SHORTER CONTRIBUTION

Note on the role of boundary friction in the wind-driven ocean circulation

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(Manuscript received November 26, 1963)

The oceans are subjected to wind stresses that drive the main currents. Because of the stratification of the water the currents decay vertically and at the bottom the stresses are quite small. It has earlier been argued that the torque due to the wind stresses must be balanced by lateral stresses at the coasts, to prevent the oceanic bodies of water from accelerating in rotation. MORGAN (1956) has, however, pointed out that the equation expressing conservation of angular momentum does not require such stresses. This conservation law can only be applied in an inertial system, i.e. one has to consider angular momentum with respect to the earth's axis of rotation. In this system one has contributions from the pressure forces of the coasts as well. If the water level varies along the coast in a suitable way, one may balance the wind stress torque by this effect alone.

More specific information about the role of the boundary friction can be obtained from another equation derived for the vertical component of the vorticity. In this equation the pressure term is eliminated. Assume for the sake of simplicity that the ocean is a body of uniform incompressible water and that the depth is constant. The governing equations are

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} + 2\boldsymbol{\Omega}\cdot\mathbf{v} = -\frac{1}{\varrho_0}\nabla p + \mathbf{g} + \frac{1}{\varrho_0}\nabla\mathbf{F} \tag{1}$$

$$\nabla\cdot\mathbf{v} = 0 \tag{2}$$

where $\mathbf{v}$ is the velocity, p the pressure, ϱ_0 the density, $\mathbf{F}$ the friction force, $\mathbf{g}$ the acceleration of gravity, and $\boldsymbol{\Omega}$ the angular velocity of the earth. $\mathbf{F}$ can be expressed in terms of a stress tensor $\boldsymbol{\tau}$, that includes both laminar and turbulent friction,

$$\mathbf{F} = \nabla\cdot\boldsymbol{\tau}. \tag{3}$$

Applying the curl operator one obtains for the relative vorticity vector $\mathbf{Z}$:

$$\frac{\partial \mathbf{Z}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{Z} - \{(\mathbf{Z}+2\boldsymbol{\Omega})\cdot\nabla\}\mathbf{v} = \frac{1}{\varrho_0}\nabla\cdot\mathbf{F}. \tag{4}$$

Let $\hat{k}$ denote a vertical unit vector, directed upward. By scalar multiplication of (4) by $\hat{k}$ one derives for the vertical vorticity component Z_k:

$$\frac{\partial Z_k}{\partial t} + \mathbf{v}\cdot\nabla(Z_k + 2\Omega_k) - (\mathbf{Z}+2\boldsymbol{\Omega})\,\nabla v_k = \frac{1}{\varrho_0}\hat{k}\cdot(\nabla\cdot\mathbf{F}) \tag{5}$$

$$\frac{\partial Z_k}{\partial t} + \nabla\cdot\{\mathbf{v}(Z_k+2\Omega_k) - v_k(\mathbf{Z}+2\boldsymbol{\Omega})\} = \frac{1}{\varrho_0}\hat{k}\cdot(\nabla\cdot\mathbf{F}).$$

In a steady state one finds by integration over the entire volume of water V (n denoting the normal outward to the boundary surface S):

$$\iint_S \{v_n(Z_k+2\Omega_k) - v_k(Z_n+2\Omega_n)\}dS = \frac{1}{\varrho_0}\iiint_V \hat{k}\cdot\nabla\cdot\mathbf{F}\,dV. \tag{6}$$

At all boundaries $v_n = 0$, hence the first term in the surface integral vanishes. At rigid boundaries $v_k = 0$, for a non-slipping fluid. At the free surface one gets a contribution from the term $v_k Z_n$, but because the free surface is closely horizontal the contribution is very small, about 10^{-4} of the wind stress term in the

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right hand side. To a very close approximation one must thus require

$$\iiint_V \hat{k}\cdot(\nabla\cdot\mathbf{F})\,dV=0. \quad (7)$$

If one divides the ocean in horizontal slices, each with an area A and thickness dz, one can write (7)

$$\int_{-h}^{0} dz \iint_A \hat{k}\cdot(\nabla\cdot\mathbf{F})\,dV=\int_{-h}^{0} dz \oint_B F_s\,ds=0 \quad (7\,a)$$

where h is the depth ($z=0$ is the upper boundary), B denotes the coastal boundary, and s denotes the tangential direction of this boundary. One can also write (7a) as

$$\oint \overline{F}_s\,ds=0 \quad (7\,b)$$

where the bar denotes a vertical integration. If one introduces the horizontal coordinates λ (longitude) and φ (latitude) one has

$$\mathbf{F}=\frac{\partial \boldsymbol{\tau}_z}{\partial z}+\frac{1}{R\cos\varphi}\left\{\frac{\partial}{\partial\varphi}(\cos\varphi\,\boldsymbol{\tau}_\varphi)+\frac{\partial}{\partial\lambda}\boldsymbol{\tau}_\lambda\right\}$$

where R is the radius of the earth, assumed large compared to h. Vertical integration and insertion in (7b) gives

$$\oint \tau_{zs}^{\text{wind}}+\frac{1}{R\cos\varphi}\oint_B\left\{\frac{\partial}{\partial\varphi}(\cos\varphi\,\overline{\tau}_{\varphi s})+\frac{\partial}{\partial\lambda}\overline{\tau}_{\lambda s}\right\}ds=0. \quad (8)$$

The first term represents the integrated wind stress curl, by the circulation theorem,[1] the second integral gives the balancing lateral terms. Now these terms depend only on the horizontal *variation* of the stresses. Thus the boundary stresses themselves may well be zero. This is of interest since it makes it possible to achieve a steady state with a non-viscous model, in which the wind stresses are replaced by equivalent body forces. If such a model becomes unstable and exhibits non-linearly interacting waves or large-scale turbulent motion, internal Reynolds stresses arise that can balance the wind stress. At the boundary the Reynolds stress is zero, since the normal velocity vanishes.

