Finite amplitude convection in a conditionally unstable stagnant air mass

By S. M. A. HAQUE, *Florida State University*[1]

(Manuscript received November 5, 1963)

ABSTRACT

The steady-state non-linear equations of motion and heat transport for a conditionally unstable stagnant air mass between two plane horizontal boundaries are solved as a power series of a parameter of instability for the region of vertically upward motion of a disturbance symmetric about a vertical axis. For such a disturbance, which is known from previous studies to have the dimension of a typical cumulus cloud, it is shown that the magnitudes of the different velocity components in the state of maximum development will depend on the assumed values of the kinematic coefficient of eddy viscosity and the Prandtl number. Examples are given with some assumed values of these parameters in which the first term of the power series is calculated. The calculated values of the maximum vertical velocity seem to be of the same order of magnitude as in a typical cumulus cloud of moderate intensity. It is also seen that the disturbance will develop a component of the tangential velocity independent of height, which will be of the second order.

1. Introduction

The question of initiation of thermal convection in the atmosphere has been considered by HAQUE (1952), KUO (1961*a*), LILLY (1960) and SYONO (1953). These investigations have examined the problem of initiation of convection in a conditionally unstable air mass under different types of initial and boundary conditions. One of the results which follow from their works is that in a conditionally unstable stagnant air mass in which there is no variation of temperature and humidity in the horizontal direction, convection will start in the form of cells having the dimension of a cumulus cloud. Another result is that at the initiation of convection, the regions of updraft will be interspersed with regions of downdraft, and the regions of downdraft will be much larger in area than the regions of updraft.

The above investigations only give us the mechanisms which lead to the initiation of convection and the distribution of the various parameters of convection when their amplitude is infinitesimal. Since these analyses are carried out on the basis of linearised equations in which the effects of all non-linear interactions are ignored, they cannot give us any information about the parameters of convection when they assume finite amplitudes as they must when the process starts and develops. On the other hand, the problem of finite amplitude steady thermal convection in a liquid heated from below has been considered by MALKUS and VERONIS (1958), KUO and PLATZMANN (1960) and KUO (1961*b*). These investigations have attempted to derive the parameters of finite amplitude convection by expanding these parameters as Fourier series of the appropriate dimension and by assuming that the amplitude of each of the Fourier components can be expanded as a power series of a function of the excess of steady state instability over marginal instability. For instability which is only slightly in excess of the marginal instability, the first few terms of the power series give us a good approximation of the finite amplitude parameters of convection. It therefore suggests itself that the methods of finite amplitude convection may be applied to atmospheric convection. This is particularly so because in the atmosphere the development will in general be gradual and if the maximum instability does not depart appreciably from the marginal instability, then the evaluation of only a few terms of the expansion will give us useful results. The atmospheric convection problem is however

[1] On leave of absence from the University of Dacca, East Pakistan. Present Address: University of Hawaii.

more complicated because of the presence of moisture and eddy turbulence and the difficulty in choosing suitable boundary conditions. Moreover, any convection in the atmosphere will be essentially unsteady, but it can be seen that the state of maximum development will be quasi-steady and the solution of steady-state equations will give us an insight into the state of maximum development. In what follows we choose the simple case of development of convection as an axi-symmetric cell about the vertical axis in a conditionally unstable stagnant air mass. This corresponds roughly to the development of an isolated cumulus cloud.

2. Equations of convection

The equation of motion in the vector form is taken as

$$\frac{d\mathbf{V}}{dt} + 2\boldsymbol{\omega}\times\mathbf{V} = -\frac{1}{\varrho}\nabla P + \mathbf{g} + \nu\nabla^2\mathbf{V} \tag{1}$$

where $\mathbf{V}$ is the velocity, $\boldsymbol{\omega}$ is the earth's angular velocity, P is the pressure, ϱ is the density, $\mathbf{g}$ is the earth's gravity and ν is the kinematic coefficient of eddy diffusion for momentum. Following HAQUE (1952), we introduce the quantity S by the equation

$$S = \frac{1}{\gamma}\log P - \log \varrho \tag{2}$$

and call it the entropy. When air can be regarded as dry, i.e. when there is no release of the latent heat of condensation, S will be a constant for dry-adiabatic motion if γ is the ratio of the specific heats. For wet-adiabatic motion as well, S will be a constant if γ is regarded as a variable. But the range of variation of γ for the various wet-bulb potential temperatures encountered in the atmosphere will be small, and for our purposes we shall regard γ as a constant for wet-adiabatic motion as well. The important thing to remember is that the values of γ will be different according as whether the motion is dry or saturated.

The equation for the variation of S may then be written as

$$\frac{dS}{dt} = \varkappa\nabla^2 S \tag{3}$$

where $\varkappa$ is the coefficient for the eddy diffusion of S. We also neglect the effect of compressibility in the equation of continuity and write it in the form

$$\nabla\cdot\mathbf{V} = 0 \tag{4}$$

We now assume that the air mass was initially at rest with entropy uniform in the horizontal direction. Owing to gradual increase of entropy downwards, the air mass ultimately became unstable and initial convection developed according to the linearised theory (KUO, 1961*a*). If the process of gain of entropy at the bottom and loss of entropy at the top still continues, equilibrium will be reached at a certain stage and steady convection will prevail. We shall assume that at this state, the entropy difference between the top and the bottom is not appreciably greater than in the state of marginal stability. If we assume that the instability which develops is symmetrical about the vertical axis, and the radial, tangential and vertical components of velocity are v_r, $v_{\tilde{\omega}}$ and v_z, and $S = S_0 + Bz + s$, where S_0 is the prescribed entropy at the bottom, $B = (S_1 - S_0)/h$, where S_1 is the prescribed entropy at the top and h is the total height, then the equations of motion, variation of entropy and continuity, namely (1), (3) and (4), in the steady state can be put in the forms:

$$v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{dz} - \frac{v_{\tilde{\omega}}^2}{r} - f v_{\tilde{\omega}} = -\frac{1}{\varrho}\frac{\partial P}{\partial r} + \nu\left(\nabla^2 - \frac{1}{r^2}\right)v_r \tag{5}$$

$$v_r\frac{\partial v_{\tilde{\omega}}}{\partial r} + v_z\frac{\partial v_{\tilde{\omega}}}{\partial z} + \frac{v_r v_{\tilde{\omega}}}{r} + f v_r = \nu\left(\nabla^2 - \frac{1}{r^2}\right)v_{\tilde{\omega}} \tag{6}$$

$$v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z} = -\frac{1}{\varrho}\frac{\partial P}{\partial z} - g + \nu\nabla^2 v_z \tag{7}$$

$$v_r\frac{\partial s}{\partial r} + v_z\left(B + \frac{\partial s}{\partial z}\right) = \varkappa\nabla^2 s \tag{8}$$

$$\frac{\partial v_z}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r v_r) = 0 \tag{9}$$

From (9), we introduce the stream function Ψ, where

$$v_z = \frac{1}{r}\frac{\partial\Psi}{\partial r}, \quad v_r = -\frac{1}{r}\frac{\partial\Psi}{\partial z}. \tag{10}$$

We can visualize a situation in which the entropy at the bottom is S_0 and the entropy at the top is S_1, and the atmosphere is at rest. This state of rest will, of course, be unstable and the variation of entropy with height will be linear. Let P_0 and ϱ_0 denote the pressure and density in this state, and let $P = P_0 + p$ and $\varrho = \varrho_0 + \varrho'$, so that p and ϱ' give us the departures of pressure and density in the state of steady convection from the unstable state of rest. With these assumptions, we easily get from (2)

$$S_0 + Bz = \frac{1}{\gamma} \log P_0 - \log \varrho_0 \tag{11}$$

and
$$s = \frac{p}{\gamma P_0} - \frac{\varrho'}{\varrho_0}. \tag{12}$$

The hydrostatic relation gives us

$$0 = -g - \frac{1}{\varrho_0}\frac{\partial P_0}{\partial z}. \tag{13}$$

Now we have

$$-\frac{1}{\varrho}\frac{\partial P}{\partial z} = -\frac{1}{\varrho_0 + \varrho'}\frac{\partial}{\partial Z}(P_0 + p)$$

$$= -\frac{1}{\varrho_0}\frac{\partial P_0}{\partial z} + \frac{\varrho'}{\varrho_0^2}\frac{\partial P_0}{dz} - \frac{1}{\varrho_0}\frac{\partial p}{\partial z},$$

neglecting the second order term

$$= g - g\frac{\varrho'}{\varrho_0} - \frac{1}{\varrho_0}\frac{\partial p}{\partial z}$$

$$= g - g\left(\frac{p}{\gamma P_0} - s\right) - \frac{1}{\varrho_0}\frac{\partial p}{\partial z},$$

substituting from (12)

$$= g - \frac{1}{\varrho_0}\frac{\partial p}{\partial z} + \frac{p}{\varrho_0^2}\frac{\partial \varrho_0}{\partial z} + gs + \frac{p}{\varrho_0}\left(\frac{1}{\gamma P_0}\frac{\partial P_0}{\partial z} - \frac{1}{\varrho_0}\frac{\partial \varrho_0}{\partial z}\right),$$

using (13)

$$= g - \frac{1}{\varrho_0}\frac{\partial p}{\partial z} + gs + \frac{p}{\varrho_0}\left(B + \frac{1}{\varrho_0}\frac{\partial \varrho_0}{\partial z}\right), \text{ using (11).}$$

