) \tag{4}$$

The variable part of the Coriolis parameter, βy, will be neglected except when differentiated with respect to y. We use also a vorticity equation obtained from cross differentiation of (1) and (2). If we neglect $\partial v/\partial x - \partial u/\partial y$ compared to f^*, it is

$$\frac{D\zeta}{Dt} + f^*\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v + P = 0 \tag{5}$$

where

$$P = \frac{\partial}{\partial x}\left(\frac{1}{\varrho}\frac{\partial}{\partial x}\varrho[\sigma_x\sigma_y] + \frac{1}{\varrho}\frac{\partial}{\partial y}\varrho[\sigma_y^2]\right) - \frac{\partial}{\partial y}\left(\frac{1}{\varrho}\frac{\partial}{\partial x}\varrho[\sigma_x^2] + \frac{1}{\varrho}\frac{\partial}{\partial y}\varrho[\sigma_x\sigma_y]\right).$$

The continuity equation, (3) may be used to eliminate the term $\varrho(\partial u/\partial x + \partial v/\partial y)$ in (5) in order to arrive at a potential vorticity equation which is

$$\frac{D\zeta}{Dt} - f^*\frac{D\ln\varrho}{Dt} + \beta v + P = 0 \tag{6}$$

If one assumes certain scales of the motions, it follows (DICKINSON, 1963)

that
$$\left.\begin{aligned} \frac{Du}{Dt} &< f^* v \\ \frac{Dv}{Dt} &< fu \end{aligned}\right\} \tag{7}$$

$$\frac{1}{\varrho}\frac{D\varrho}{Dt} < \left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial v}{\partial y}\right| \tag{8}$$

It then is a consequence of (7) that equations (1) and (2) have the approximate solution

$$\left.\begin{aligned} v &\simeq 1/f^*\left(\frac{\partial \phi'}{\partial x} + \frac{1}{\varrho}\frac{\partial}{\partial x}\varrho[\sigma_x^2] + \frac{1}{\varrho}\frac{\partial}{\partial y}\varrho[\sigma_x\sigma_y]\right) \\ u &\simeq -1/f\left(\frac{\partial \phi'}{\partial y} + \frac{1}{\varrho}\frac{\partial}{\partial x}\varrho[\sigma_x\sigma_y] + \frac{1}{\varrho}\frac{\partial}{\partial y}\varrho[\sigma_y^2]\right) \end{aligned}\right\} \tag{9}$$

which we call the galactostrophic velocities. Also from (8) it follows that the flow is approximately nondivergent so

$$\left.\begin{aligned} v &\simeq \frac{\partial \psi}{\partial x} \\ u &\simeq -\frac{\partial \psi}{\partial y} \end{aligned}\right\} \tag{10}$$

It is consistent with our approximations to use (8) to obtain the nondivergent part of the flow, while the divergent flow which is smaller is obtained from the continuity equation. In the following, we shall neglect $[\sigma_x\sigma_y]$ and neglect differentiation of $[\sigma_x^2]$ and $[\sigma_y^2]$. We shall write the density ϱ as $\varrho_0 + \gamma y + \varrho'$. Then it follows from (9) that

$$\zeta_g = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$$
$$= \frac{1}{f}\left(\frac{f}{f^*}\frac{\partial^2 \phi'}{\partial x^2} + \frac{\partial^2 \phi'}{\partial y^2} + \frac{f}{f^*}\frac{[\sigma_x^2]}{\varrho_0}\frac{\partial^2 \varrho'}{\partial x^2} + \frac{[\sigma_y^2]}{\varrho_0}\frac{\partial^2 \varrho'}{\partial y^2}\right). \tag{11}$$

The approximate potential vorticity equation then is

$$\frac{\partial q}{\partial t} + U\frac{\partial q}{\partial x} + J(\psi, q) + \left(\beta - \frac{f^*\gamma}{\varrho_0}\right)\frac{\partial \psi}{\partial x}$$
$$+ ([\sigma_y^2] - [\sigma_x^2])\, 1/\varrho_0 \frac{\partial^2 \varrho'}{\partial x \partial y} = 0 \tag{12}$$

where $q = \zeta_g - f^*\varrho'/\varrho_0$

$$J(\psi, q) = \frac{\partial \psi}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial q}{\partial x}.$$

