Note on the stability of water stratified by both salt and heat

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ABSTRACT

The stability of water stratified by both salt and heat is considered. A new criterium complementing the one derived earlier by STERN (1960) is found. It is demonstrated that, contrary to ordinary gravitational convection, convection against a stable density gradient will generally occur on a scale limited by internal parameters and not by the dimensions of the vessel. This limits the transporting capacity of the convection and thus reduces its importance. The characteristic scale that will dominate the instability is derived.

Introduction

The stability of water stratified by both heat and salt has already been considered in the case when the salt concentration increases upwards (STERN, 1960; STOMMEL, ARONS, BLANCHARD, 1956). They found that the system can be unstable even in cases where the density increases in the direction of gravity, the instability being due to the difference in the diffusivity of heat and salt. In this paper an additional case of instability is demonstrated together with the one already found by Stommel *et al.* Special emphasis is given to a comparison of the behavior of the frequency relation in these cases and the ordinary gravitationally unstable case.

This work was stimulated by a note from H. Stommel (private communication Jan. 1964) in which he reported on observations of the tendency for a salt stratification to break up into horizontal layers of constant salt concentration when heated from below.

List of symbols

$\mathbf{v} = (u, v, w)$ = velocity vector
P = pressure
ϱ = density
ν = kinematic viscosity
g = acceleration of gravity
T' = temperature
S' = salinity
$T = \alpha_1 T'$ = density anomaly due to heat
$S = -\alpha_2 S'$ = density anomaly due to salt
$\varkappa_1$ = diffusion coefficient of heat
$\varkappa_2$ = diffusion coefficient of salt
β_1 = gradient of density anomaly due to heat
β_2 = gradient of density anomaly due to salt
$p = \sigma + i\lambda$ = time exponent of disturbance
$\mathbf{k} = (k_x, k_y, k_z)$ = wavenumber vector of disturbance

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$\mathbf{r} = (x, y, z)$

$k^2 = k_x^2 + k_y^2 + k_z^2 = k_1^2 + k_z^2$

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Derivation of the frequency relations

In this section we shall derive the relations governing the time development of disturbances on a basic stratification of salt and heat.

The equations of motion and diffusion are within the Boussinesq approximation:

$$\frac{d}{dt}\mathbf{v} = -\frac{1}{\varrho_0}\nabla P + \nu\nabla^2\mathbf{v} - g\frac{\varrho}{\varrho_0}\hat{z}, \qquad (1\,a)$$

$$\frac{d}{dt}T' = \varkappa_1\nabla^2 T', \qquad (1\,b)$$

$$\frac{d}{dt}S' = \varkappa_2\nabla^2 S', \qquad (1\,c)$$

$$\nabla \cdot \mathbf{v} = 0, \tag{1d}$$

$$\varrho = \varrho_0 (1 - \alpha_1 T' + \alpha_2 S'). \tag{1e}$$

To stress the symmetry between the appearance of salt and temperature we introduce:

$$T = \alpha_1 T' \quad \text{and} \quad S = -\alpha_2 S'$$

which gives $\quad \varrho = \varrho_0(1 - (T + S))$.

The fluid is assumed infinite in all directions and the ground state is given by

$$T_1 = \beta_1 z, \quad S_1 = \beta_2 z, \quad u_1 = v_1 = w_1 = 0,$$

$$\frac{1}{\varrho_0}\frac{\partial P_1}{\partial z} = -g(1 - (T_1 + S_1)), \quad \frac{\partial P_1}{\partial x} = \frac{\partial P_1}{\partial y} = 0.$$

Introduce $u = u_1 + u'$ etc. in $(1a-e)$, neglect terms non-linear in the primed quantities and drop the prims.

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2\right) u = -\frac{1}{\varrho_0}\frac{\partial P}{\partial x}, \tag{2a}$$

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2\right) v = -\frac{1}{\varrho_0}\frac{\partial P}{\partial y}, \tag{2b}$$

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2\right) w = -\frac{1}{\varrho_0}\frac{\partial P}{\partial z} + g(T + S), \tag{2c}$$

$$\left(\frac{\partial}{\partial t} - \varkappa_1 \nabla^2\right) T = -w\beta_1, \tag{2d}$$

$$\left(\frac{\partial}{\partial t} - \varkappa_2 \nabla^2\right) S = -w\beta_2, \tag{2e}$$

$$\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v + \frac{\partial}{\partial z} w = 0. \tag{2f}$$

When the linear set of equations $(2a–f)$ are solved for w we get

$$\left(\frac{\partial}{\partial t} - \varkappa_1 \nabla^2\right)\left(\frac{\partial}{\partial t} - \varkappa_2 \nabla^2\right)\left(\frac{\partial}{\partial t} - \nu \nabla^2\right)\nabla^2 w$$
$$= -g\nabla_1^2\left(\left(\frac{\partial}{\partial t} - \varkappa_2 \nabla^2\right) w\beta_1 + \left(\frac{\partial}{\partial t} - \varkappa_1 \nabla^2\right) w\beta_2\right). \tag{3}$$

We consider solutions of the form

$$w = \exp(pt + i\mathbf{k}\cdot\mathbf{r}),$$

where $p = \sigma + i\lambda$, $\mathbf{k} = (k_x, k_y, k_z)$, $\mathbf{r} = (x, y, z)$ and $\sigma, \lambda, k_x, k_y, k_z$ are real quantities.