Since in the atmosphere, B has a smaller order of magnitude than $(1/\varrho_0)(\partial \varrho_0/\partial z)$, we write

$$-\frac{1}{\varrho}\frac{\partial P}{\partial z} = g + gs - \frac{\partial}{\partial z}\left(\frac{p}{\varrho_0}\right). \tag{14}$$

Substituting (14) in (7), we get

$$v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z} = gs - \frac{\partial}{\partial z}\left(\frac{p}{\varrho_0}\right) + \nu\nabla^2 v_z. \tag{15}$$

Neglecting the variation of density in the horizontal direction we can write (5) in the form:

$$v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z} - \frac{v_{\tilde\omega}^2}{r} - f v_{\tilde\omega} = -\frac{\partial}{\partial r}\left(\frac{p}{\varrho_0}\right) + \nu\left(\nabla^2 - \frac{1}{r^2}\right)v_r. \tag{16}$$

We can eleminate p/ϱ_0 between (15) and (16), and then by introducing (10), we get

$$\left(-\frac{1}{r}\frac{\partial\Psi}{\partial z}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial\Psi}{\partial r}\frac{\partial}{\partial z} + \frac{1}{r^2}\frac{\partial\Psi}{\partial z}\right)\left(\frac{1}{r}\frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r^2}\frac{\partial\Psi}{\partial r} + \frac{1}{r}\frac{\partial^2\Psi}{\partial z^2}\right)$$

$$+\frac{\partial}{\partial z}\left(\frac{v_{\tilde\omega}^2}{r} + f v_{\tilde\omega}\right) - g\frac{\partial s}{\partial r} - \nu\left(\nabla^2 - \frac{1}{r^2}\right)\left(\frac{1}{r}\frac{\partial^2\Psi}{\partial z^2}\right)$$

$$-\nu\frac{\partial}{\partial r}\nabla^2\left(\frac{1}{r}\frac{\partial\Psi}{\partial r}\right) = 0. \tag{17}$$

Introducing Ψ, (6) and (8) take the forms:

$$-\frac{1}{r}\frac{\partial\Psi}{\partial z}\frac{\partial v_{\tilde\omega}}{\partial r} + \frac{1}{r}\frac{\partial\Psi}{\partial r}\frac{\partial v_{\tilde\omega}}{\partial z} - \frac{f}{r}\frac{\partial\Psi}{\partial z} - \frac{1}{r^2}\frac{\partial\Psi}{\partial z}v_{\tilde\omega}$$

$$= \nu\left(\nabla^2 - \frac{1}{r^2}\right)v_{\tilde\omega} \tag{18}$$

$$-\frac{1}{r}\frac{\partial\Psi}{\partial z}\frac{\partial s}{\partial r} + \frac{1}{r}\frac{\partial\Psi}{\partial r}\frac{\partial s}{\partial z} + \frac{B}{r}\frac{\partial\Psi}{\partial r} = \varkappa\nabla^2 s. \tag{19}$$

We want to find solutions for Ψ, $v_{\tilde\omega}$ and s which satisfy (17), (18), (19) and the necessary boundary conditions.

3. The boundary conditions and the nature of the solutions

In the atmosphere it is difficult to adopt suitable boundary conditions for the region of convection, and even when some boundary conditions have been chosen, it is difficult to find solutions which will satisfy all the boundary conditions and the differential equations. In the case of the linearised equations, the simplest assumption is to assume that the region is bounded by two rigid horizontal boundaries, one at the top and the other at the bottom. For such boundary conditions, complete solutions for non-viscous and adiabatic equations have been found by HAQUE (1952). When eddy diffusion of momentum and heat are taken into account, solutions of the linearised equations can be found, which are very similar in form to the non-viscous solutions, and which satisfy the differential equations, but they do not satisfy the additional boundary conditions (the non-slip conditions) which must be satisfied at rigid boundaries for viscous flows. These solutions cannot thus be regarded as complete solutions of the linearized viscous differential equations and the rigid boundary conditions, but since the additional viscous boundary conditions only affect the nature of the solutions in the immediate neighborhood of the boundaries and does not materially alter the nature of the solutions away from the boundaries, we may take these solutions as approximate solutions which give us the broad features of the motion in regions not very close to the boundaries. As a matter of fact, these solutions satisfy the boundary conditions appropriate to a free surface at the bottom and at the top, and in accepting these solutions we are actually assuming that the rigid surfaces at the top and the bottom can be replaced by free surfaces. Such an assumption will be quite valid in our case, specially in view of the difficulty in formulating appropriate boundary conditions for the atmosphere, as long as we remember that the solutions will not be appropriate in the neighborhood of a possible lower rigid boundary where the non-slip condition will materially affect the nature of the solutions. For the non-linear viscous and adiabatic equations (17), (18) and (19) too, we shall adopt the same boundary conditions at the bottom and at the top, and the same restriction on the nature of the solutions will hold. This is done particularly because we can find approximate solutions of the non-linear equations under these boundary conditions and because the solutions will give us a reasonably valid description of the motion in all regions except the immediate neighborhood of a possible lower rigid boundary.

4. The method of solution of the equations of convection

We now assume that equations (17), (18) and (19) admit of solutions of the forms:

$$\Psi = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \Psi_{mk} \sin \frac{m\pi z}{h} r J_1(n_k r) \tag{20}$$

$$v_{\tilde{\omega}} = \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \Phi_{mk} \cos \frac{m\pi z}{h} J_1(n_k r) \tag{21}$$

$$s = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} S_{mk} \sin \frac{m\pi z}{h} J_0(n_k r) \tag{22}$$

where Ψ_{mk}, Φ_{mk}, and S_{mk} are independent of the space coordinates r and z, h is the height, and the n_k's are real numbers whose values will be given later on.

If we now substitute (20), (21) and (22) in (17), (18), (19), differentiate the equation resulting from (19) with respect to r and then equate the coefficients of $\sin (m\pi z/h) J_1(n_k r)$ and $\cos (m\pi z/h) J_1(n_k r)$ for the different combinations of the values of m and k, we shall get two types of terms in the resulting equations. One set of terms will be linear in the Ψ's, Φ's and S's and the other will be quadratic in the Ψ's, Φ's and the S's. We may say that the latter set of terms arises out of the non-linearity of the equations (17), (18), and (19) and represents interactions between the different components of the harmonics of (20), (21) and (22). If we now put the linear terms on the left-hand side and the interaction terms on the right-hand side, the resulting sets of equations take the following forms:

$$\nu \left[n_k^2 + \left(\frac{m\pi}{h} \right)^2 \right]^2 \Psi_{mk} - g n_k S_{mk} + f \frac{m\pi}{h} \Phi_{mk} = B_{mk} + C_{mk} \tag{23}$$

$$\nu \left[n_k^2 + \left(\frac{m\pi}{h} \right)^2 \right] \Phi_{mk} - \frac{f m\pi}{h} \Psi_{mk} = D_{mk} \tag{24}$$

$$n_k \varkappa \left[n_k^2 + \left(\frac{m\pi}{h} \right)^2 \right] S_{mk} + B n_k^2 \Psi_{mk} = E_{mk} \tag{25}$$

where the B's represent the interactions of Ψ's with themselves, C's represent the interactions of Φ's with themselves, D's represent the interaction of Φ's with the Ψ's, and E's represent the interaction of Ψ's with S's. If we eliminate S_{mk} and Φ_{mk} from (23) with the help of (24) and (25), we get

$$\left[\nu^2\left\{n_k^2+\left(\frac{m\pi}{h}\right)^2\right\}^3+\left(\frac{\nu}{\varkappa}\right)Bgn_k^2+f^2\left(\frac{m\pi}{h}\right)^2\right]\Psi_{mk}$$
$$=\nu\left\{n_k^2+\left(\frac{m\pi}{h}\right)^2\right\}(B_{mk}+C_{mk})+g\left(\frac{\nu}{\varkappa}\right)E_{mk}-\frac{fm\pi}{h}D_{mk}. \tag{26}$$