To exhibit a simple solution to (11), assume the initial condition

$$\varrho' = a_0 e^{ikx + ily}$$

If we Fourier analyse (12) and neglect y-variation of $U(y)$ (this is equivalent to assuming our system is so constrained that the other exponential modes generated by nonlinear interaction of the initial mode with $U(y)$ do not occur), then (12) will have the solution

$$\varrho' = a_0 \exp\left[ik\left(x - \left[U_0 + \frac{\beta - \gamma f^*/\varrho_0}{k^2 + l^2 + A(k,l)}\right]t\right) - klB(k,l)ft + ily\right] \tag{13}$$

where $A(k,l)$ and $B(k,l)$ are positive constants.

$$A(k,l) = (ff^*)\left[\frac{f}{f^*}k^2[\sigma_x^2] + l^2[\sigma_y^2] - \frac{2\pi G\varrho_0}{(k^2+l^2)^{\frac{1}{2}}}\left(\frac{f}{f^*}k^2 + l^2\right)\right]^{-1}$$

$$B(k,l) = ([\sigma_y^2] - [\sigma_x^2])\left[\frac{f}{f^*}k^2[\sigma_x^2] + l^2[\sigma_y^2] - \frac{2\pi G\varrho_0}{(k^2+l^2)^{\frac{1}{2}}}\left(\frac{f}{f^*}k^2 + l^2\right) + ff^*\right]^{-1}$$

We use the fact that (4) may be reduced to $\phi' = -2\pi G(k^2+l^2)^{-\frac{1}{2}}\varrho'$. For no y dependence, our solution reduces to a galactic analogue to Rossby waves (Rossby, 1939).

From theory and observation it is known that $[\sigma_y^2] > [\sigma_x^2]$ and in our galaxy in the solar neighborhood $[\sigma_y^2] \div [\sigma_x^2] \simeq 2$. Within the framework of our simplified analysis, this leads to decay of leading arms ($kl > 0$) and growth of trailing arms ($kl < 0$) in the limit of large wave numbers when the approximate potential vorticity is opposite in sign to ϱ'. This difference in sign between q and ϱ is equivalent within the accuracy of our approximations to gravitational stability of the high frequency modes as discussed by Toomre (1964).

4. Discussion

The usefulness of our expansion depends on whether or not the large scale irregular mass and motion fields in a spiral galaxy are similar to meteorological motions in that the time rates of change, moving with the fluid flow (substantial derivative), are smaller than the Coriolis frequency. Such is only plausible if rapid gravitational instabilities of the type considered by Toomre do not occur except possibly in the past history of the galaxy.

Rossby waves may be residual motions excited during a highly irregular galactic birth, or they may possibly be excited by slow instabilities that occur when the galaxy is stable to the oscillations with frequencies near the Coriolis frequency or less. Qualitatively our results show that the anisotropic pressure may act to decrease the potential vorticity in trailing arms and increase it in leading arms. A resultant increase of kinetic energy of the fluid eddy motions occurs at the expense of the kinetic energy of the random star motions. If the galaxy is gravitationally stable, the resultant flow will increase the mass in trailing arms and decrease the mass of leading arms. Perhaps arms start to form in such an orientation and then are tightened into circular rings by the winding process of differential rotation which has been discussed recently by Oort (1962).

We note in closing that for highly asymmetric galaxies such as barred spirals, one may generalize our expansion by use of a gradient wind rather than a galactostrophic wind. Prendergast (1962) suggests such a balance approximation.

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REFERENCES

Charney, J. G., 1948, On the scale of atmospheric motions, *Geophys. Publ.* **17**, No. 2, 17 p.

Dickinson, R. E., 1963, Analysis of large scale disturbances in spiral galaxies. *Geofisica Pura e Applicata*, **56**, pp. 174–184.

Oort, J. H., 1962, Spiral structure. *The Distribution and Motion of Interstellar Matter in Galaxies* (ed. L. Woltjer). W. A. Benjamin, New York, pp. 234–245.

Phillips, N. A., 1963, Geostrophic Motion. *Reviews of Geophysics*, **1**, 2, pp. 123–176.

Prendergast, K. H. 1962, The motion of gas in barred spiral galaxies. *The Distribution and Motion of Interstellar Matter in Galaxies* (ed. L. Woltjer), W. A. Benjamin, New York, pp. 217–221.

Rossby, C. G., 1939, Relation between variations of the intensity of the zonal circulation of the atmosphere and the displacements of the semipermanent centers of action. *J. Marine Res.*, **2**, pp. 38–55.

Starr, V. P., 1963, Galactic dynamics in the light of meteorological theory. *Geofisica Pura e Applicata*, **56**, 174–184.

Thompson, P. D., 1961, *Numerical Weather Analysis and Prediction.* Macmillan, New York, 170 p.

Toomre, A., 1964, On gravitational instability in spiral galaxies. I, Disk of stars. *Ap. J.* 140.