Inserting in (3) we get after a slight simplification

$$(\sigma + i\lambda + \varkappa_1 k^2)(\sigma + i\lambda + \varkappa_2 k^2)(\sigma + i\lambda + \nu k^2)$$
$$= -g\frac{k_1^2}{k^2}(\beta_1 + \beta_2)\left(\sigma + i\lambda + k^2\frac{\varkappa_2\beta_1 + \varkappa_1\beta_2}{\beta_1 + \beta_2}\right). \tag{4}$$

For the moment we set:

$$\sigma + \varkappa_1 k^2 = a, \quad \sigma + \varkappa_2 k^2 = b, \quad \sigma + \nu k^2 = c$$

$$-gk_1^2/k^2(\beta_1 + \beta_2) = A_1, \quad \sigma + k^2\frac{\varkappa_2\beta_1 + \varkappa_1\beta_2}{\beta_1 + \beta_2} = A_2$$

and obtain from (4)

$$(i\lambda + a)(i\lambda + b)(i\lambda + c) = A_1(i\lambda + A_2).$$

Equalizing real and imaginary parts gives

$$\begin{cases} -\lambda^3 + (ab + bc + ac)\lambda = A_1\lambda, \\ -(a + b + c)\lambda^2 + abc = A_1A_2. \end{cases}$$

This system of equations yields two different cases:

$$\text{A.} \quad \begin{cases} \lambda = 0, \\ abc = A_1A_2; \end{cases}$$

$$\text{B.} \quad \begin{cases} \lambda^2 = ab + bc + ac - A_1, \\ -(a + b + c)(ab + bc + ac) + abc = \\ \qquad = -(a + b + c)A_1 + A_1A_2 \end{cases}$$

or, when reinserting for a, b, etc.

$$\text{A.}\begin{cases} \lambda = 0, \\ (\sigma + \varkappa_1 k^2)(\sigma + \varkappa_2 k^2)(\sigma + \nu k^2) = \\ \quad = -g\frac{k_1^2}{k^2}((\varkappa_2\beta_1 + \varkappa_1\beta_2)k^2 + \sigma(\beta_1 + \beta_2)); \end{cases} \tag{5}$$

$$\text{B.}\begin{cases} \lambda^2 = (\sigma + \varkappa_1 k^2)(\sigma + \varkappa_2 k^2) + (\sigma + \varkappa_1 k^2)(\sigma + \nu k^2) + \\ \quad + (\sigma + \varkappa_2 k^2)(\sigma + \nu k^2) + g\frac{k_1^2}{k^2}(\beta_1 + \beta_2), \\ (3\sigma + (\varkappa_1 + \varkappa_2 + \nu)k^2)((\sigma + \varkappa_1 k^2)(\sigma + \varkappa_2 k^2) + \\ \quad + (\sigma + \varkappa_1 k^2)(\sigma + \nu k^2) + (\sigma + \varkappa_2 k^2)(\sigma + \nu k^2)) - \\ \quad - (\sigma + \varkappa_1 k^2)(\sigma + \varkappa_2 k^2)(\sigma + \nu k^2) = \\ \quad = -g\frac{k_1^2}{k^2}(((\varkappa_1 + \nu)\beta_1 + (\varkappa_2 + \nu)\beta_2)k^2 + \\ \quad + 2\sigma(\beta_1 + \beta_2)). \end{cases} \tag{6}$$

When $\beta_2 = 0$ equation (4) gives

$$(\sigma + i\lambda + \varkappa_1 k^2)(\sigma + i\lambda + \nu k^2) = -g\frac{k_1^2}{k^2}\beta_1.$$

By the same procedure as when $\beta_2 \neq 0$ we get two cases describing the direct and oscillating mode.

$$\text{C.} \quad \begin{cases} \lambda = 0, \\ (\sigma + \varkappa_1 k^2)(\sigma + \nu k^2) = -g\dfrac{k_1^2}{k^2}\beta_1; \end{cases} \qquad (7)$$

$$\text{D.} \quad \begin{cases} \lambda^2 = (\sigma + \varkappa_1 k^2)(\sigma + \nu_2 k^2) + g\dfrac{k_1^2}{k^2}\beta_1 \\ 2\sigma + \varkappa_1 k^2 + \nu_2 k^2 = 0. \end{cases} \qquad (8)$$

Examination of the frequency relations

Equations (5)–(8) describe the development of the direct and oscillating mode with and without salt in the basic stratification.