The above equation shows that when we consider the problem of initiation of convection and ignore the non-linear interaction terms on the right-hand side, the marginal value of B for the growth of cells having the characteristic values m and k is given by

$$-\frac{\nu}{\varkappa}Bg=\left[\nu^2\left\{n_k^2+\left(\frac{m\pi}{h}\right)^2\right\}^3+\left(\frac{fm\pi}{h}\right)^2\right]/n_k^2 \tag{27}$$

(27) shows that the value of B is numerically a minimum when $m=1$, and corresponding to this value of m, the value of n_k which gives the numerically minimum value of B is given by $n_k=(1/\sqrt{2})\pi/h$. In the atmosphere, $(fm\pi/h)^2$ is of a much smaller order of magnitude than the other term in (27), and in calculating the value of n_k which gives the minimum numerical value of B, the term $(fm\pi/h)^2$ has been neglected. Since in the atmosphere convection will initiate from the state of marginal instability which becomes numerically a minimum for $m=1$ and $n_k=(1/\sqrt{2})(\pi/h)$, we shall take these values of m and n_k as the first harmonic of the convection cells, i.e., we shall assume that $n_1=(1\sqrt{2})(\pi/h)$. Let a be the radial length such that n_1a is the first zero of the Bessel function of the first order; then we shall choose n_2, n_3, ... in such a way that n_2a, n_3a ... are the second, third, ... zeros of Bessel's function of the first order.

The marginal value of B for which initiation of convection takes place will be denoted by B_0, and its value from what has been discussed above is given by

$$-\frac{\nu}{k}B_0g=\left[\nu^2\left\{n_1^2+\left(\frac{\pi}{h}\right)^2\right\}^3+\left(\frac{f\pi}{h}\right)^2\right]/n_1^2. \tag{28}$$

We now assume that $B-B_0$ is numerically a small quantity compared with either B or B_0, and put

$$\eta=\left(\frac{B-B_0}{B}\right)^{\frac{1}{2}} \tag{29}$$

Following the method of Kuo (1961*b*), we now assume that Ψ_{mk}, Φ_{mk}, and S_{mk} can be expressed in power series in terms of the powers of η, such that

$$\Psi_{mk}(\eta)=\Psi_{mk,r}\eta^r+\Psi_{mk,r+2}\eta^{r+2}+\Psi_{mk,r+4}\eta^{r+4}+\dots \tag{30}$$

with similar expressions for Φ_{mk} and S_{mk}, where $\Psi_{mk,r}$ is independent of η, and r is called the order of magnitude of $\Psi_{mk}(\eta)$. For Ψ_{11}, Φ_{11}, and S_{11}, we shall take $r=1$; the order of magnitude of Ψ_{mk}, Φ_{mk}, and S_{mk} for other values of m and k will be greater than 1, as they arise by non-linear interaction from the B's, C's, etc.

In theory, we substitute the expansions (30) in the equations (23), (24), (25) and equate the coefficients of the different powers of η to find the values Ψ_{mkl} etc., for different values of m, k, l. In practice, suppose we are interested in calculating Ψ_{mk}'s etc., up to the 2*s*th power of η. In this case, putting

$$\lambda=-\frac{\nu}{\varkappa}Bg \tag{31}$$

we have $$\eta=\left(\frac{\lambda-\lambda_0}{\lambda}\right)^{\frac{1}{2}}$$

giving $$\lambda=\frac{\lambda_0}{1-\eta^2}=\lambda_0(1+\eta^2+\eta^4+\dots).$$

Instead of the above infinite series, we put

$$\lambda=\lambda_0+\lambda_{0S}(\eta^2+\eta^4+\dots+\eta^{2s}) \tag{32}$$

which gives $$\lambda_{0S}=\frac{\lambda_0}{1-\eta^{2s}}. \tag{33}$$

In theory we shall regard λ_{0S} as a constant, and in calculations its value will be given by (33). This is the process adopted by Kuo (1961*b*).

5. Calculation of B_{mk}, C_{mk}, D_{mk}, and E_{mk}

If we substitute (20), (21) and (22) in (17), (18), and (19), and differentiate with respect to r the equation resulting from (19), we find that the non-linear terms involve products of the trigonometric functions and products of Bessel's function of the form $(1/r)J_1(n_ir)$, $J_1(n_jr)$ and $J_0(n_ir)\ J_1(n_jr)$. To evaluate B_{mk}, C_{mk}, D_{mk} and

E_{mk}, it is necessary to express these products of Bessel's functions as a Fourier series in terms of $J_1(n_1 r)$, $J_1(n_2 r)$, $J_1(n_3 r)$, ... in the interval (o, a). Let us assume that

$$\frac{1}{r} J_1(n_i r) J_1(n_j r) = \sum_{k=1}^{\infty} \gamma_{ijk} J_1(n_k r) \tag{34}$$

and
$$J_0(n_i r) J_1(n_j r) = \sum_{k=1}^{\infty} \delta_{ijk} J_1(n_k r) \tag{35}$$

where it can be easily seen that

$$\gamma_{ijk} = \frac{2}{a J_0^2(j_k)} \int_0^1 J_1(j_i \zeta) J_1(j_j \zeta) J_1(j_k \zeta) d\zeta \tag{36}$$

and
$$\delta_{ijk} = \frac{2}{J_0^2(j_k)} \int_0^1 \zeta J_0(j_i \zeta) J_1(j_j \zeta) J_1(j_k \zeta) d\zeta \tag{37}$$

where j_i is the ith zero of $J_1(\zeta)$.

The values of γ_{ijk} and δ_{ijk}, calculated on a desk calculator, for several values of the suffixes, are given in Appendix A. No degree of accuracy is claimed for these coefficients, but they are accurate enough for the calculations that we make later.

The values of B_{mk}, C_{mk}, D_{mk}, and E_{mk}, for $m = 1, 2, 3, 4$ and $k = 1$ are given in Appendix B in the form of infinite series. The values of these quantities for $k \neq 1$, can also be written down easily from the corresponding expression for $k = 1$, simply by substituting γ_{ijk} and δ_{ijk} for γ_{ij1} and δ_{ij1} respectively.

6. Calculation of the spectral functions

Let us put

$$\lambda_{mk} = \frac{\nu^2 \left[n_k^2 + \left(\frac{m\pi}{h} \right)^2 \right]^3 + f^2 \left(\frac{m\pi}{h} \right)^2}{n_k^2} \tag{38}$$

Equation (26) then takes the form

$$(\lambda_{mk} - \lambda) \Psi_{mk} = \nu \frac{n_k^2 + \left(\frac{m\pi}{h} \right)^2}{n_k^2} (B_{mk} + C_{mk}) + \frac{g}{n_k^2} \left(\frac{\nu}{\varkappa} \right) E_{mk} - \frac{(fm\pi)/h}{n_k^2} D_{mk} \tag{39}$$

Making substitution from (32) in (39), we get

$$(\lambda_{mk} - \lambda_0) \Psi_{mk} = \lambda_0 S \Psi_{mk} \sum_{j=1}^{s} \eta^{2j} + \nu \frac{n_k^2 + \left(\frac{m\pi}{h} \right)^2}{n_k^2} (B_{mk} + C_{mk}) + \frac{g}{n_k^2} \left(\frac{\nu}{\varkappa} \right) E_{mk} - \frac{(fm\pi)/h}{n_k^2} D_{mk}. \tag{40}$$