One may easily construct solutions with zero boundary stresses and a net vorticity generation. As example, consider the model given by MUNK (1950). The governing equation is

$$A_H\nabla^4\psi-\beta(\partial\psi/\partial x)=\operatorname{curl}_z\boldsymbol{\tau}_z^{\text{wind}}=T(y) \quad (9)$$

The boundaries are at $x=0$, and $x=a$. ψ is the transport stream function. A_H is the coefficient of horizontal eddy viscosity, and $\beta=(2\Omega\cos\varphi)/R$, both assumed constant in the problem. x and y are quasi-rectangular coordinates, x running eastward and y northward. In a slip-case the boundary conditions are $\psi=\partial^2\psi/\partial x^2=0$ at $x=0$, a. The solution is derived as in Munk's case. If the boundary layer is thin $((\beta/A_H)^{\frac{1}{3}}a\gg 1)$, one can split up the solution in one interior solution

$$\psi_i=\frac{1}{\beta}(a-x)\,T(y) \quad (10\,a)$$

and a western boundary solution

$$\psi_b=\frac{a}{\beta}T(y)\times\left\{1-\frac{2}{\sqrt{3}}e^{-\frac{1}{2}(\beta/A_H)^{\frac{1}{3}}}\times\sin\left[\frac{\sqrt{3}}{2}\left(\frac{\beta}{A_H}\right)^{\frac{1}{3}}x-\frac{\pi}{3}\right]\right\}. \quad (10\,b)$$

One can easily verify that the last solution satisfies the boundary layer equation

$$A_H\frac{\partial^4\psi}{\partial x^4}-\beta\frac{\partial\psi}{\partial x}=0 \quad (11)$$

and the boundary conditions $\psi=\partial^2\psi/\partial x^2=0$ at $x=0$, further it joins the interior solution.

In Fig. 1 the ψ-profiles of this solution and the original solution by Munk are compared. In the latter case the boundary conditions are $\psi=\partial\psi/\partial x=0$ at $x=0$ and a. In the slip case one gets a stronger boundary current and countercurrent, the changes are, however, not very great. The stress condition at the wall is obviously not very critical. In the fig re is also for comparison shown the ψ-profile in the case where the boundary stress has the same magnitude as in Munk's solution, but with an

[1] The circulation theorem seems to give rise to a paradox, that should be examined further. In a viscous fluid all velocity components vanish at the wall, hence also the vertical shear vanish there. It follows that the wind-stress must be zero at the wall, and by the circulation theorem the integrated wind-stress curl is identically zero.

opposite sign. It may be remarked in this connection that recent computations of the momentum transport across the Gulf Stream by WEBSTER (1961) indicate that the Reynolds stresses are, in fact, opposite to the direction expected for a laminar motion, and one may have a boundary stress that accelerates the Gulf Stream. The indication of a countercurrent close to the shore fits in with this result.

The lateral terms in the vorticity balance equation (8) represent diffusion of vorticity across the boundary. It is of interest to note that this diffusion does not require a solid wall, or any kind of "recipient" at the boundary. As example, consider an infinite circular-cylindric body of a viscous fluid, with free boundaries, in vacuum. If we subject this fluid column to axially symmetric forces that constantly generate vorticity one may still arrive at a steady state. The only requirement is that the net torque of the forces vanish, so that total angular momentum is conserved. Let the tangential velocity be u, and the applied force $F_\varphi(r)$, the equation of motion for the column is

$$\frac{\partial u}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \tau_{r\varphi}) + F_\varphi \tag{12}$$

where
$$\tau_{r\varphi} = \mu r \frac{\partial}{\partial r}\left(\frac{u}{r}\right). \tag{13}$$

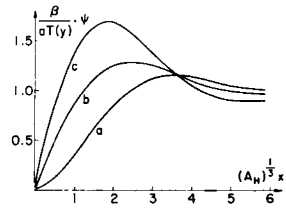

FIG. 1. Profiles of the stream function across the western boundary current in Munk's solution (a), in the slip-case (b), and in the case of a boundary stress opposite to that in Munk's solution (c).

In a steady state $\partial u/\partial t = 0$, and $\tau_{r\varphi} = -\int_0^r F_\varphi r^2 dr$. At the free surface, $r = a$, the stress should vanish, thus $\int_0^a F_\varphi r^2 dr = 0$. This is identical with the requirement that the total angular momentum is conserved. The vorticity equation is, in the steady state

$$\mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \zeta}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r}(rF\varphi) = 0 \tag{14}$$

where $\zeta = 1/r \; \partial(ru)/\partial r$. The last term in (14) represents the generation of vorticity. Its net value, obtained by integration over the cross-section, is $2\pi a(F_\varphi)_{r=a}$. One may easily construct cases where the net torque vanishes while the net vorticity generation does not. One case is, as example, $F_\varphi = C(r - \frac{3}{4}a)$, that leads to the solution $u = -C/8\mu(r^3 - 2ra)$. In all these cases a true steady state exists. The vorticity generated is diffused out to the free boundary, which acts as a vorticity sink.

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