We shall consider three aspects of equations (5)–(8).

I. Marginal states (e.g. when $\sigma = 0$),
II. Behavior for small k (e.g. large scales),
III. Behavior for intermediate k.

First, we note that equation (8) always gives negative σ. This means that the oscillating mode on a simple heat stratification is always damped. Thus we need not discuss this case further.

I. *The marginal states are obtained by inserting $\sigma = 0$ in equations* (5)–(7).

Thus we obtain:

$$\text{A.} \qquad \frac{(\varkappa_2\beta_1 + \varkappa_1\beta_2)gk_1^2}{\varkappa_1\varkappa_2\nu k^6} = -1, \qquad (9)$$

$$\text{B.} \qquad \frac{((\varkappa_1+\nu)\beta_1 + (\varkappa_2+\nu)\beta_2)gk_1^2}{\sum \cdot k^6} = -1, \qquad (10)$$

where

$$\sum = 2\varkappa_1\varkappa_2\nu + \varkappa_1^2(\nu + \varkappa_2) + \varkappa_2^2(\nu + \varkappa_1) + \nu^2(\varkappa_1 + \varkappa_2),$$

$$\text{C.} \qquad \frac{\beta_1 g k_1^2}{\varkappa_1 \nu k^6} = -1. \qquad (11)$$

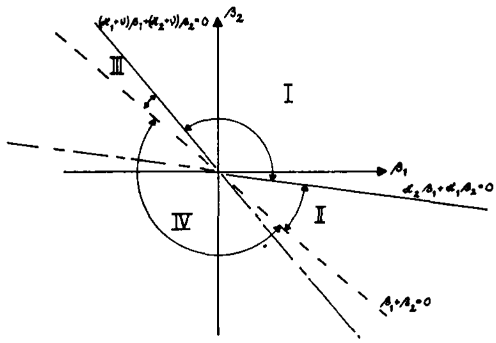

FIG. 1. Limits of stability in the "β-plane". β_1 represents density gradient due to heat and β_2 density gradient due to salt. In regions I–III density decreases upwards. ($\beta_1 + \beta_2 > 0$.) *Region I:* Stable within linear theory. *Region II:* Direct mode instability (case A). *Region III:* Oscillating mode instability (case B). *Region IV:* Instability of essentially the same type as in ordinary thermal convection.

From these conditions can be seen that when a certain linear combination of the gradients β_1 and β_2 is negative there will always exist marginal stability for some value of k (assuming $k_1 \sim k$). Smaller values of k will give positive σ or instability.

The conditions for instability at least for some value of k are thus

$$\text{A.} \qquad \varkappa_2\beta_1 + \varkappa_1\beta_2 < 0, \qquad (12)$$

$$\text{B.} \qquad (\varkappa_1 + \nu)\beta_1 + (\varkappa_2 + \nu)\beta_2 < 0, \qquad (13)$$

$$\text{C.} \qquad \beta_1 < 0. \qquad (14)$$

These conditions are illustrated in Fig. 1.

Equation (14) states the obvious requirement that density increasing upwards gives instability in a simple heat stratification. Equations (12) and (13), however, show that instability can occur even if $\beta_1 + \beta_2 > 0$ as soon as $\varkappa_1 \neq \varkappa_2$. Because $\varkappa_1 \gg \varkappa_2$ equation (12) predicts instability essentially up to the limit $\beta_2 = 0$. Thus a system with the salt concentration increasing upwards can hardly be stabilized by a temperature gradient at all. The instability governed by equation (13) is more restricted because of the appearance of ν in the condition. For temperatures of about 20°C we have $\nu \simeq 7\varkappa_1$ and the unstable region thus extends only a small distance into the region where $\beta_1 + \beta_2 > 0$ (see Fig. 1). Equation (13) is, however, very interesting in that the relative magnitude of three diffusion coefficients enters into the condition.

II. *We now proceed to determine the behavior of σ for small values of k when the conditions* (12), (13) *or* (14) *are fulfilled.*

If we neglect the left-hand side of equations (5) and (6) we get an upper bound on σ which behaves as k^2. This implies that all terms on the left side behave as k^6 for small k, while the right hand side behaves as k^2 (assuming $k_1 \sim k$). Thus the behaviour of σ for small k is obtained by neglecting the left sides of (5) and (6). In equation (7) on the other hand we must keep the term σ^2 on the left side for small k.