Expanding Ψ_{mk}, B_{mk}, etc. in powers of η, and equating the coefficients of η^{r+2j} from both sides, we have

$$(\lambda_{mk} - \lambda_0) \Psi_{mk,\, r+2j} = \lambda_0 S \sum_{p=0}^{j-1} \Psi_{mk,\, r+2p} + \nu \frac{n_k^2 + \left(\frac{m\pi}{h} \right)^2}{n_k^2} (B_{mk,\, r+2j} + C_{mk,\, r+2j}) + \frac{g}{n_k^2} \left(\frac{\nu}{\varkappa} \right) E_{mk,\, r+2j} - \frac{(fm\pi)/h}{n_k^2} D_{mk,\, r+2j} \tag{41}$$

where $B_{mk,\, r+2j}$ etc. are the coefficients of η^{r+2j}, when B_{mk} etc. are expanded in ascending powers of η. If we now put $m = k = 1$, $r = 1$, $j = 0$, in (41), we get

$$(\lambda_{11} - \lambda_0) \Psi_{111} = 0 \tag{42}$$

as B_{11} etc., being interaction terms, are necessarily of an order higher than unity. Since $\Psi_{111} \neq 0$, (42) gives

$$\lambda_{11} = \lambda_0 = \left[\nu^2 \left\{ n_1^2 + \left(\frac{\pi}{h} \right)^2 \right\}^3 + \left(\frac{f\pi}{h} \right)^2 \right] \Big/ n_1^2 \tag{43}$$

a result which we have already deduced in (28). Now, putting $m = k = 1$, $r = 1$ and $j = p$ in (41), we get

$$\lambda_0 S \sum_{j=1}^{p} \Psi_{11,\, 2j-1} + \nu \frac{n_1^2 + (\pi/h)^2}{n_1^2} (B_{11,\, 2p+1} + C_{11,\, 2p+1}) - \frac{f\pi/h}{n_1^2} D_{11,\, 2p+1} + \frac{g}{n_1^2} \left(\frac{\nu}{\varkappa} \right) E_{11,\, 2p+1} = 0 \tag{44}$$

Putting $p = 1$ in the above, we get

$$\lambda_0 S \Psi_{111} + \nu \frac{n_1^2 + (\pi/h)^2}{n_1^2} (B_{113} + C_{113}) - \frac{f\pi/h}{n_1^2} D_{113} + \frac{g}{n_1^2} \left(\frac{\nu}{\varkappa} \right) E_{113} = 0 \tag{45}$$

Equation (45) enables us to determine Ψ'_{111}; but before we are able to do so, it is necessary to calculate B_{113} etc. from Appendix B.

If we put $m=1$, $r=1$, $j=0$, $k\neq 1$ in (41), we get

$$(\lambda_{1k}-\lambda_0)\Psi'_{1k1}=0. \tag{46}$$

Since $\lambda_{1k}\neq\lambda_0$ for $k\neq 1$, we get from (46) that

$$\Psi'_{1k1}=0\,(k\neq 1). \tag{47}$$

From equation (B. 1) we can easily see that

$$B_{113}=\frac{\pi}{h}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\Psi'_{1i1}\Psi'_{2j2}[(2\alpha_{1i}^2\gamma_{1j1}-\alpha_{2j}^2\gamma_{ji1}) - (n_i\delta_{ij1}+\tfrac{1}{2}n_j\delta_{ji1})(\alpha_{1i}^2-\alpha_{2j}^2)]. \tag{48}$$

Using (47), (48) reduces to

$$B_{113}=\frac{\pi}{h}\Psi'_{111}\sum\Psi'_{2j2}[(2\alpha_{11}^2\gamma_{1j1}-\alpha_{2j}^2\gamma_{j11}) - (n_1\delta_{1j1}+\tfrac{1}{2}n_j\delta_{j11})(\alpha_{11}^2-\alpha_{2j}^2)]. \tag{49}$$

In exactly the same way, using equations (B. 5), (B. 10), and (B. 14), we get

$$C_{113}=-\frac{\pi}{h}\frac{f\pi/h\nu}{n_1^2+(\pi/h)^2}\Psi'_{111}\sum\frac{2\pi f/h\nu}{n_j^2+(2\pi/h)^2}\gamma_{1j1}\Psi'_{2j2} \tag{50}$$

$$D_{113}=\frac{\pi}{h}\Psi'_{111}\sum\phi_{0j2}n_j\delta_{j11}+\frac{\pi}{h}\frac{f\pi}{h\nu}\Psi'_{111}\sum\Psi'_{2j2}\left\{\frac{1}{n_1^2+(\pi/h)^2}+\frac{2}{n_j^2+(2\pi/h)^2}\right\}(n_1\delta_{1j1}+\tfrac{1}{2}n_j\delta_{j11}) \tag{51}$$

$$\begin{aligned}E_{113}=\frac{\pi}{h}\Bigg[&\Psi'_{111}\sum\Bigg\{-\frac{B_0n_j}{\varkappa[n_j^2+(2\pi/h)^2]}\Psi'_{2j2} + \frac{\pi}{h}\Psi'^2_{111}\frac{B_0n_1^2\gamma_{11j}}{n_j\varkappa^2[n_j^2+(2\pi/h)^2][n_1^2+(\pi/h)^2]}\Bigg\}\\&\{(n_1^2+\tfrac{1}{2}n_j^2)\delta_{j11}+\tfrac{3}{2}n_1n_j\delta_{1j1}-n_j\gamma_{1j1}\}\\&+\Psi'_{111}\frac{B_0n_1}{\varkappa[n_1^2+(\pi/h)^2]}\sum\Psi'_{2j2}\\&\{(n_1^2+\tfrac{1}{2}n_j^2)\delta_{1j1}+\tfrac{3}{2}n_1n_j\delta_{1j1}-2n_1\gamma_{j11}\}\Bigg].\end{aligned} \tag{52}$$

In deducing the above expressions for C_{113}, D_{113} and E_{113}, we have made use of the following relations:

$$\phi_{111}=\frac{f\pi/h\nu}{n_1^2+(\pi/h)^2}\Psi'_{111};\ \phi_{1k1}=0\,(k\neq 1) \tag{53}$$

$$S_{111}=-\frac{B_0n_1}{\varkappa[n_1^2+(\pi/h)^2]}\Psi'_{111};\ S_{1k1}=0\,(k\neq 1) \tag{54}$$

$$\phi_{2k2}=\frac{2\pi f/h\nu}{n_k^2+(2\pi/h)^2}\Psi'_{2k2} \tag{55}$$

$$\varkappa\left[n_k^2+\left(\frac{2\pi}{h}\right)^2\right]S_{2k2}=-B_0n_k\Psi'_{2k2}+\frac{\pi}{h}\frac{B_0n_1^2\gamma_{11k}}{n_k\varkappa[n_1^2+(\pi/h)^2]}\Psi'^2_{111} \tag{56}$$

$$B_{2k2}=-(\pi/h)\,\alpha_{11}^2\gamma_{11k}\Psi'^2_{111} \tag{57}$$

$$C_{2k2}=-\frac{\pi}{h}\frac{(f\pi/h\nu)^2}{[n_1^2+(\pi/h)^2]^2}\Psi'^2_{111} \tag{58}$$

$$D_{2k2}=0 \tag{59}$$

$$E_{2k2}=\frac{\pi}{h}\frac{B_0n_1^2\gamma_{11k}}{\varkappa[n_1^2+(\pi/h)^2]}\Psi'^2_{111} \tag{60}$$

which can be obtained from equations (24), (25), (B.2), (B.6), (B.11), and (B.15).

If we put $m=2$, $r=0$, $j=1$ in (41) and make the necessary substitutions, we get

$$(\lambda_{2k}-\lambda_0)\Psi'_{2k2}=\frac{\pi}{h}\frac{n_1^2\gamma_{11k}}{\nu n_k^2[n_1^2+(\pi/h)^2]}\left[-\lambda_{11}\frac{n_k^2+(2\pi/h)^2}{n_1^2+(\pi/h)^2}-\sigma\lambda_0\right]\Psi'^2_{111} \tag{61}$$

where $\sigma=\nu/\varkappa$, is the Prandtl number. From (24) and (B.9), putting $n=0$ we get

$$\nu n_j^2\phi_{0j2}=\frac{\pi}{h}\phi_{111}\Psi'_{111}n_1\delta_{11j} \tag{62}$$

If we now make substitutions in (45) and cancel the non-zero factor Ψ'_{111}, we get the following relation:

$$\lambda_0 s \Psi_{111}^{-2}\left(\frac{\pi}{h}\right)^{-2} + \sigma\frac{B_0 g}{\varkappa^2 \alpha_{11}^2}\sum\frac{\gamma_{11j}}{n_j \alpha_{2j}^2}$$

$$\times \left\{(n_1^2+\tfrac{1}{2}n_j^2)\,\delta_{j11}+\tfrac{3}{2}n_1 n_j \delta_{ij1}-n_j\gamma_{1j1}\right\}$$

$$+\sum\frac{\gamma_{11j}}{(\lambda_{2j}-\lambda_0)\,n_j^2}\left\{-\lambda_{11}\frac{\alpha_{2j}^2}{\alpha_{11}^2}-\sigma\lambda_0\right\}\{(2\alpha_{11}^2\gamma_{1j1}$$

$$-\alpha_{2j}^2\gamma_{j11})-(n_1\delta_{1j1}+\tfrac{1}{2}n_j\delta_{j11})(\alpha_{11}^2-\alpha_{2j}^2)\}$$

$$-2\frac{(f\pi/h\nu)^2}{\alpha_{11}^2}\sum\frac{\gamma_{11j}\gamma_{1j1}}{n_j^2(\lambda_{2j}-\lambda_0)}\left(-\frac{\lambda_{11}}{\alpha_{11}^2}-\frac{\sigma\lambda_0}{\alpha_{2j}^2}\right)$$

$$-\frac{(f\pi/h\nu)^2}{n_1\alpha_{11}^2}\sum\frac{\delta_{j11}\delta_{11j}}{n_j}-\frac{(f\pi/h\nu)^2}{\alpha_{11}^2}\sum\frac{\gamma_{11j}}{n_j^2(\lambda_{2j}-\lambda_0)}$$

$$\times\left(-\lambda_{11}\frac{\alpha_{2j}^2}{\alpha_{11}^2}-\sigma\lambda_0\right)\left(\frac{1}{\alpha_{11}^2}+\frac{2}{\alpha_{2j}^2}\right)(n_1\delta_{1j1}+\tfrac{1}{2}n_j\delta_{j11})$$

$$-\sigma\frac{B_0 g}{\varkappa\nu\alpha_{11}^2}\sum\frac{\gamma_{11j}}{(\lambda_{2j}-\lambda_0)\,n_j}\left\{-\frac{\lambda_{11}}{\alpha_{11}^2}-\frac{\sigma\lambda_0}{\alpha_{2j}^2}\right\}\{(n_1^2$$

$$+\tfrac{1}{2}n_j^2)\,\delta_{j11}+\tfrac{3}{2}n_1 n_j\delta_{1j1}-n_j\gamma_{1j1}\}+\sigma\frac{B_0 g}{\varkappa\nu}\left(\frac{n_1}{\alpha_{11}^4}\right)$$

$$\times\sum\frac{\gamma_{11j}}{n_j^2(\lambda_{2j}-\lambda_0)}\left\{-\lambda_{11}\frac{\alpha_{2j}^2}{\alpha_{11}^2}-\sigma\lambda_0\right\}\{(n_1^2+\tfrac{1}{2}n_j^2)\,\delta_{1j1}$$

$$+\tfrac{3}{2}n_1 n_j\delta_{j11}-2n_1\gamma_{j11}\}=0. \tag{63}$$

The above relation gives us the value of Ψ_{111} in terms of other known quantities. We have adopted the general method adopted in KUO (1961*b*) to deduce (63) and the same method can be used to deduce Ψ_{112}, Ψ_{212}, etc. But since the expressions for these quantities will necessarily be more complicated, and their contribution to Ψ will be small for small η, we shall only calculate Ψ_{111}.

7. Examples and comments

From the theory of initiation of convective disturbances in the atmosphere (KUO, 1961*a*), we know that convection will start when the value of λ_0 is given by (43). The horizontal scale of the disturbance is given by n_1 and we choose the value of n_1 which minimizes λ_0. Since the effect of the term in (43) containing the Coriolis parameter will be small, λ_0 will be a minimum when

$$n_1 = \frac{1}{\sqrt{2}}\left(\frac{\pi}{h}\right). \tag{64}$$

We shall assume this value of n_1 in calculating Ψ_{111} from (63). Since $n_1 a = 3.832$, the first zero of Bessel's function of order 1, (64) shows that we shall have

$$a = 1.7256\,h. \tag{64a}$$

We, therefore, see from (63) that the value of Ψ_{111} will depend on h, the height of the convective layer, and the parameters ν and σ.

We have few guides for the choice of a value of ν appropriate to a convective air mass. The choice of a constant value is by itself an approximation. The value of ν will probably increase with h, but the choice of a constant value will give us a good approximation when the range of variation of h is small. We know, however, from JORDAN (1958) and KUO (1961*a*) that the observed instability in a tropical air mass corresponds to the decrease of potential wet-bulb temperature by only a few degrees Centigrade, and the height of the unstable mass is of the order of a few kilometers. With values of h and λ_0 of these orders of magnitude the value $\nu = 10^7$ cm^2 sec^{-1} seems to work out rather well in (43), and has been taken for our illustrations. This is also the value of ν adopted by KUO (1961*a*). For this value of ν and $\sigma = 1$, Table 1 gives the values of the differences in potential wet-bulb temperatures between the top and bottom of a column of saturated air mass of height h, which will lead to the initiation of convective motion according to equation (43). For values of σ other than 1, the corresponding values of $\Delta\theta$ will be found by multiplying the values in Table 1 by the reciprocal of σ.

TABLE 1.

H (km)	$\Delta\theta$ (°C)
1	19.78
2	2.46
3	0.73
4	0.31
5	0.16

We have even less guidance for the choice of σ, the Prandtl number appropriate to a convective air mass, than in the choice of ν, the kinematic coefficient of eddy viscosity. The value $\sigma = 1$ is usually chosen, as in KUO (1961*a*), which assumes that the kinematic coefficients of eddy transport of momentum and of heat and moisture for neutralizing the gradient in potential wet-bulb temperature are equal. But since in the atmosphere we find rather considerable gradients in momentum, but only small variations in the potential wet-bulb temperature of a saturated air mass, it is quite possible that an appropriate value of σ for an unstable air mass will be rather less than 1. We give below three examples, in the first two of which we take $\sigma = 1$ and in the third, we take $\sigma = 0.1$.

Example 1. $h = 2$ km, $\sigma = 1$, $\nu = \varkappa = 10^7$ cm^2 sec^{-1}. From (43) we get $\lambda_0 = -B_0 g = 4.11 \times 10^{-5}$. This corresponds to an initial potential wet-bulb temperature difference of 2.5°C to start convection. Substituting in (63) and reducing, we get

$$4.07 \times 10^9 \lambda_{0S} - 14.93 \times 10^{-12} \Psi_{111}^2 = 0.$$

Since we are retaining only one term in the expansion (30), we put $s = 1$ in (32). That gives $\lambda_{01} = \lambda_0/(1-\eta^2)$ and reduces the above equation to

$$\Psi_{111} = 1.06 \times 10^8 \left(\frac{\lambda}{\lambda_0} - 1\right)^{\frac{1}{2}}$$

Example 2. $h = 1.5$ km, $\sigma = 1$, $\nu = \varkappa = 10^7$ cm^2 sec^{-1}. From (43) we get $\lambda_0 = -B_0 g = 1.29 \times 10^{-4}$. This corresponds to an initial potential wet-bulb temperature difference of 5.8°C to start convection. Following the same procedure as in Example 1, we get

$$\Psi_{111} = 1.06 \times 10^8 \left(\frac{\lambda}{\lambda_0} - 1\right)^{\frac{1}{2}}.$$

Example 3. $h = 5$ km, $\sigma = 0.1$, $\nu = 10^7$ cm^2 sec^{-1}, $\varkappa = 10^8$ cm^2 sec^{-1}. From (43), we get $\lambda_0 = -\sigma B_0 g = 1.06 \times 10^{-6}$, which corresponds to a potential wet-bulb temperature difference of 1.6°C to start convection. Following the procedure of Example 1, we get

$$\Psi_{111} = 2.16 \times 10^8 \left(\frac{\lambda}{\lambda_0} - 1\right)^{\frac{1}{2}}.$$