Thus for enough small k we have:

$$\text{A.} \qquad \sigma \simeq \sigma_0 = -\frac{\varkappa_2\beta_1 + \varkappa_1\beta_2}{\beta_1+\beta_2}k^2, \tag{15}$$

$$\text{B.} \qquad \sigma \simeq \sigma_0 = -\frac{(\varkappa_1+\nu)\beta_1 + (\varkappa_2+\nu)\beta_2}{2(\beta_1+\beta_2)}k^2, \tag{16}$$

$$\text{C.} \qquad \sigma \simeq \sigma_0 = -\frac{1}{2}\frac{k_1}{k}\sqrt{-g\beta_1}. \tag{17}$$

Equations (15), (16) and (17) show the important fact that for small k, σ is independent of k in case C, while it behaves as k^2 in cases A and B. This fact gives a justification for our treatment in cases A and B in that the disturbances with large scales can be neglected. The behaviour will thus not be influenced by boundaries enough far away. Case C, however, can certainly not be fully examined without the inclusion of boundaries.

III. *For increasing values of k equations (15) and (16) become invalid and σ will reach a maximum at a certain scale.*

This scale might have some relevance for the finite amplitude situation occurring after instability and we shall try to determine it. The simplest and also the most relevant case from a physical viewpoint is when the situation is not very far from the marginal state. This means that σ_0 as defined in (15) or (16) is not too large. Specifically we can assume $\sigma_0 < \varkappa_1 k^2$ when treating case A and $\sigma_0 < \nu k^2$ when treating case B. From equations (15) and (16) can be seen that these assumptions are valid if we are not close to the limit $\beta_1 + \beta_2 = 0$. For other cases the behaviour will be essentially the same, the only difference being a flattening of the maximum for larger σ_0. Furthermore we know that $\nu > \varkappa_1 > \varkappa_2$ and for simplicity we neglect $\varkappa_1$ when compared to ν and $\varkappa_2$ when compared to $\varkappa_1$ in the left side of equations (5) and (6). Using these simplifications equations (5) and (6) give:

$$\text{A.} \quad \sigma \simeq \frac{-gk_1^2(\varkappa_2\beta_1+\varkappa_1\beta_2) - \varkappa_1\varkappa_2\nu k^6}{gk_1^2/k^2(\beta_1+\beta_2)+\varkappa_1\nu k^4}, \tag{18}$$

$$\text{B.} \quad \sigma \simeq \frac{-gk_1^2((\varkappa_1+\nu)\beta_1 + (\varkappa_2+\nu)\beta_2) - \nu^2\varkappa_1 k^6}{gk_1^2/k^2\, 2(\beta_1+\beta_2)+\nu^2 k^4}. \tag{19}$$

The scale of maximum σ will either be determined by the dissipation term in the nominator or denominator. The lowest value of k obtained will be valid. Thus in each case we get two criteria:

$$\text{A.} \qquad \frac{(\varkappa_2\beta_1+\varkappa_1\beta_2)gk_1^2}{\varkappa_1\varkappa_2\nu k^6} = -1 \tag{20a}$$

$$\text{or} \qquad \frac{(\beta_1+\beta_2)gk_1^2}{\varkappa_1\nu k^6} = 1, \tag{20b}$$

$$\text{B.} \qquad \frac{((\varkappa_1+\nu)\beta_1+(\varkappa_2+\nu)\beta_2)gk_1^2}{\nu^2\varkappa_1 k^6} = -1 \tag{21a}$$

$$\text{or} \qquad \frac{(\beta_1+\beta_2)gk_1^2}{\nu^2 k^6} = 1. \tag{21b}$$

The criteria (20a) and (21a) are valid close to the limits defined by (12) and (13), and (20b) and (21b) apply to more unstable cases. However, (20a) is valid within a much smaller range than (21a) because of the appearance of $\varkappa_2$ in the denominator, $\varkappa_2$ being much smaller than $\varkappa_1 (\varkappa_2/\varkappa_1 = 10^{-2})$.

Relation to experiments

No attempt is made here to compare the theoretical predictions about marginal stability and characteristic scales with experiments because no conclusive experimental results have as yet been reported. There are certain severe difficulties to obtain the physical situation to which the theory applies. Experiments reported on the case with a destabilizing salt gradient (case A above) have been made with the initial situation rather like a step in at least the salt distribution. The theory certainly does not apply to this case. The case with a destabilizing heat gradient seems easier to realize by just heating from below on a prefixed salt gradient. However, difficulties arise from the fact that it takes a considerable amount of time to bring the temperature gradient into the liquid. Furthermore, extreme precautions must be taken to eliminate horizontal disturbances from breaking up the stratification into a stepwise structure. Some work in the laboratory at the institution has demonstrated these difficulties.

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