In order to make the results of the examples more comprehensible, let us calculate the values

TABLE 2. *Values of $(v_z)_{max}$ in km hr^{-1} for different values of $((\lambda/\lambda_0) - 1)$ for the three examples.*

	Values of $\lambda/\lambda_0 - 1$				
	0.1	0.2	0.3	0.4	0.5
Example 1	13.59	19.12	23.36	26.76	30.16
Example 2	18.09	25.43	31.09	35.61	40.13
Example 3	11.06	15.55	19.01	21.77	24.54

of the maximum vertical velocities arising out of Stoke's stream functions deduced in the examples. From (10) we can calculate the maximum value of v_z if Ψ is known. Applying to the examples, we find that

$$(v_z)_{max} = 1.18 \times 10^3 \left(\frac{\lambda}{\lambda_0} - 1\right)^{\frac{1}{2}} \text{ in Example 1}$$

$$= 1.57 \times 10^3 \left(\frac{\lambda}{\lambda_0} - 1\right)^{\frac{1}{2}} \text{ in Example 2}$$

$$= 9.6 \times 10^2 \left(\frac{\lambda}{\lambda_0} - 1\right)^{\frac{1}{2}} \text{ in Example 3.}$$

The values of $(v_z)_{max}$ for different values of $((\lambda/\lambda_0) - 1)$ for the three examples are given in Table 2.

We can also calculate from (10) the maximum value of v_r that will occur. However, it can be easily seen that $|(v_r)_{max}|/(v_z)_{max} = 0.82$. The maximum v_z will occur in the center of the region of the updraft and the maximum v_r will occur near the outer boundary of this region, but their magnitudes will be of the same order.

The value of $v_{\tilde{\omega}}$ can be calculated from (21) and (53). However, if we compare $(v_{\tilde{\omega}})_{max}$ with $(v_z)_{max}$, we find that the ratio of $(v_{\tilde{\omega}})_{max}$ to $(v_z)_{max}$ in the three examples are respectively, 0.01, 0.006 and 0.07. The Coriolis parameter, therefore, does not play an effective part in an isolated convection cell of this type and does not give rise to any appreciable cross-radial velocity. It may be noted that the Coriolis parameter does not materially affect the value of Ψ_{111} as given by (62); the terms containing the Coriolis parameter are of a much smaller order of magnitude than other terms.

It may be noted that $v_{\tilde{\omega}}$, the tangential component of velocity, will have a component which is independent of height. This is evident from (24), where by putting $m = 0$, we find that ϕ_{0j} will have a non-zero value. However, the order of ϕ_{0j} will be 2, and since we are interested only in quantities of the first order, this has not been calculated.

We have so far examined the question of development of convection cells in a conditionally unstable stagnant air mass. The method can be applied for any arbitrary difference in the potential wet-bulb temperature between the top and the bottom of the convective layer, but since we have calculated only the first order terms mainly because of the complexity in the calculation of the higher order terms, the results are applicable only when the actual difference between the potential wet-bulb temperatures of the top and the bottom of the convective layer does not exceed appreciably the corresponding marginal difference which is necessary for starting convection. It would seem that this would be the case for isolated convective clouds of moderate intensity. The results obtained from the examples are, of course, dependent on the values chosen for ν and σ, but the values in Table 2 show that the values chosen for ν and σ in the examples are perhaps not altogether unreasonable. It may be mentioned that the value chosen for σ has appreciable effect on equation (43), i.e. it affects the conditions which will initiate the process of convection, but it has very little effect on equation (63), i.e., it does not appreciably affect the values of the variables of state and motion in the fully developed state of convection.

It should be noted that our observations apply only to the region where upward motion takes place, i.e., from $r = 0$ to the value of r whose approximate value is given by $J_0(n_1 r) = 0$. Beyond this value of r, there will be downward motion and although the series expansions (34) and (35) have been made between $r = 0$ and $r = a$, the physical results are only applicable in the region of upward motion.

REFERENCES

HAQUE, S. M. A., 1952, The initiation of cyclonic circulation in a vertically unstable stagnant air mass. *Quart. J. R. Meteor. Soc.*, **78**, pp. 394–406.

JORDAN, C. L., 1958, Mean soundings for West Indies Area. *J. Meteor.*, **15**, pp. 91–97.

KUO, H. L., 1961*a*, Convection in conditionally unstable atmosphere. *Tellus*, **13**, pp. 441–459.

KUO, H. L., 1961*b*, Solution of the non-linear equations of cellular convection and heat transport. *J. Fluid Mech.*, **10**, pp. 611–634.

KUO, H. L., and PLATZMANN, G. W., 1961, A normal mode non-linear solution of the Rayleigh convection problem. *Beit. Phy. frei Atmos.*, **33**, 137–168.

LILLY, D. K., 1960, On the theory of disturbances in a conditionally unstable atmosphere. *Mon. Wea. Rev.*, U.S.W.B., **88**(1), pp. 1–17.

MALKUS, W. V. R., and VERONIS, G., 1958, Finite amplitude cellular convection. *J. Fluid. Mech.*, **4**, pp. 225–260.

SYONO, S., 1953, On the formation of tropical cyclones. *Tellus*, **5**, pp. 179–195.

Appendix A

Values of $\gamma_{ijk}\, a$

	k							
ij	1	2	3	4	5	6	7	8
11	1.0148	0.5441	0.0456	0.0214				
12	0.3021	1.0728	0.5229	−0.0512				
22	0.5813	0.8159	1.0797	0.4650				
13	−0.0175	0.3620	1.0861	0.4922				
14	0.0063	−0.0271	0.3766	1.0860	0.4905			
23	0.2010	0.7475	0.8994	1.0470	0.4496			
24	0.0150	0.2463	0.8010	0.9237	0.9909	0.5438		
33	0.4175	0.6227	0.9161	0.9677	1.1738	0.4334		
34	0.1448	0.5545	0.7403	0.9690	0.9873	0.9994	0.4278	
44	0.3194	0.4892	0.7413	0.8524	1.0199	1.0025	0.9990	0.4603

Values of δ_{ijk}

	k							
ij	1	2	3	4	5	6	7	8
11	0.1763	0.4388	0.0129	−0.0016				
12	0.2436	0.2567	0.4150	0.0068				
21	−0.1805	0.0831	0.3759	0.1178				
22	0.0461	0.0779	0.1609	0.3407				
13	0.0050	0.2873	0.2731	0.3958	0.0172			
31	−0.0062	−0.2169	0.0532	0.3504	0.0107			
14	0.0005	0.0036	0.3028	0.2782	0.3895	0.0148		
41	0.0011	−0.0082	−0.2351	0.0422	0.3383	0.0116		
23	0.1445	0.1114	0.1131	0.1751	0.3236	0.0131		
32	0.1204	−0.0082	0.0495	0.1337	0.3115	0.0116		
24	0.0346	0.1805	0.1340	0.1235	0.1783	0.3128	0.0168	
42	−0.0045	−0.1518	−0.0265	0.0386	0.1195	0.2970	0.0147	
33	0.0204	0.0343	0.0612	0.0856	0.1461	0.2911	0.0114	
34	0.1030	0.0708	0.0655	0.0793	0.2004	0.1502	0.2823	
43	−0.0904	−0.0183	0.0146	0.0443	0.0738	0.1324	0.2761	
44	0.0124	0.0205	0.0339	0.0440	0.0610	0.0865	0.1400	0.2723

Appendix B

Values of B_{mk}, C_{mk}, D_{mk} and E_{mk}

$$\alpha_{mk}^2 = (m\pi/h)^2 + n_k^2$$

$$B_{11} = \frac{\pi}{h}\sum\sum \Psi'_{1i}\Psi'_{2j}\left\{(2\alpha_{1i}^2\gamma_{ij1} - \alpha_{2j}^2\gamma_{ji1}) - (n_i\delta_{ij1} + \tfrac{1}{2}n_j\delta_{ji1})(\alpha_{1i}^2 - \alpha_{2j}^2)\right\}$$

$$+ \frac{\pi}{h}\sum\sum \Psi'_{2i}\Psi'_{3j}\left\{(3\alpha_{2i}^2\gamma_{ij1} - 2\alpha_{3j}^2\gamma_{ji1}) - (\tfrac{3}{2}n_i\delta_{ij1} + n_j\delta_{ji1})(\alpha_{2i}^2 - \alpha_{3j}^2)\right\}$$

$$+ \frac{\pi}{h}\sum\sum \Psi'_{3i}\Psi'_{4j}\left\{(4\alpha_{3i}^2\gamma_{ij1} - 3\alpha_{4j}^2\gamma_{ji1}) - (2n_i\delta_{ij1} + \tfrac{3}{2}n_j\delta_{ji1})(\alpha_{3i}^2 - \alpha_{4j}^2)\right\}$$

$$+ \frac{\pi}{h}\sum\sum \Psi'_{4i}\Psi'_{5j}\left\{(5\alpha_{4i}^2\gamma_{ij1} - 4\alpha_{5j}^2\gamma_{ji1}) - (\tfrac{5}{2}n_i\delta_{ij1} + 2n_j\delta_{ji1})(\alpha_{4i}^2 - \alpha_{5j}^2)\right\}$$

$$+ \dots\dots\dots \tag{B.1}$$

$$B_{21} = \frac{\pi}{h}\sum\sum \Psi'_{1i}\Psi'_{ij}\left\{-\alpha_{1i}^2\gamma_{ij1} + \tfrac{1}{2}\alpha_{1i}^2(n_i\delta_{ij1} - n_j\delta_{ii1})\right\}$$

$$+ \frac{\pi}{h}\sum\sum \Psi'_{1i}\Psi'_{3j}\left\{(3\alpha_{1i}^2\gamma_{ij1} - \alpha_{3j}^2\gamma_{ij1}) - (\tfrac{3}{2}n_i\delta_{ij1} + \tfrac{1}{2}n_j\delta_{ji1})(\alpha_{1i}^2 - \alpha_{3j}^2)\right\}$$

$$+ \frac{\pi}{h}\sum\sum \Psi'_{2i}\Psi'_{4j}\left\{(4\alpha_{2i}^2\gamma_{ij1} - 2\alpha_{4j}^2\gamma_{ji1}) - (2n_i\delta_{ij1} + n_j\delta_{ji1})(\alpha_{2i}^2 - \alpha_{4j}^2)\right\}$$

$$+ \frac{\pi}{h}\sum\sum \Psi'_{3i}\Psi'_{5j}\left\{(5\alpha_{3i}^2\gamma_{ij1} - 3\alpha_{5j}^2\gamma_{ji1}) - (\tfrac{5}{2}n_i\delta_{ij1} + \tfrac{3}{2}n_j\delta_{ji1})(\alpha_{3i}^2 - \alpha_{5j}^2)\right\}$$

$$+ \dots\dots\dots \tag{B.2}$$

$$B_{31}=\frac{\pi}{h}\sum\sum\Psi_{1i}\Psi_{2j}\{-(2\alpha_{1i}^2\gamma_{ij1}+\alpha_{2j}^2\gamma_{ji1})+(n_i\delta_{ij1}-\tfrac{1}{2}n_j\delta_{ji1})(\alpha_{1i}^2-\alpha_{2j}^2)\}$$
$$+\frac{\pi}{h}\sum\sum\Psi_{1i}\Psi_{4j}\{(4\alpha_{1i}^2\gamma_{ij1}-\alpha_{4j}^2\gamma_{ji1})-(2n_i\delta_{ij1}+\tfrac{1}{2}n_j\delta_{ji1})(\alpha_{1i}^2-\alpha_{4j}^2)\}$$
$$+\frac{\pi}{h}\sum\sum\Psi_{2i}\Psi_{5j}\{(5\alpha_{2i}^2\gamma_{ij1}-2\alpha_{5j}^2\gamma_{ji1})-(\tfrac{5}{2}n_i\delta_{ij1}+n_j\delta_{ji1})(\alpha_{2i}^2-\alpha_{5j}^2)\}$$
$$+\;.\;.\;.\;.\;.\;.\;.\;. \tag{B.3}$$

$$B_{41}=\frac{\pi}{h}\sum\sum\Psi_{1i}\Psi_{3j}\{-(3\alpha_{1i}^2\gamma_{ij1}+\alpha_{3j}^2\gamma_{ji1})+(\tfrac{3}{2}n_i\delta_{ij1}-\tfrac{1}{2}n_j\delta_{ji1})(\alpha_{1i}^2-\alpha_{3j}^2)\}$$
$$+\frac{\pi}{h}\sum\sum\Psi_{2i}\Psi_{2j}\{-2\alpha_{2i}^2\gamma_{ij1}+\alpha_{2i}^2(n_i\delta_{ij1}-n_j\delta_{ji1})\}$$
$$+\frac{\pi}{h}\sum\sum\Psi_{1i}\Psi_{5j}\{(5\alpha_{1i}^2\gamma_{ij1}-\alpha_{5i}^2\gamma_{ji1})-(\tfrac{5}{2}n_i\delta_{ij1}+\tfrac{1}{2}n_j\delta_{ji1})(\alpha_{1i}^2-\alpha_{5j}^2)\}$$
$$+\;.\;.\;.\;.\;.\;.\;.\;. \tag{B.4}$$

$$C_{11}=-\frac{\pi}{h}\sum\sum\gamma_{ij1}(\phi_{1i}\phi_{2j}+\phi_{2i}\phi_{3j}+\phi_{3i}\phi_{4j}+\phi_{4i}\phi_{5j}+\ldots) \tag{B.5}$$

$$C_{21}=-\frac{\pi}{h}\sum\sum\gamma_{ij1}(\phi_{1i}\phi_{1j}+2\phi_{1i}\phi_{3j}+2\phi_{2i}\phi_{4j}+2\phi_{3i}\phi_{5j}+\ldots) \tag{B.6}$$

$$C_{31}=-\frac{\pi}{h}\sum\sum\gamma_{ij1}(3\phi_{1i}\phi_{2j}+3\phi_{1i}\phi_{4j}+3\phi_{2i}\phi_{5j}+\ldots) \tag{B.7}$$

$$C_{41}=-\frac{\pi}{h}\sum\sum\gamma_{ij1}(4\phi_{1i}\phi_{3j}+2\phi_{2i}\phi_{2j}+4\phi_{1i}\phi_{5j}+\ldots) \tag{B.8}$$

$$D_{01}=\frac{\pi}{h}\sum\sum(n_i\delta_{ij1}+n_j\delta_{ji1})(\tfrac{1}{2}\phi_{1i}\Psi_{1j}+\phi_{2i}\Psi_{2j}+\tfrac{3}{2}\phi_{3i}\Psi_{3j}+\ldots) \tag{B.9}$$

$$D_{11}=\frac{\pi}{h}\sum\sum\{n_i\delta_{ij1}\phi_{0i}\Psi_{1j}+(n_i\delta_{ij1}+\tfrac{1}{2}n_j\delta_{ji1})\phi_{1i}\Psi_{2j}$$
$$+(\tfrac{1}{2}n_i\delta_{ij1}+n_j\delta_{ji1})\phi_{2i}\Psi_{1j}+(\tfrac{3}{2}n_i\delta_{ij1}+n_j\delta_{ji1})\phi_{2i}\Psi_{3j}$$
$$+(n_i\delta_{ij1}+\tfrac{3}{2}n_j\delta_{ji1})\phi_{3i}\Psi_{2j}+(2n_i\delta_{ij1}+\tfrac{3}{2}n_j\delta_{ji1})\phi_{3i}\Psi_{4j}$$
$$+(\tfrac{3}{2}n_i\delta_{ij1}+2n_j\delta_{ji1})\phi_{4i}\Psi_{3j}+(\tfrac{5}{2}n_i\delta_{ij1}+2n_j\delta_{ji1})\phi_{4i}\Psi_{5j}$$
$$+(2n_i\delta_{ij1}+\tfrac{5}{2}n_j\delta_{ji1})\phi_{5i}\Psi_{3j}+\ldots\} \tag{B.10}$$

$$D_{21}=\frac{\pi}{h}\sum\sum\{(n_i\delta_{ij1}-n_j\delta_{ji1})\phi_{1i}\Psi_{1j}+2n_i\delta_{ij1}\phi_{0i}\Psi_{2j}+(\tfrac{3}{2}n_i\delta_{ij1}+\tfrac{1}{2}n_j\delta_{ji1})\phi_{1i}\Psi_{3j}$$
$$+(\tfrac{1}{2}n_i\delta_{ij1}+\tfrac{3}{2}n_j\delta_{ji1})\phi_{3i}\Psi_{1j}+(2n_i\delta_{ij1}+n_j\delta_{ji1})\phi_{2i}\Psi_{4j}$$
$$+(n_i\delta_{ij1}+2n_j\delta_{ji1})\phi_{4i}\Psi_{2j}+(\tfrac{5}{2}n_i\delta_{ij1}+\tfrac{3}{2}n_j\delta_{ji1})\phi_{3i}\Psi_{5j}$$
$$+(\tfrac{3}{2}n_i\delta_{ij1}+\tfrac{5}{2}n_j\delta_{ji1})\phi_{5i}\Psi_{3j}+\ldots\} \tag{B.11}$$

$$D_{31} = \frac{\pi}{h}\sum\sum\Big\{(n_i\delta_{ij1} - \tfrac{1}{2}n_j\delta_{ji1})\,\phi_{1i}\Psi'_{2j} + (\tfrac{1}{2}n_i\delta_{ij1} - n_j\delta_{ji1})\,\phi_{2i}\Psi'_{1j}$$
$$+ 3n_i\delta_{ij1}\phi_{0i}\Psi'_{3j} + (2n_i\delta_{ij1} + \tfrac{1}{2}n_j\delta_{ji1})\,\phi_{1i}\Psi'_{4j}\Big\}$$
$$+ (\tfrac{1}{2}n_i\delta_{ij1} + 2n_j\delta_{ji1})\,\phi_{4i}\Psi'_{1j} + (\tfrac{5}{2}n_i\delta_{ij1} + n_j\delta_{ji1})\,\phi_{2i}\Psi'_{5j}$$
$$+ (n_i\delta_{ij1} + \tfrac{5}{2}n_j\delta_{ji1})\,\phi_{5i}\Psi'_{2j} + \dots\Big\} \qquad \text{(B.12)}$$

$$D_{41} = \frac{\pi}{2}\sum\sum\Big\{(\tfrac{3}{2}n_i\delta_{ij1} - \tfrac{1}{2}n_j\delta_{ji1})\,\phi_{1i}\Psi'_{3j} + (\tfrac{1}{2}n_i\delta_{ij1} - \tfrac{3}{2}n_j\delta_{ji1})\,\phi_{3i}\Psi'_{1j}$$
$$+ (n_i\delta_{ij1} - n_j\delta_{ji1})\,\phi_{2i}\Psi'_{2j} + 4n_i\delta_{ij1}\phi_{0i}\Psi'_{4j}$$
$$+ (\tfrac{5}{2}n_i\delta_{ij1} + \tfrac{1}{2}n_j\delta_{ji1})\,\phi_{1i}\Psi'_{5j} + (\tfrac{1}{2}n_i\delta_{ij1} + \tfrac{5}{2}n_j\delta_{ji1})\,\phi_{5i}\Psi'_{1j} + \dots\Big\} \qquad \text{(B.13)}$$

$$E_{11} = \frac{\pi}{h}\sum\sum\Big\{(\tfrac{1}{2}\Psi'_{1i}S_{2j} - \Psi'_{2i}S_{1j})\,(n_j^2\delta_{ji1} + n_in_j\delta_{ij1} - 2n_j\gamma_{ij1})$$
$$+ (\Psi'_{1i}S_{2j} - \tfrac{1}{2}\Psi'_{2i}S_{1j})\,(n_i^2\delta_{ji1} + n_in_j\delta_{ij1})$$
$$+ (\Psi'_{2i}S_{3j} - \tfrac{3}{2}\Psi'_{3i}S_{2j})\,(n_j^2\delta_{ji1} + n_in_j\delta_{ij1} - 2n_j\gamma_{ij1})$$
$$+ (\tfrac{3}{2}\Psi'_{2i}S_{3j} - \Psi'_{3i}S_{2j})\,(n_i^2\delta_{ji1} + n_in_j\delta_{ij1}) + \dots\Big\} \qquad \text{(B.14)}$$

$$E_{21} = \frac{\pi}{h}\sum\sum\Big\{\Psi'_{12}\cdot S_{1j}\cdot[\tfrac{1}{2}\delta_{ij1}(n_j^2 - n_i^2) - n_j\cdot\gamma_{ij1}]$$
$$+ (\tfrac{1}{2}\Psi'_{1i}S_{3j} - \tfrac{3}{2}\Psi'_{3i}S_{1j})\,(n_j^2\delta_{ji1} + n_in_j\delta_{ij1} - 2n_j\gamma_{ij1})$$
$$+ (\tfrac{3}{2}\Psi'_{1i}S_{3j} - \tfrac{1}{2}\Psi'_{3i}S_{1j})\,(n_i^2\delta_{ji1} + n_in_j\delta_{ij1})$$
$$+ (\Psi'_{2i}S_{4j} - 2\Psi'_{4i}S_{2j})\,(n_j^2\delta_{ji1} + n_in_j\delta_{ij1} - 2n_j\gamma_{ij1})$$
$$+ (2\Psi'_{2i}S_{4j} - \Psi'_{4i}S_{2j})\,(n_i^2\delta_{ji1} + n_in_j\delta_{ij1}) + \dots\Big\} \qquad \text{(B.15)}$$

$$E_{31} = \frac{\pi}{h}\sum\sum\Big\{(\tfrac{1}{2}\Psi'_{1j}S_{2j} + \Psi'_{2j}S_{1j})\,(n_j^2\,\delta_{ji1} + n_in_j\,\delta_{ij1} - 2n_j\gamma_{ij1})$$
$$- (\Psi'_{1i}S_{2j} + \tfrac{1}{2}\Psi'_{2i}S_{1j})\,(n_i^2\delta_{ji1} + n_in_j\delta_{ij1}) + (\tfrac{1}{2}\Psi'_{1i}S_{4j} - 2\Psi'_{4i}S_{1j})$$
$$\times (n_j^2\delta_{ji} + n_in_j\delta_{ij} - 2n_j\delta_{ij1}) + (2\Psi'_{1i}S_{4j} - \tfrac{1}{2}\Psi'_{4i}S_{1j})\,(n_i^2\delta_{ji} + n_in_j\delta_{ij})$$
$$+ (\Psi'_{2i}S_{5j} - \tfrac{5}{2}\Psi'_{5i}S_{2j})\,(n_j^2\delta_{ji1} + n_in_j\delta_{ij1} - 2n_j\gamma_{ij1})$$
$$+ (\tfrac{5}{2}\Psi'_{2i}S_{5j} - \Psi'_{5i}S_{2j})\,(n_i^2\delta_{ji1} + n_in_j\delta_{ij1}) + \dots\Big\} \qquad \text{(B.16)}$$

$$E_{41} = \frac{\pi}{h}\sum\sum\Big\{(\tfrac{1}{2}\Psi'_{1i}S_{3j} + \tfrac{3}{2}\Psi'_{3i}S_{1j})\,(n_j^2\delta_{ij1} + n_in_j\delta_{ij1} - 2n_j\gamma_{ij1})$$
$$- (\tfrac{3}{2}\Psi'_{1i}S_{3j} + \tfrac{1}{2}\Psi'_{3i}S_{1j})\,(n_i^2\delta_{ij1} + n_in_j\delta_{ij1})$$
$$+ \Psi'_{2i}S_{2j}[(n_j^2 - n_i^2)\,\delta_{ji1} - 2n_j\gamma_{ij1}]$$
$$+ (\tfrac{1}{2}\Psi'_{1i}S_{5j} - \tfrac{5}{2}\Psi'_{5i}S_{1j})\,(n_j^2\delta_{ji1} + n_in_j\delta_{ij1} - 2n_j\delta_{ij1})$$
$$+ (\tfrac{5}{2}\Psi'_{1i}S_{5j} - \tfrac{1}{2}\Psi'_{5i}S_{1j})\,(n_i^2\delta_{ji1} + n_in_j\delta_{ij1}) + \dots\Big\} \qquad \text{(B.17)}$$